Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2024, 12, 3114-3134
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Multiple Solutions for a Class of Singular Boundary Value Problems of Hadamard Fractional Differential Systems with *p*-Laplacian Operator

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How to cite this paper: Wang, C. and Liu, Y.S. (2024) Multiple Solutions for a Class of Singular Boundary Value Problems of Hadamard Fractional Differential Systems with *p*-Laplacian Operator. *Journal of Applied Mathematics and Physics*, **12**, 3114-3134.
https://doi.org/10.4236/jamp.2024.129187

Received: July 30, 2024
Accepted: September 17, 2024
Published: September 20, 2024

Abstract

This paper discusses the existence and multiplicity of positive solutions for a class of singular boundary value problems of Hadamard fractional differential systems involving the *p*-Laplacian operator. First, for the sake of overcoming the singularity, sequences of approximate solutions to the boundary value problem are obtained by applying the fixed point index theory on the cone. Next, it is demonstrated that these sequences of approximate solutions are uniformly bounded and equicontinuous. The main results are then established through the Ascoli-Arzelà theorem. Ultimately, an instance is worked out to test and verify the validity of the main results.

Keywords

Multiple Solutions, Fixed Point Index Theory, Nonlinear Fractional Differential Systems, Hadamard Fractional Derivative

1. Introduction

During the last few decades, fractional calculus has garnered significant attention. In particular, the study of fractional differential systems has become increasingly popular and important due to its widespread applications in various fields of science and engineering. So, numerous monographs have been published focusing on the fractional differential systems [1]-[3]. Moreover, it is well acknowledged that Leibenson [4] introduced the following differential equation with the *p*-Laplacian operator to investigate the turbulent flow in a porous medium

$$\left(\phi\left(u'(t)\right)\right)' = f\left(t,u(t)\right).$$

This has led to a surge of interest in boundary value problems associated with the p-Laplacian, owing to their implications in both theoretical and practical aspects of mathematics and physics [5]. Recently, lots of scholars discussed the existence and multiplicity of solutions for boundary value problems of fractional differential equation with p-Laplacian operator. For details, see [6] [7].

As we know, the Riemann-Liouville derivative stands out as one of the earliest fractional derivatives which have attracted considerable attention [8] [9]. For instance, [10] investigated the following Riemann-Liouville fractional differential system with integral boundary conditions:

$$\begin{cases} D_{0^+}^{\vartheta_1}\left(\phi_p\left(D_{0^+}^{\vartheta_2}u(t)\right)\right) = f\left(t,u(t),D_{0^+}^{\vartheta_2}u(t)\right), t\in(0,1); \\ D_{0^+}^{\vartheta_1-2}\left(\phi_p\left(D_{0^+}^{\vartheta_2}u(0)\right)\right) = D_{0^+}^{\vartheta_1-1}\left(\phi_p\left(D_{0^+}^{\vartheta_2}u(1)\right)\right) = \int_0^1 g(s)\phi_p\left(D_{0^+}^{\vartheta_2}u(s)\right)\mathrm{d}s; \\ D_{0^+}^{\vartheta_2}u(0) = 0, u(0) = 0, D_{0^+}^{\vartheta_2-2}u(0) = D_{0^+}^{\vartheta_2-1}u(1) = \int_0^1 h(s)u(s)\mathrm{d}s, \end{cases}$$

where $2<\vartheta_1,\vartheta_2\le 3$, $5<\vartheta_1+\vartheta_2\le 6$, $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville fractional derivative of order α, and $f\in C\left([0,1]\times\mathbb{R}^+\times\mathbb{R},\mathbb{R}^+\right)$. By applying the properties of Green's function and Bai-Ge's fixed point theorem, the existence of multiple positive solutions was obtained.

Another type of fractional derivative, known as the Hadamard fractional derivative, was proposed by Jacques Hadamard [11] in 1982. It differs from Riemann-Liouville derivative by incorporating a logarithmic function in the integral kernel. Recently, a great number of works focused on fractional differential systems with Hadamard fractional derivatives. For example, in [12], Alesem studied the following fractional differential system with Hadamard fractional derivative:

$$\begin{cases} {}^H D_{1^+}^{\beta_1}\left(\phi_p\left({}^H D_{1^+}^{\alpha_1}u(t)\right)\right) = \lambda^{p-1}f\left(t,u(t),v(t),w(t)\right), t\in(1,e); \\ {}^H D_{1^+}^{\beta_2}\left(\phi_p\left({}^H D_{1^+}^{\alpha_2}v(t)\right)\right) = \mu^{p-1}g\left(t,u(t),v(t),w(t)\right), t\in(1,e); \\ {}^H D_{1^+}^{\beta_3}\left(\phi_p\left({}^H D_{1^+}^{\alpha_3}w(t)\right)\right) = \nu^{p-1}h\left(t,u(t),v(t),w(t)\right), t\in(1,e), \end{cases}$$

subject to the following boundary conditions:

$$\begin{cases} u^{(j)}(1) = 0, 0\le j\le n-2, \mu_1 u^{p_1}(e) = \lambda_1 u^{p_1}(\xi); \\ \phi_p\left({}^H D_{1^+}^{\alpha_1}u(1)\right) = 0 = {}^H D_{1^+}^{p_2}\left(\phi_p\left({}^H D_{1^+}^{\alpha_1}u(e)\right)\right); \\ v^{(j)}(1) = 0, 0\le j\le m-2, \mu_1 u^{q_1}(e) = \lambda_1 u^{q_1}(\xi); \\ \phi_p\left({}^H D_{1^+}^{\alpha_2}u(1)\right) = 0 = {}^H D_{1^+}^{q_2}\left(\phi_p\left({}^H D_{1^+}^{\alpha_2}u(e)\right)\right); \\ w^{(j)}(1) = 0, 0\le j\le l-2, \mu_1 w^{r_1}(e) = \lambda_1 w^{r_1}(\xi); \\ \phi_p\left({}^H D_{1^+}^{\alpha_3}u(1)\right) = 0 = {}^H D_{1^+}^{r_2}\left(\phi_p\left({}^H D_{1^+}^{\alpha_3}w(e)\right)\right), \end{cases}$$

where $n-1<\alpha_1\le n$, $m-1<\alpha_2\le m$, $l-1<\alpha_3\le l$, $n,m,l\in\mathbb{N}$, $n,m,l\ge 3$, $\beta_i\in(1,2]$, $i=1,2,3$, $1\le p_1<\alpha_1-1$, $1\le q_1<\alpha_2-1$, $1\le r_1<\alpha_3-1$,

$p_2, q_2, r_2 \in (1,2]$, and $p_2, q_2, r_2 < \beta_i - 1$. $\lambda_1, \mu_1 > 0$ and $\xi \in (1,e)$ are constants. ${}^H D_{1^+}^{\alpha}$ denotes the Hadamard fractional derivative of order α. Several existence results of positive solutions for the above problem were studied by means of Guo-Krasnosel'skii fixed point theorem on cones.

Furthermore, singular boundary value problems are a classic branch of research and arise from various fields of thermodynamics, fluid mechanics, biomathematics, and chemistry. These singular boundary problems involve time singularities and nonlinearities having singularities in space variables. In recent years, many important results have been obtained based on theoretical developments and practical applications [13]-[16]. For instance, Wang [17] discussed a class of singular fractional differential systems with nonlocal boundary value conditions:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \ t \in (0,1); \\ u(0) = u'(0) = \cdots = u^{n-2}(0) = 0; \\ D_{0^+}^{\beta} u(1) = \int_1^{\eta} a(t) D_{0^+}^{\gamma} u(t) \mathrm{d}V(t), \end{cases}$$

where $n-1 < \alpha \le n$ $(n \ge 3)$, $0 < \beta \le 1$, $0 \le \gamma \le \alpha - 1$, $\eta \in (0,1]$, $a(\cdot) \in L^1[0,1] \cap C(0,1)$ is nonnegative, $\int_1^{\eta} a(t) t^{\alpha-\gamma-1} \mathrm{d}V(t)$ denotes the Riemann-Stieltjes integral, and V is a bounded variation function. $f(t,u)$ may be singular at $t = 0,1$ and $u = 0$. By applying the fixed point index theory, the existence and multiplicity of positive solutions were attained.

From all aforementioned analysis, we find the fact that there exist several unresolved issues presented in the field. For example, there exist few papers focusing on singular boundary value problems of Hadamard fractional differential systems involving p-Laplacian operator. Moreover, the existing literature predominantly focuses on the existence and multiplicity of nontrivial positive solutions of fractional differential systems, neglecting a broader spectrum of potential scenarios. We have also observed a gap in that no papers concern singular boundary problems of Hadamard fractional differential systems when there are some connections between the nonlinear terms in the present paper.

Motivated by the above, we discuss in this paper the existence of multiple solutions for singular boundary value problems (SBVP, for short) of Hadamard fractional differential systems with p-Laplacian operator as follows:

$$\begin{cases} {}^H D_{1^+}^{\beta_1}\left(\phi_p\left({}^H D_{1^+}^{\alpha_1} u(t)\right)\right) = f(t, u(t), v(t)), t \in (1,e); \\ {}^H D_{1^+}^{\beta_2}\left(\phi_p\left({}^H D_{1^+}^{\alpha_2} v(t)\right)\right) = g(t, u(t), v(t)), t \in (1,e); \\ u^{(j)}(1) = 0, 0 \le j \le n-2, \delta u(e) = \lambda_1 \delta u(\xi), \phi_p\left({}^H D_{1^+}^{\alpha_1} u(1)\right) = 0; \\ {}^H D_{1^+}^{\beta_1 - 2}\left(\phi_p\left({}^H D_{1^+}^{\alpha_1} u(1)\right)\right) = {}^H D_{1^+}^{\beta_1 - 1}\left(\phi_p\left({}^H D_{1^+}^{\alpha_1} u(e)\right)\right) = \int_1^e m_1(s) \phi_p\left({}^H D_{1^+}^{\alpha_1} u(s)\right) \frac{\mathrm{d}s}{s}; \\ v^{(j)}(1) = 0, 0 \le j \le m-2, \delta v(e) = \lambda_1 \delta v(\xi), \phi_p\left({}^H D_{1^+}^{\alpha_1} v(1)\right) = 0; \\ {}^H D_{1^+}^{\beta_2 - 2}\left(\phi_p\left({}^H D_{1^+}^{\alpha_2} v(1)\right)\right) = {}^H D_{1^+}^{\beta_2 - 1}\left(\phi_p\left({}^H D_{1^+}^{\alpha_2} v(e)\right)\right) = \int_1^e m_2(s) \phi_p\left({}^H D_{1^+}^{\alpha_2} v(s)\right) \frac{\mathrm{d}s}{s}, \end{cases} \tag{1.1}$$

where $n-1<\alpha_1\le n$, $m-1<\alpha_2\le m$, $n,m\in N$, $n,m\ge 3$, $\lambda_1\in(0,+\infty)$ and $\xi\in(1,e)$ are constants. ${}^{H}D_{1^+}^{\alpha}$ denotes the Hadamard fractional derivative of order α. $\phi_p(t)=|t|^{p-2}t$, $p>1$, $\phi_q=\phi_p^{-1}$, and $\frac{1}{p}+\frac{1}{q}=1$.

$m_1,m_2\in C([1,e],[0,+\infty))$, $\delta:=t\frac{\mathrm{d}}{\mathrm{d}t}$, and $I=[1,e]$.

$f,g\in C[(1,e)\times(0,+\infty)\times\mathbb{R},\mathbb{R}]$ may be singular at $t=1,e$ and $u=0$.

This manuscript has the following novel characteristics. Firstly, SBVP (1.1) is of a Hadamard fractional differential system equipping with the *p*-Laplacian operator. Secondly, the existence and multiplicity of solutions to SBVP (1.1) are considered in this manuscript when f and g are singular at $t=1,e$, and $u=0$. Thirdly, a relatively new cone is constructed to deal with the singularity of SBVP (1.1). Finally, in the resulting solution (u,v), the component u is positive, but the component v is permitted to exhibit varying signs, potentially even negative.

The rest of the present work is organized as follows. Section 2 gives some fundamental definitions and necessary lemmas. In Section 3, we investigate the existence of multiple solutions for SBVP (1.1) with the help of the properties of Green's function and the theory of fixed point index on the cone. Finally, in Section 4, a specific example is presented to support the main results.

2. Preliminaries

We begin this section by introducing some basic definitions and lemmas from fractional calculus theory.

Definition 2.1. ([18]) The left-sided Hadamard fractional integral of order $\alpha\in\mathbb{R}^+$ for a function $u:[a,+\infty)\to\mathbb{R}$ ($a\ge 0$) is defined as follows:

$$\left({}^{H}I_{a^+}^{\alpha}u\right)(t)=\frac{1}{\Gamma(\alpha)}\int_a^t(\ln t-\ln s)^{\alpha-1}\frac{u(s)}{s}\mathrm{d}s,\quad t\in[1,+\infty),$$

where $\Gamma(\cdot)$ is a gamma function.

Definition 2.2. ([18]) The left-sided Hamadard fractional derivative of order $\alpha\in\mathbb{R}^+$ for a function $u:[a,+\infty)\to\mathbb{R}\,(a\geqslant 0)$ is defined as follows:

$$\left({}^{H}D_{a^+}^{\alpha}u\right)(t)=\left(t\frac{\mathrm{d}}{\mathrm{d}t}\right)^n\frac{1}{\Gamma(n-\alpha)}\int_a^t(\ln t-\ln s)^{n-\alpha-1}\frac{u(s)}{s}\mathrm{d}s,\quad t\in[1,+\infty),$$

where $n=[\alpha]+1$, and $[\alpha]$ denotes integer part of number α.

Lemma 2.1. ([18]) Let $\alpha>0$, $u\in C[1,e]\cap L^1[1,e]$. Then the fractional differential equation ${}^{H}D_{a^+}^{\alpha}u(t)=0$ has a unique solutions

$$u(t)=c_1(\ln t)^{\alpha-1}+c_2(\ln t)^{\alpha-2}+\cdots+c_n(\ln t)^{\alpha-n},$$

where c_i is arbitrary constants, $n=[\alpha]+1$, $i=1,2,\cdots,n$.

Lemma 2.2. ([18]) Let $\alpha>0$, ${}^{H}I_{a^+}^{\alpha}$ be a Hadamard fractional derivative of order α. Assume that $u\in C[1,e]\cap L^1[1,e]$. Then

$${}^{H}I_{a^+}^{\alpha}\,{}^{H}D_{a^+}^{\alpha}u(t)=u(t)+c_1(\ln t)^{\alpha-1}+c_2(\ln t)^{\alpha-2}+\cdots+c_n(\ln t)^{\alpha-n},$$

where c_i is arbitrary constants, $n=[\alpha]+1$, $i=1,2,\cdots,n$.

Lemma 2.3. ([18]) Let $\alpha > 0, \beta > 0$ with $\alpha < \beta$, then the following equations hold:

$$\left({}^H I_{a^+}^{\alpha} (\ln t)^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln t)^{\beta+\alpha-1},$$

$$\left({}^H D_{a^+}^{\alpha} (\ln t)^{\beta-1}\right) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln t)^{\beta-\alpha-1},$$

in particular, $\left({}^H D_{a^+}^{\alpha}\right)(\ln t)^{\alpha-j} = 0$, $j = 1,2,\cdots,[\alpha]+1$.

Lemma 2.4. ([18]) Let $\alpha > 0$ and $u \in L^p(1,e)$, then

$${}^H D_{a^+}^{\alpha}\, {}^H I_{a^+}^{\alpha} u(t) = u(t).$$

Lemma 2.5. (Ascoli-Arzelà theorem [19]) $H \subset C[J,E]$ is relative compact if and only if H is equicontinuous, and for any $t \in J$, $H(t)$ is a relatively compact set in E.

Lemma 2.6. ([19]) Let X be a Banach space and P be a cone in X. Denote $P_r = \{u \in P : \|u\| < r\}$ and $\partial P_r = \{u \in P : \|u\| = r\}$ ($\forall r > 0$). Let $A: P \to P$ be a complete continuous mapping, then the following conclusions hold.

1) If $\mu Au \neq u$ for $u \in \partial P_r$ and $\mu \in (0,1]$, then $i(A, \partial P_r, P) = 1$.

2) If $\inf_{u \in \partial P_r} \|u\| > 0$ and $\mu Au \neq u$ for $u \in \partial P_r$ and $\mu \geq 1$, then $i(A, \partial P_r, P) = 1$.

3. Main Results

In this section, the solutions for SBVP (1.1) are proved to exist and be multiple by means of fixed point index theory. For this sake, we first demonstrate the properties of the Green's function linked to SBVP (1.1).

Lemma 3.1. Assume that $\Delta_1 = 1 - \lambda_1 (\ln \xi)^{\alpha_1 - 2} > 0$, $k \in C[1,e]$. Then the problem

$$\begin{cases} {}^H D_{1^+}^{\alpha_1} u(t) + k(t) = 0, \ t \in (1,e); \\ u^{(j)}(1) = 0, \ 0 \leq j \leq n-2; \\ \delta u(e) = \lambda_1 \delta u(\xi) \end{cases} \tag{3.1}$$

is equivalent to

$$u(t) = \int_1^e G_1(t,s) k(s) \frac{\mathrm{d}s}{s}, \ t \in (1,e),$$

where

$$G_1(t,s) = G_{11}(t,s) + \frac{\lambda_1 (\ln t)^{\alpha_1-1}}{\Delta_1} G_{12}(\xi,s), \tag{3.2}$$

$$G_{11}(t,s) = \frac{1}{\Gamma(\alpha_1)} \begin{cases} (\ln t)^{\alpha_1-1} (1-\ln s)^{\alpha_1-2} - (\ln t - \ln s)^{\alpha_1-1}, & 1 \leq s \leq t \leq e; \\ (\ln t)^{\alpha_1-1} (1-\ln s)^{\alpha_1-2}, & 1 \leq t \leq s \leq e, \end{cases}$$

$$G_{12}(t,s) = \frac{1}{\Gamma(\alpha_1)} \begin{cases} (\ln t)^{\alpha_1-2} (1-\ln s)^{\alpha_1-2} - (\ln t - \ln s)^{\alpha_1-2}, & 1 \leq s \leq t \leq e; \\ (\ln t)^{\alpha_1-2} (1-\ln s)^{\alpha_1-2}, & 1 \leq t \leq s \leq e. \end{cases}$$

Proof. Through the utilization of *Lemma 2.2*, (3.1) can be written as the following

$$u(t) = -{}^{H}I_{1^+}^{\alpha_1} k(t) + c_1 (\ln t)^{\alpha_1 - 1} + c_2 (\ln t)^{\alpha_1 - 2} + \cdots + c_n (\ln t)^{\alpha_1 - n}$$
$$= -\frac{1}{\Gamma(\alpha_1)} \int_1^t (\ln t - \ln s)^{\alpha_1 - 1} k(s) \frac{ds}{s} + c_1 (\ln t)^{\alpha_1 - 1} + \cdots + c_n (\ln t)^{\alpha_1 - n},$$

where $c_i \in \mathbb{R}, i = 1, 2, \cdots, n$. The boundary condition $u^{(j)}(1) = 0$ ($0 \le j \le n-2$) implies $c_2 = c_3 = \cdots = c_n = 0$. Therefore,

$$u(t) = -\frac{1}{\Gamma(\alpha_1)} \int_1^t (\ln t - \ln s)^{\alpha_1 - 1} k(s) \frac{ds}{s} + c_1 (\ln t)^{\alpha_1 - 1}.$$

By the condition $\delta u(e) = \lambda_1 \delta u(\xi)$, we have

$$c_1 = \frac{1}{\Gamma(\alpha_1)\Delta_1} \left\{ \int_1^e (1 - \ln s)^{\alpha_1 - 2} k(s) \frac{ds}{s} - \lambda_1 \int_1^\xi (\ln \xi - \ln s)^{\alpha_1 - 2} k(s) \frac{ds}{s} \right\}.$$

Hence, the unique solution of (3.1) is

$$\begin{aligned} u(t) &= -\frac{1}{\Gamma(\alpha_1)} \int_1^t (\ln t - \ln s)^{\alpha_1 - 1} k(s) \frac{ds}{s} + \frac{(\ln t)^{\alpha_1 - 1}}{\Delta_1 \Gamma(\alpha_1)} \int_1^e (1 - \ln s)^{\alpha_1 - 2} k(s) \frac{ds}{s} \\ &\quad - \frac{\lambda_1 (\ln t)^{\alpha_1 - 1}}{\Delta_1 \Gamma(\alpha_1)} \int_1^\xi (\ln \xi - \ln s)^{\alpha_1 - 2} k(s) \frac{ds}{s} \\ &= -\frac{1}{\Gamma(\alpha_1)} \int_1^t (\ln t - \ln s)^{\alpha_1 - 1} k(s) \frac{ds}{s} + \frac{(\ln t)^{\alpha_1 - 1}}{\Gamma(\alpha_1)} \int_1^e (1 - \ln s)^{\alpha_1 - 2} k(s) \frac{ds}{s} \\ &\quad - \frac{\lambda_1 (\ln t)^{\alpha_1 - 1}}{\Delta_1 \Gamma(\alpha_1)} \int_1^\xi (\ln \xi - \ln s)^{\alpha_1 - 2} k(s) \frac{ds}{s} \\ &\quad + \frac{\lambda_1 (\ln t)^{\alpha_1 - 1}}{\Delta_1 \Gamma(\alpha_1)} \int_1^e (\ln \xi)^{\alpha_1 - 2} (1 - \ln s)^{\alpha_1 - 2} k(s) \frac{ds}{s} \\ &= \int_1^e G_1(t, s) k(s) \frac{ds}{s}. \end{aligned}$$

The proof is finished. □

Let $\alpha_1 = \frac{5}{2}$, $\lambda = 1$ and $\xi = \frac{3}{2}$, the representation of the Green's function $G_1(t, s)$ is exhibited by **Figure 1**.

Lemma 3.2. Assume that $\Delta_2 = 1 - \frac{1}{\Gamma(\beta_1)} \int_1^e \left((\ln s)^{\beta_1 - 1} + (\ln s)^{\beta_1 - 2} (\beta_1 - 1) \right) m_1(s) \frac{ds}{s} > 0$, $h \in C[1, e]$. Then the problem

$$\begin{cases} {}^{H}D_{1^+}^{\beta_1} \left(\phi_p \left({}^{H}D_{1^+}^{\alpha_1} u(t) \right) \right) = h(t), \ t \in (1, e); \\ u^{(j)}(1) = 0, 0 \le j \le n-2, \delta u(e) = \lambda_1 \delta u(\xi), \phi_p \left({}^{H}D_{1^+}^{\alpha_1} u(1) \right) = 0; \\ {}^{H}D_{1^+}^{\beta_1 - 2} \left(\phi_p \left({}^{H}D_{1^+}^{\alpha_1} u(1) \right) \right) = {}^{H}D_{1^+}^{\beta_1 - 1} \left(\phi_p \left({}^{H}D_{1^+}^{\alpha_1} u(e) \right) \right) = \int_1^e m_1(s) \phi_p \left({}^{H}D_{1^+}^{\alpha_1} u(s) \right) \frac{ds}{s} \end{cases} \tag{3.3}$$

is equivalent to

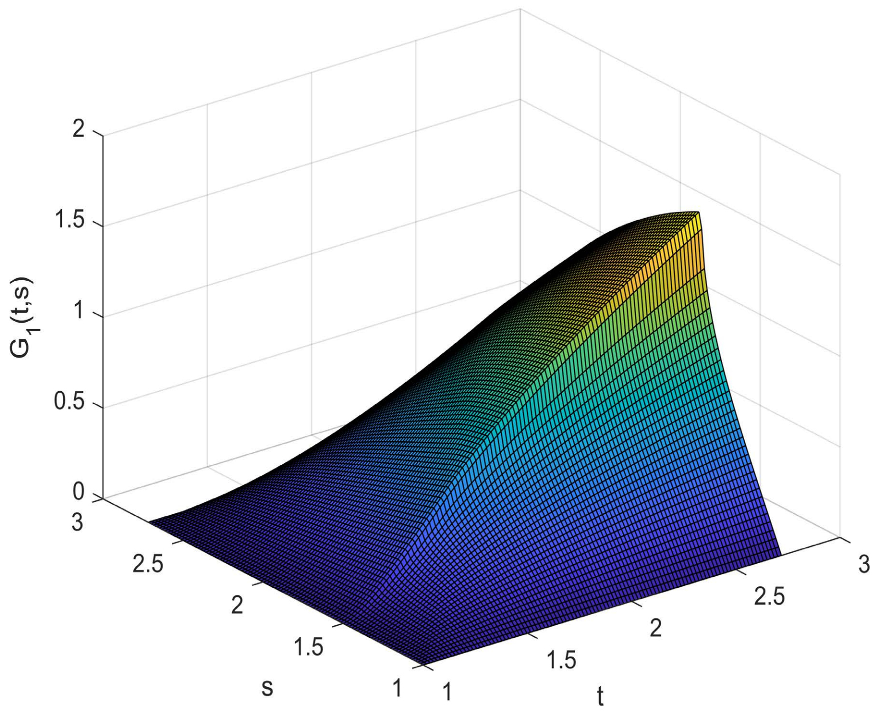

Figure 1. $G_1(t,s)$ for $\alpha=\frac{5}{2}$.

$$u(t)=\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)h(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s},\ t\in(1,e),$$

where $G_1(t,s)$ is defined by (3.2),

$$H_1(t,s)=H_{11}(t,s)+\frac{(\ln t)^{\beta_1-1}+(\ln t)^{\beta_1-2}(\beta_1-1)}{\Delta_2\Gamma(\beta_1)}\int_1^e H_{11}(t,s)m_1(t)\frac{\mathrm{d}t}{t},\quad (3.4)$$

$$H_{11}(t,s)=\frac{1}{\Gamma(\beta_1)}\begin{cases}(\ln t)^{\beta_1-1}-(\ln t-\ln s)^{\beta_1-1}, & 1\le s\le t\le e;\\ (\ln t)^{\beta_1-1}, & 1\le t\le s\le e.\end{cases}$$

Proof. Applying the Hadamard fractional integral of order β_1 to both sides of

$${}^HD_{1^+}^{\beta_1}\left(\phi_p\left({}^HD_{1^+}^{\alpha_1}u(t)\right)\right)=h(t),$$

it follows from *Lemma 2.2* that

$$\phi_p\left({}^HD_{1^+}^{\alpha_1}u(t)\right)={}^HI_{1^+}^{\beta_1}h(t)+d_1(\ln t)^{\beta_1-1}+d_2(\ln t)^{\beta_1-2}+d_3(\ln t)^{\beta_1-3}.$$

This together with $\phi_p\left({}^HD_{1^+}^{\alpha_1}u(1)\right)=0$ means $d_3=0$. Consequently,

$$\phi_p\left({}^HD_{1^+}^{\alpha_1}u(t)\right)={}^HI_{1^+}^{\beta_1}h(t)+d_1(\ln t)^{\beta_1-1}+d_2(\ln t)^{\beta_1-2}.\quad (3.5)$$

In light of *Lemma 2.3*, it follows from boundary conditions of (3.3) that

$$c_1=-\frac{1}{\Gamma(\beta_1)}\int_1^e h(s)\frac{\mathrm{d}s}{s}+\frac{1}{\Gamma(\beta_1)}\int_1^e m_1(s)\phi_p\left({}^HD_{1^+}^{\alpha_1}u(s)\right)\frac{\mathrm{d}s}{s}.\quad (3.6)$$

$$c_2=\frac{1}{\Gamma(\beta_1-1)}\int_1^e m_1(s)\phi_p\left({}^H D_{1^+}^{\alpha_1}u(s)\right)\frac{ds}{s}. \tag{3.7}$$

Putting (3.6) and (3.7) into (3.5), one attains

$$\begin{aligned}\phi_p\left({}^H D_{1^+}^{\alpha_1}u(t)\right)&=\frac{1}{\Gamma(\beta_1)}\int_1^t(\ln t-\ln s)^{\beta_1-1}h(s)\frac{ds}{s}\\&\quad-\frac{(\ln t)^{\beta_1-1}}{\Gamma(\beta_1)}\left[\int_1^e h(s)\frac{ds}{s}-\int_1^e m_1(s)\phi_p\left({}^H D_{1^+}^{\alpha_1}u(s)\right)\frac{ds}{s}\right]\\&\quad+\frac{(\ln t)^{\beta_1-2}}{\Gamma(\beta_1-1)}\int_1^e m_1(s)\phi_p\left({}^H D_{1^+}^{\alpha_1}u(s)\right)\frac{ds}{s}\\&=-\int_1^e H_{11}(t,s)h(s)\frac{ds}{s}\\&\quad+\frac{(\ln t)^{\beta_1-1}+(\ln t)^{\beta_1-2}(\beta_1-1)}{\Gamma(\beta_1)}\int_1^e m_1(s)\phi_p\left({}^H D_{1^+}^{\alpha_1}u(s)\right)\frac{ds}{s},\end{aligned}$$

where

$$\begin{aligned}&\int_1^e m_1(s)\phi_p\left({}^H D_{1^+}^{\alpha_1}u(s)\right)\frac{ds}{s}\\&=-\int_1^e m_1(s)\int_1^e H_{11}(s,\tau)h(\tau)\frac{d\tau}{\tau}\frac{ds}{s}\\&\quad+\frac{(\ln\tau)^{\beta_1-1}+(\ln\tau)^{\beta_1-2}(\beta_1-1)}{\Gamma(\beta_1)}\int_1^e m_1(\tau)\phi_p\left({}^H D_{1^+}^{\alpha_1}u(\tau)\right)\frac{d\tau}{\tau}\frac{ds}{s}.\end{aligned}$$

By simple calculations, the following result can be obtained

$$\int_1^e m_1(s)\phi_p\left({}^H D_{1^+}^{\alpha_1}u(s)\right)\frac{ds}{s}=\frac{1}{\Delta_2}\int_1^e m_1(s)\left[-\int_1^e H_{11}(s,\tau)h(\tau)\frac{d\tau}{\tau}\right]\frac{ds}{s}.$$

As a result,

$$\begin{aligned}&\phi_p\left({}^H D_{1^+}^{\alpha_1}u(t)\right)\\&=-\int_1^e H_{11}(t,s)h(s)\frac{ds}{s}\\&\quad+\frac{(\ln t)^{\beta_1-1}+(\ln t)^{\beta_1-2}(\beta_1-1)}{\Gamma(\beta_1)\Delta_2}\int_1^e\left[-\int_1^e H_{11}(s,\tau)m_1(\tau)\frac{d\tau}{\tau}\right]h(s)\frac{ds}{s}\\&=-\int_1^e H_1(t,s)h(s)\frac{ds}{s}.\end{aligned}$$

Finally, combining with *Lemma 3.1*, the unique solution of (3.3) is as follows:

$$u(t)=\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)h(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}.$$

The proof is completed. □

Let $\beta_1=\frac{5}{2}$ and $m_1(t)=\frac{1}{2}$, the representation of the Green's function $H_1(t,s)$ is exhibited by **Figure 2**.

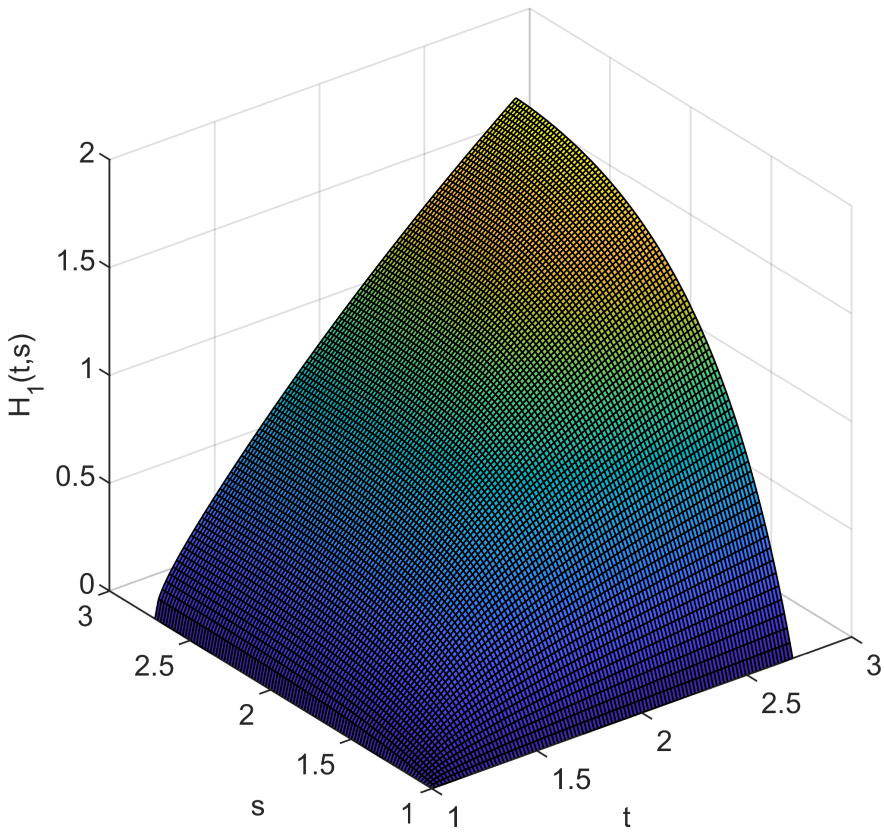

Figure 2. $H_1(t,s)$ for $\beta=\frac{5}{2}$.

Lemma 3.3. Let $G_1(t,s)$ be defined by (3.2), then

1) $G_1(t,s)$ is continuous on $[1,e]\times[1,e]$;

2) $G_1(t,s)\ge 0$, for $t,s\in[1,e]$;

3) $G_1(t,s)\le G_1(e,s)$, for $t,s\in[1,e]$;

4) $G(t,s)\ge(\ln t)^{\alpha_1-1}G_1(e,s)$, for $t,s\in[1,e]$.

Proof. Firstly, it is obvious that (1) holds. Now, we prove that (2) is true. It is easy to check that $G_1(t,s)\ge 0$ for $1\le t\le s\le e$. If $1\le s\le t\le e$, then

$$\begin{aligned}G_{11}(t,s)&=\frac{1}{\Gamma(\alpha_1)}\left[(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-2}-(\ln t-\ln s)^{\alpha_1-1}\right]\\&=\frac{(\ln t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[(1-\ln s)^{\alpha_1-2}-\left(1-\frac{\ln s}{\ln t}\right)^{\alpha_1-1}\right]\\&\ge\frac{(\ln t)^{\alpha_1-1}}{\Gamma(\alpha_1)}\left[(1-\ln s)^{\alpha_1-2}-(1-\ln s)^{\alpha_1-1}\right]\\&\ge 0.\end{aligned}$$

Similarly, $G_{12}(\xi,s)\ge 0$ for $1\le s\le\xi\le e$. The previous analysis together with (3.2) means that $G_1(t,s)\ge 0$ for $1\le s\le t\le e$.

In addition, it is evident to see that $\frac{\partial G_{11}(t,s)}{\partial t}\ge 0$ through simple calculations for $1 \le t \le s \le e$. If $1 \le s \le t \le e$, we have

$$\begin{aligned}\frac{\partial G_{11}(t,s)}{\partial t} &= \frac{\alpha_1 - 1}{t\Gamma(\alpha_1)}\left[(\ln t)^{\alpha_1-2}(1-\ln s)^{\alpha_1-2} - (\ln t - \ln s)^{\alpha_1-2}\right] \\ &\ge \frac{(\alpha_1-1)(\ln t)^{\alpha_1-2}}{t\Gamma(\alpha_1)}\left[(1-\ln s)^{\alpha_1-2} - (1-\ln s)^{\alpha_1-2}\right] \\ &= 0.\end{aligned}$$

Besides, for $s \le \xi$,

$$\begin{aligned}G_{12}(\xi,s) &= \frac{1}{\Gamma(\alpha_1)}\left[(\ln \xi)^{\alpha_1-2}(1-\ln s)^{\alpha_1-2} - (\ln \xi - \ln s)^{\alpha_1-2}\right] \\ &\ge \frac{(\ln \xi)^{\alpha_1-2}}{\Gamma(\alpha_1)}\left[(1-\ln s)^{\alpha_1-2} - (1-\ln s)^{\alpha_1-2}\right] \\ &\ge 0.\end{aligned}$$

So,

$$\frac{\partial G_1(t,s)}{\partial t} = \frac{\partial G_{11}(t,s)}{\partial t} + \frac{\lambda_1(\alpha_1-1)(\ln t)^{\alpha_1-2}}{t\Delta_1}G_{12}(\xi,s) \ge 0,$$

which implies that $G_1(\cdot,s)$ is nondecreasing on $[1,e]$. That is to say, (3) holds.

Let $1 \le t \le s \le e$, then

$$\frac{G_{11}(t,s)}{G_{11}(e,s)} = \frac{(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-2}}{(1-\ln s)^{\alpha_1-2}} = (\ln t)^{\alpha_1-1}.$$

If $1 \le s \le t \le e$, we have

$$\begin{aligned}\frac{G_{11}(t,s)}{G_{11}(e,s)} &= \frac{(\ln t)^{\alpha_1-1}(1-\ln s)^{\alpha_1-2} - (\ln t - \ln s)^{\alpha_1-1}}{(1-\ln s)^{\alpha_1-2} - (1-\ln s)^{\alpha_1-1}} \\ &\ge \frac{(\ln t)^{\alpha_1-1}\left[(1-\ln s)^{\alpha_1-2} - (1-\ln s)^{\alpha_1-1}\right]}{(1-\ln s)^{\alpha_1-2} - (1-\ln s)^{\alpha_1-1}} \\ &\ge (\ln t)^{\alpha_1-1}.\end{aligned}$$

Hence, $G_{11}(t,s) \ge (\ln t)^{\alpha_1-1} G_{11}(e,s)$ for $t,s \in [1,e]$. Immediately

$$\begin{aligned}G_1(t,s) &= G_{11}(t,s) + \frac{\lambda_1(\ln t)^{\alpha_1-1}}{\Delta_1}G_{12}(\xi,s) \\ &\ge (\ln t)^{\alpha_1-1}G_{11}(e,s) + \frac{\lambda_1(\ln t)^{\alpha_1-1}}{\Delta_1}G_{12}(\xi,s) \\ &= (\ln t)^{\alpha_1-1}G_1(e,s).\end{aligned}$$

The proof is completed. □

Lemma 3.4. Let $H_1(t,s)$ be defined by (3.4), then

1) $H_1(t,s)$ is continuous on $[1,e]\times[1,e]$;
2) $H_1(t,s) \ge 0$, for $t,s \in [1,e]$;
3) $H_1(t,s) \le H_1(e,s)$, for $t,s \in [1,e]$;
4) $H_1(t,s) \ge (\ln t)^{\beta_1-1} H_1(e,s)$, for $t,s \in [1,e]$.

Proof. It is evident that (1), (2) and (3) hold. Next, we verify that (4) holds. Let $1\le s\le t\le e$, then

$$\frac{H_{11}(t,s)}{H_{11}(e,s)}=\frac{(\ln t)^{\beta_1-1}-(\ln t-\ln s)^{\beta_1-1}}{1-(1-\ln s)^{\beta_1-1}}$$
$$\ge(\ln t)^{\beta_1-1}\frac{1-(1-\ln s)^{\beta_1-1}}{1-(1-\ln s)^{\beta_1-1}}$$
$$=(\ln t)^{\beta_1-1}.$$

Moreover, combining with (3.4), we immediately get

$$H_1(t,s)=H_{11}(t,s)+\frac{(\ln t)^{\beta_1-1}+(\ln t)^{\beta_1-2}(\beta_1-1)}{\Delta_2\Gamma(\beta_1)}\int_1^e H_{11}(t,s)m_1(t)\frac{dt}{t}$$
$$\ge(\ln t)^{\beta_1-1}H_{11}(e,s)+\frac{(\ln t)^{\beta_1-1}+(\ln t)^{\beta_1-1}(\beta_1-1)}{\Delta_2\Gamma(\beta_1)}\int_1^e H_{11}(t,s)m_1(t)\frac{dt}{t}$$
$$=(\ln t)^{\beta_1-1}H_1(e,s).$$

This implies that (4) holds. The proof is finished. □

Furthermore, we consider the following SBVP:

$$\begin{cases} {}^{H}D_{1^+}^{\beta_2}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(t)\right)\right)=\rho(t),\ t\in(1,e); \\ v^{(j)}(1)=0,0\le j\le m-2,\delta v(e)=\lambda_1\delta v(\xi),\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(1)\right)=0; \\ {}^{H}D_{1^+}^{\beta_2-2}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(1)\right)\right)={}^{H}D_{1^+}^{\beta_2-1}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(e)\right)\right)=\int_1^e m_2(s)\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(s)\right)\frac{ds}{s}. \end{cases} \tag{3.8}$$

By employing the same approach used in the proof of *Lemma 3.1* and *Lemma 3.2*, the following lemmas can be acquired.

Lemma 3.5. Assume that $\rho\in C(1,e)$, $\Delta_3=1-\lambda_1(\ln\xi)^{\alpha_2-2}>0$, and $\Delta_4=1-\frac{1}{\Gamma(\beta_2)}\int_1^e\left((\ln s)^{\beta_2-1}+(\ln s)^{\beta_2-2}(\beta_2-1)\right)m_2(s)\frac{ds}{s}>0$. Then the problem (3.8) is equivalent to

$$v(t)=\int_1^e G_2(t,s)\phi_q\left(\int_1^e H_2(s,\tau)\rho(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s},\ t\in(1,e),$$

where

$$G_2(t,s)=G_{21}(t,s)+\frac{\lambda_1(\ln t)^{\alpha_2-1}}{\Delta_3}G_{22}(\xi,s), \tag{3.9}$$

$$G_{21}(t,s)=\frac{1}{\Gamma(\alpha_2)}\begin{cases}(\ln t)^{\alpha_2-1}(1-\ln s)^{\alpha_2-2}-(\ln t-\ln s)^{\alpha_2-1}, & 1\le s\le t\le e;\\ (\ln t)^{\alpha_2-1}(1-\ln s)^{\alpha_2-2}, & 1\le t\le s\le e,\end{cases}$$

$$G_{22}(t,s)=\frac{1}{\Gamma(\alpha_2)}\begin{cases}(\ln t)^{\alpha_2-2}(1-\ln s)^{\alpha_2-2}-(\ln t-\ln s)^{\alpha_2-2}, & 1\le s\le t\le e;\\ (\ln t)^{\alpha_2-2}(1-\ln s)^{\alpha_2-2}, & 1\le t\le s\le e,\end{cases}$$

$$H_2(t,s)=H_{21}(t,s)+\frac{(\ln t)^{\beta_2-1}+(\ln t)^{\beta_2-2}(\beta_2-1)}{\Delta_4\Gamma(\beta_2)}\int_1^e H_{21}(t,s)m_2(t)\frac{\mathrm{d}t}{t}, \quad (3.10)$$

and

$$H_{21}(t,s)=\frac{1}{\Gamma(\beta_2)}\begin{cases}(\ln t)^{\beta_2-1}-(\ln t-\ln s)^{\beta_2-1}, & 1\le s\le t\le e;\\ (\ln t)^{\beta_2-1}, & 1\le t\le s\le e.\end{cases}$$

The following conclusion can be established immediately.

Lemma 3.6. Assume that $\Delta_1,\Delta_2,\Delta_3,\Delta_4>0$. Then there exist positive constants N_1 and N_2 such that

$$G_2(t,s)\le N_1G_1(t,s),\ H_2(t,s)\le N_2H_1(t,s),\ t,s\in[1,e].$$

Proof. Let

$$N_1=\sup_{\substack{1<s,t\le e\\ s\ne e}}\frac{G_2(t,s)}{G_1(t,s)},\ N_2=\sup_{1<s,t\le e}\frac{H_2(t,s)}{H_1(t,s)}.$$

It is not difficult to confirm $N_1,N_2<+\infty$ according to the continuity of $G_i(\cdot,\cdot)$ and $H_i(\cdot,\cdot)$ on $(1,e)\times(1,e)$. In addition, (3.2), (3.4), (3.9) and (3.10) imply that

$$G_1(1,s)=G_2(1,s)=G_1(t,1)=G_2(t,1)=G_1(t,e)=G_2(t,e)=0,$$

$$H_1(1,s)=H_2(1,s)=H_1(t,1)=H_2(t,1)=0,\ \text{for } t,s\in[1,e].$$

Hence, it is easy to see that *Lemma 3.6* holds. The proof is completed. □

Suppose that the following condition holds in the sequel.

(C_1) $f\in C[(1,e)\times(0,+\infty)\times\mathbb{R},\mathbb{R}^+]$, $g\in C[(1,e)\times(0,+\infty)\times\mathbb{R},\mathbb{R}]$, and there exists $N_3>0$ such that

$$|g(t,u,v)|<N_3f(t,u,v),\ (t,u)\in(1,e)\times(0,+\infty),|v|\le Nu,$$

where $N=N_1(N_2N_3)^{q-1}$.

Let $X=C(I)$ be a Banach space with the norm $\|u\|=\max_{t\in I}\{|u(t)|\}$ for $u\in X$. Let $Y=X\times X$, then Y is a Banach space with the norm $\|(u,v)\|_Y=\max\{N\|u\|,\|v\|\}$ for $(u,v)\in Y$. Define a set $P\subset Y$ by

$$P=\left\{(u,v)\in Y:u(t)\ge(\ln t)^{\alpha_1-1}\|u\|,|v(t)|\le Nu(t),t\in I\right\}.$$

We can easily confirm that P is a cone of Y. As a matter of convenience, let

$$\Omega_r=\left\{(u,v)\in P:\|(u,v)\|_Y<r\right\}=\left\{(u,v)\in P:\|u\|<\frac{r}{N}\right\}.$$

Then

$$\partial\Omega_r=\left\{(u,v)\in P:\|(u,v)\|_Y=r\right\}=\left\{(u,v)\in P:\|u\|=\frac{r}{N}\right\},$$

$$\bar{\Omega}_r=\left\{(u,v)\in P:\|(u,v)\|_Y\le r\right\}=\left\{(u,v)\in P:\|u\|\le\frac{r}{N}\right\}.$$

Moreover, in order to overcome the singularity linked to SBVP (1.1), consider the following approximate boundary value problem:

$$\begin{cases} {}^{H}D_{1^+}^{\beta_1}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_1}u(t)\right)\right)=f_n\left(t,u(t),v(t)\right),t\in(1,e); \\ {}^{H}D_{1^+}^{\beta_2}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(t)\right)\right)=g_n\left(t,u(t),v(t)\right),t\in(1,e); \\ u^{(j)}(1)=0,0\le j\le n-2,\delta u(e)=\lambda_1\delta u(\xi),\phi_p\left({}^{H}D_{1^+}^{\alpha_1}u(1)\right)=0; \\ {}^{H}D_{1^+}^{\beta_1-2}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_1}u(1)\right)\right)={}^{H}D_{1^+}^{\beta_1-1}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_1}u(e)\right)\right)=\int_1^e m_1(s)\phi_p\left({}^{H}D_{1^+}^{\alpha_1}u(s)\right)\frac{\mathrm{d}s}{s}; \\ v^{(j)}(1)=0,0\le j\le m-2,\delta v(e)=\lambda_1\delta v(\xi),\phi_p\left({}^{H}D_{1^+}^{\alpha_1}v(1)\right)=0; \\ {}^{H}D_{1^+}^{\beta_2-2}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(1)\right)\right)={}^{H}D_{1^+}^{\beta_2-1}\left(\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(e)\right)\right)=\int_1^e m_2(s)\phi_p\left({}^{H}D_{1^+}^{\alpha_2}v(s)\right)\frac{\mathrm{d}s}{s}, \end{cases} \tag{3.11}$$

where

$$f_n(t,u,v)=f\left(t,u+\frac{1}{n},v\right),\ g_n(t,u,v)=g\left(t,u+\frac{1}{n},v\right),\ n\in\mathbb{N}.$$

Similarly, one can see that (u_n,v_n) is a solution of SBVP (3.11) if and only if (u_n,v_n) is a fixed point of the operator A_n, which is defined as follows:

$$A_n(u,v)(t)=\left(A_{1n}\left(u(t),v(t)\right),A_{2n}\left(u(t),v(t)\right)\right),\ t\in I,(u,v)\in P,n\in\mathbb{N},$$

where

$$A_{1n}(u,v)(t)=\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)f_n\left(\tau,u(\tau),v(\tau)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s},$$

$$A_{2n}(u,v)(t)=\int_1^e G_2(t,s)\phi_q\left(\int_1^e H_2(s,\tau)g_n\left(\tau,u(\tau),v(\tau)\right)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}.$$

Therefore, we mainly discuss that A_n has a nontrivial fixed point (u_n,v_n) in the following work.

For convenience, define a functional $\Theta:L^1(I)\to\mathbb{R}^+$ as follows:

$$\Theta(\varphi)=\max_{t\in I}\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)\varphi(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s},\ \text{for } \varphi\in L^1(I).$$

In addition, for any $R_1\ge r_1>0$, set

$$I_{[r_1,R_1](t)}:=\left(\left(\frac{r_1}{N}(\ln t)^{\alpha_1-1}\right),\frac{R_1}{N}+1\right]\times[-R_1,R_1].$$

Let us list some assumptions which will be used later.

(C_2) For any $R_1\ge r_1>0$, there exists $\Psi_{r_1,R_1}\in L^1(I)\cap C(1,e)$ such that

$$f(t,u,v)\le\Psi_{r_1,R_1}(t),(t,u,v)\in(1,e)\times I_{[r_1,R_1](t)}.$$

(C_3) There exist $R>r>0$ and $\Phi_r\in L^1(I)$ such that

1) $f(t,u,v)\ge\Phi_r(t)$, for $(t,u,v)\in(1,e)\times I_{[r,r](t)}$,

2) $\Theta(\Psi_{R,R})<\frac{R}{N}$, $\Theta(\Phi_r)>\frac{r}{N}$.

(C_4) There exists $[a,b]\subset(1,e)$ such that

$$\lim_{\substack{|v|<Nu\\ u\to+\infty}}\min_{t\in[a,b]}\frac{f(t,u,v)}{u^{p-1}}=+\infty.$$

In what follows, some lemmas will be shown, which are important in this paper. Firstly, for any fixed $n\in\mathbb{N}$, the complete continuity of the operator $A_n:\Omega_{R_1}\setminus\bar{\Omega}_{r_1}\to P$ is demonstrated by means of Ascoli-Arzelà theorem.

Lemma 3.7 Assume that (C_1) and (C_2) holds. Then $A_n:\Omega_{R_1}\setminus\bar{\Omega}_{r_1}\to P$ is a completely continuous operator for $\forall\, R_1\geqslant r_1>0$.

Proof. For any $n\in\mathbb{N}$, $(u,v)\in\Omega_{R_1}\setminus\bar{\Omega}_{r_1}$, we have

$$\frac{r_1}{N}(\ln t)^{\alpha_1-1}<u(t)+\frac{1}{n}\le\frac{R_1}{N}+\frac{1}{n},$$

and

$$|v(t)|\le Nu(t)\le R_1,\, t\in I,$$

which means $(u(t),v(t))\in I_{[r_1,R_1](t)}$ for $t\in I$. In view of (C_2), there exists $\Psi_{r_1,R_1}\in L^1(I)\cap C(1,e)$ such that

$$f(t,u(t),v(t))\le\Psi_{r_1,R_1}(t),\ \text{for } t\in(1,e).$$

Therefore,

$$\begin{aligned}\|A_n(u,v)\|_Y&\le\|A_{1n}(u,v)\|_Y\\&\le\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)f_n(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\&\le\int_1^e G_1(e,s)\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{r_1,R_1}(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\&<+\infty,\end{aligned}$$

which implies that A_n is well defined. For $\forall(u,v)\in\Omega_{R_1}\setminus\bar{\Omega}_{r_1}$, by ($C_1$), one has

$$\begin{aligned}u(t)&=\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)f_n(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\&\ge(\ln t)^{\alpha_1-1}\int_1^e G_1(e,s)\phi_q\left(\int_1^e H_1(s,\tau)f_n(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\&\ge(\ln t)^{\alpha_1-1}\|u\|,\ \forall t\in I,n\in\mathbb{N},\end{aligned}$$

and

$$\begin{aligned}|v(t)|&=\int_1^e G_2(t,s)\phi_q\left(\int_1^e H_2(s,\tau)g_n(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\&\le N_1\int_1^e G_1(t,s)\phi_q\left(N_2\int_1^e H_1(s,\tau)N_3f_n(\tau,u(\tau),v(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\&=Nu(t),\ \forall t\in I,n\in\mathbb{N}.\end{aligned}$$

This implies that $A_n\left(\Omega_{R_1}\setminus\bar{\Omega}_{r_1}\right)\subset P$.

Based on the aforementioned demonstration, it is clear that $A_n\left(\Omega_{R_1}\setminus\bar{\Omega}_{r_1}\right)$ is bounded. Since $G_1(\cdot,\cdot)$ is continuous on $[1,e]\times[1,e]$, it follows that $G_1(\cdot,\cdot)$ is uniformly continuous on $[1,e]\times[1,e]$. Hence, for any $\varepsilon>0$, there exists $\delta>0$ such that, for any $t_1,t_2\in[1,e]$, $s\in[1,e]$, if $|t_1-t_2|<\delta$, then

$$|G_1(t_1,s)-G_1(t_2,s)|<\frac{\varepsilon}{\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{r_1,R_1}(\tau)\frac{d\tau}{\tau}\right)+1},$$

For any $n \in \mathbb{N}$, $(u,v) \in \Omega_{R_1} \setminus \bar{\Omega}_{r_1}$, it follows from C_1 that

$$\begin{aligned}
&\left|A_{1n}(u,v)(t_1) - A_{1n}(u,v)(t_2)\right| \\
&\le \int_1^e \left|G_1(t_1,s) - G_1(t_2,s)\right| \phi_q\left(\int_1^e H_1(s,\tau) f_n(\tau,u(\tau),v(\tau)) \frac{d\tau}{\tau}\right)\frac{ds}{s} \\
&\le \int_1^e \left|G_1(t_1,s) - G_1(t_2,s)\right| \phi_q\left(\int_1^e H_1(s,\tau) \Psi_{r_1,R_1}(\tau) \frac{d\tau}{\tau}\right)\frac{ds}{s} \\
&\le \int_1^e \frac{\varepsilon}{\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{r_1,R_1}(\tau)\frac{d\tau}{\tau}\right)+1} \phi_q\left(\int_1^e H_1(s,\tau) \Psi_{r_1,R_1}(\tau) \frac{d\tau}{\tau}\right)\frac{ds}{s} \\
&< \varepsilon.
\end{aligned}$$

Furthermore, in the same way as above, for any $\varepsilon^* > 0$, there exists $\delta^* > 0$ such that, for any $(u,v) \in \Omega_{R_1} \setminus \bar{\Omega}_{r_1}$ and $t_1^*, t_2^* \in [1,e]$ with $\left|t_1^* - t_2^*\right| < \delta^*$, one has $\left|A_{2n}(u,v)(t_1^*) - A_{2n}(u,v)(t_2^*)\right| < \varepsilon^*$. That is to say, $A_n\left(\Omega_{R_1} \setminus \bar{\Omega}_{r_1}\right)$ is equicontinuous on $[1,e]$. The above analysis together with Ascoli-Arzelà theorem suggests that A_n is relatively compact from $\Omega_{R_1} \setminus \bar{\Omega}_{r_1}$ to P.

At the end, according to the continuity of f and g, it is not hard to confirm that A_n is continuous on $\Omega_{R_1} \setminus \bar{\Omega}_{r_1}$. Hence, A_n is a completely continuous operator. The proof is finished. □

In the sequel, the following lemma is presented to prove that for any fixed $n \in \mathbb{N}$, approximate boundary value problem (3.11) has at least two nontrivial solutions by applying the fixed point index theory.

Lemma 3.8. Assume that (C_1) - (C_4) are satisfied. Then there exists $R' > R$ such that for $\forall n \in \mathbb{N}$, the operator A_n has at least two nontrivial solutions in $\Omega_{R'} \setminus \bar{\Omega}_R$ and $\Omega_R \setminus \bar{\Omega}_r$ respectively, where R is given by (C_3).

Proof. (1) In view of *Lemma 2.6*, we start by proving that $\inf_{(u,v)\in\partial\Omega_r} \left\|A_n(u,v)\right\|_Y > 0$, and $(u,v) \ne \mu A_n(u,v)$, for $\forall (u,v) \in \partial\Omega_r$, $\mu \ge 1$, and $n \in \mathbb{N}$. Assuming the contrary, there exists $\mu_1 \ge 1$ and $(u_1,v_1) \in \partial\Omega_r$ such that $(u_1,v_1) = \mu_1 A_n(u_1,v_1)$. Therefore,

$$\begin{aligned}
u_1(t) = \mu_1 A_{1n}(u_1,v_1)(t) &\ge A_{1n}(u_1,v_1)(t) \\
&= \int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau) f_n(\tau,u_1(\tau),v_1(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\
&\ge \int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)\Phi_r(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}, t \in I.
\end{aligned}$$

The definition of functional Θ together with (C_3) implies

$$\left\|u_1\right\| \ge \Theta(\Phi_r) > \frac{r}{N},$$

which contradicts with $(u_1,v_1) \in \partial\Omega_r$. By incorporating *Lemma 2.6*, we can conclude that

$$i(A_n, \Omega_r, P) = 0, \text{ for } \forall n \in \mathbb{N}. \tag{3.12}$$

(2) Next, we show

$$(u_2,v_2) \ne \mu_2 A_n(u_2,v_2), \text{ for } (u_2,v_2) \in \partial\Omega_R, \mu_2 \in (0,1], \text{ and } n \in \mathbb{N}.$$

In fact, if there exists $\mu_2 \in (0,1]$ and $(u_2,v_2) \in \partial\Omega_R$ such that $(u_2,v_2) = \mu_2 A_n(u_2,v_2)$, then

$$\begin{aligned} u_2(t) &= \mu_2 A_{1n}(u_2,v_2)(t) \le A_{1n}(u_2,v_2)(t) \\ &= \int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau) f_n(\tau,u_2(\tau),v_2(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\le \int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{R,R}(\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}, t \in I. \end{aligned} \tag{3.13}$$

Take the maximum for both sides of (3.13) in $[1,e]$. Then it is evident that

$$\|u_2\| \le \Theta(\Psi_{R,R}) < \frac{R}{N}.$$

In light of *Lemma 2.6*, this implies that

$$i(A_n,\Omega_R,P) = 1, \text{ for } \forall n \in \mathbb{N}. \tag{3.14}$$

(3) In the following, we choose a constant $M > 0$ as follows:

$$M^{q-1} > \left((\ln a)^{(\alpha_1-1)q}\int_a^b G_1(e,s)\phi_q\left(\int_a^b H_1(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\right)^{-1}.$$

From (C_4), there exists a constant $L > 0$ such that

$$f(t,u,v) \ge Mu^{p-1}, \text{ for } t \in [a,b], u \ge L, |v| \le Nu.$$

Let $R' > \max\left\{R, \frac{LN}{(\ln a)^{\alpha_1-1}}\right\}$. Next, we claim that

$$(u,v) \ne \mu A_n(u,v), \text{ for } \forall (u,v) \in \partial\Omega_{R'}, \mu \ge 1, \text{ and } n \in \mathbb{N}.$$

As a matter of fact, if it is not true, then there exists $\mu_3 \ge 1$ and $(u_3,v_3) \in \partial\Omega_{R'}$ such that $(u_3,v_3) = \mu_3 A_n(u_3,v_3)$. Since

$$u_3(t) \ge (\ln t)^{\alpha_1-1}\|u_3\| \ge (\ln a)^{\alpha_1-1}\frac{R'}{N} > L, \text{ for } \forall t \in [a,b],$$

$$\begin{aligned} u_3(t) &= \mu_3 A_{1n}(u_3,v_3)(t) \ge A_{1n}(u_3,v_3)(t) \\ &= \int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau) f_n(\tau,u_3(\tau),v_3(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge \int_a^b (\ln t)^{\alpha_1-1} G_1(e,s)\phi_q\left(\int_a^b H_1(s,\tau)\left(M\left(u_3(\tau)+\frac{1}{n}\right)^{p-1}\right)\frac{d\tau}{\tau}\right)\frac{ds}{s} \\ &\ge M^{q-1}(\ln a)^{(\alpha_1-1)q}\frac{R'}{N}\int_a^b G_1(e,s)\phi_q\left(\int_a^b H_1(s,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}, \text{ for } t \in I. \end{aligned}$$

According to the definition of M, we have

$$\|u_3\| > \frac{R'}{N},$$

which is in contradiction with $(u_3,v_3) \in \partial\Omega_{R'}$. This means that

$$i(A_n,\Omega_{R'},P) = 0, \text{ for } \forall n \in \mathbb{N}. \tag{3.15}$$

In view of (3.12), (3.14) and (3.15), combining with the additivity of the fixed point index, we can obtain

$$i\left(A_n,\Omega_{R'}\setminus\bar{\Omega}_R,P\right)=i\left(A_n,\Omega_{R'},P\right)-i\left(A_n,\Omega_R,P\right)=-1;$$
$$i\left(A_n,\Omega_R\setminus\bar{\Omega}_r,P\right)=i\left(A_n,\Omega_R,P\right)-i\left(A_n,\Omega_r,P\right)=1.$$

That is to say, for $\forall n\in\mathbb{N}$, there exist $(u_n,v_n)\in\Omega_{R'}\setminus\bar{\Omega}_R$ and $(\bar{u}_n,\bar{v}_n)\in\Omega_R\setminus\bar{\Omega}_r$ such that $(u_n,v_n)=A_n(u_n,v_n)$ and $(\bar{u}_n,\bar{v}_n)=A_n(\bar{u}_n,\bar{v}_n)$. The proof is completed. □

The following theorem, as our main result in this paper, is proved in light of Ascoli-Arzelà theorem.

Theorem 3.1. Assume that (C_1) - (C_4) hold. Then SBVP (1.1) has at least two nontrivial solutions.

Proof. It is obvious that $\{(u_n,v_n)\}_{n\in N}$ and $\{(\bar{u}_n,\bar{v}_n)\}_{n\in N}$ are bounded.

Firstly, we prove that $\{u_n\}_{n\in N}$ is equicontinuous on $[1,e]$. Since $G_1(\cdot,\cdot)$ is uniformly continuous on $[1,e]\times[1,e]$, for any $\varepsilon>0$, there exists $\delta>0$ such that, for any $t_1,t_2\in[1,e]$, $s\in[1,e]$, if $|t_1-t_2|<\delta$, then one has

$$\left|G_1(t_1,s)-G_1(t_2,s)\right|<\frac{\varepsilon}{\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{R',R}(\tau)\frac{\mathrm{d}\tau}{\tau}\right)+1}.$$

In the same way as in *Lemma 3.6*, immediately

$$\begin{aligned}
&\left|u_n(t_1)-u_n(t_2)\right|\\
&\le\int_1^e\left|G_1(t_1,s)-G_1(t_2,s)\right|\phi_q\left(\int_1^e H_1(s,\tau)f_n(\tau,u(\tau),v(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\
&\le\int_1^e\left|G_1(t_1,s)-G_1(t_2,s)\right|\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{R',R}(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\
&\le\int_1^e\frac{\varepsilon}{\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{R',R}(\tau)\frac{\mathrm{d}\tau}{\tau}\right)+1}\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{R',R}(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\
&<\varepsilon.
\end{aligned}$$

By following a similar process as above, we deduce that $\{v_n\}_{n\in N}$ is equicontinuous on $[1,e]$. This means that the set $\{(u_n,v_n)\}_{n\in N}$ is an equicontinuous family on $[1,e]$. Similarly, the set $\{(\bar{u}_n,\bar{v}_n)\}_{n\in N}$ is equicontinuous on $[1,e]$.

In addition, according to the Ascoli-Arzelà theorem, the sets $\{(u_n,v_n)\}_{n\in N}$ and $\{(\bar{u}_n,\bar{v}_n)\}_{n\in N}$ are relatively compact. As a result, there exist convergent subsequences. Without loss of generality, suppose that $\{(u_n,v_n)\}_{n\in N}$ and $\{(\bar{u}_n,\bar{v}_n)\}_{n\in N}$ themselves converge to (u_0,v_0) and $(\bar{u}_0,\bar{v}_0)$ respectively. That is,

$$(u_n,v_n)\to(u_0,v_0),\ \text{as } n\to+\infty;$$
$$(\bar{u}_n,\bar{v}_n)\to(\bar{u}_0,\bar{v}_0),\ \text{as } n\to+\infty.$$

Hence, it follows from Lebesgue dominated convergence theorem that (u_0,v_0) and $(\bar{u}_0,\bar{v}_0)$ are nontrivial solutions to SBVP (1.1).

Finally, we demonstrate $(u_0,v_0)\neq(\bar{u}_0,\bar{v}_0)$. We need to prove only that SBVP (1.1) has no solutions on $\partial\Omega_R$. Assuming the opposite, there exists $(u',v')\in\partial\Omega_R$ such that

$$u'(t) = \int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau) f(\tau, u'(\tau), v'(\tau))\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}$$
$$\le \int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{R,R}(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}, t\in I,$$

which implies $\frac{R}{N} = \|u'\| \le \Theta(\Psi_{R,R}) < \frac{R}{N}$. This leads to a contradiction. The proof of this theorem is completed. □

4. An Example

In this section, an example is presented to illustrate the main results.

Example 4.1. Consider the following SBVP of Hadamard fractional differential systems:

$$\begin{cases} {}^H D_{1^+}^{\frac{5}{2}}\left(\phi_p\left({}^H D_{1^+}^{\frac{5}{2}} u(t)\right)\right) = f(t, u(t), v(t)), t\in(1,e); \\ {}^H D_{1^+}^{\frac{5}{2}}\left(\phi_p\left({}^H D_{1^+}^{\frac{5}{2}} v(t)\right)\right) = g(t, u(t), v(t)), t\in(1,e); \\ u^{(j)}(1) = 0, 0\le j\le 1, \delta u(e) = \delta u\left(\frac{3}{2}\right), \phi_p\left({}^H D_{1^+}^{\alpha_1} u(1)\right) = 0; \\ {}^H D_{1^+}^{\frac{1}{2}}\left(\phi_p\left({}^H D_{1^+}^{\alpha_1} u(1)\right)\right) = {}^H D_{1^+}^{\frac{3}{2}}\left(\phi_p\left({}^H D_{1^+}^{\alpha_1} u(e)\right)\right) = \frac{1}{2}\int_1^e \phi_p\left({}^H D_{1^+}^{\alpha_1} u(s)\right)\frac{\mathrm{d}s}{s}; \\ v^{(j)}(1) = 0, 0\le j\le 1, \delta v(e) = \delta v\left(\frac{3}{2}\right), \phi_p\left({}^H D_{1^+}^{\alpha_1} v(1)\right) = 0; \\ {}^H D_{1^+}^{\frac{1}{2}}\left(\phi_p\left({}^H D_{1^+}^{\frac{3}{2}} v(1)\right)\right) = {}^H D_{1^+}^{\beta_2 - 1}\left(\phi_p\left({}^H D_{1^+}^{\alpha_2} v(e)\right)\right) = \frac{1}{2}\int_1^e \phi_p\left({}^H D_{1^+}^{\alpha_2} v(s)\right)\frac{\mathrm{d}s}{s}, \end{cases} \tag{4.1}$$

where $u\in\mathbb{R}^+$, $\frac{1}{2} < |v| < 1$.

Conclusion: SBVP (4.1) has at least two nontrivial solutions.

Proof. SBVP (4.1) can be regarded as the form of SBVP (1.1), where

$$\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = \frac{5}{2}, m = n = 3, \xi = \frac{3}{2}, m_1(t) = m_2(t) = \frac{1}{2}, p = \frac{5}{4},$$

$$f(t,u,v) = \frac{v}{10\sqrt{\ln t(1-\ln t)}}\left(u^{\frac{1}{2}} + \frac{1}{u}\right),$$

and

$$g(t,u,v) = \frac{v\cos t}{10\sqrt{\ln t(1-\ln t)}}\left(u^{\frac{1}{2}} + \frac{\sin t}{u}\right).$$

Clearly, (C_1) holds. Moreover, from *Lemma 3.1* and *Lemma 3.2*, one can get the following expressions:

$$G_1(t,s) = G_{11}(t,s) + \frac{(\ln t)^{\frac{3}{2}}}{1-\left(\ln\frac{3}{2}\right)^{\frac{1}{2}}} G_{12}\left(\frac{3}{2}, s\right), \tag{4.2}$$

$$G_{11}(t,s)=\frac{4}{3\sqrt{\pi}}\begin{cases}(\ln t)^{\frac{3}{2}}(1-\ln s)^{\frac{1}{2}}-(\ln t-\ln s)^{\frac{3}{2}}, & 1\le s\le t\le e;\\ (\ln t)^{\frac{3}{2}}(1-\ln s)^{\frac{1}{2}}, & 1\le t\le s\le e,\end{cases}$$

$$G_{12}(t,s)=\frac{4}{3\sqrt{\pi}}\begin{cases}(\ln t)^{\frac{1}{2}}(1-\ln s)^{\frac{1}{2}}-(\ln t-\ln s)^{\frac{1}{2}}, & 1\le s\le t\le e;\\ (\ln t)^{\frac{1}{2}}(1-\ln s)^{\frac{1}{2}}, & 1\le t\le s\le e,\end{cases}$$

$$\Delta_2=1-\frac{2}{3\sqrt{\pi}}\int_1^e\left((\ln s)^{\frac{3}{2}}+\frac{3}{2}(\ln s)^{\frac{1}{2}}\right)\frac{\mathrm{d}s}{s}=1-\frac{14}{15\sqrt{\pi}}>0,$$

$$H_1(t,s)=H_{11}(t,s)+\frac{(\ln t)^{\frac{3}{2}}+\frac{3}{2}(\ln t)^{\frac{1}{2}}}{2\Delta_2\Gamma\left(\frac{5}{2}\right)}\int_1^e H_{11}(t,s)\frac{\mathrm{d}t}{t},$$

and

$$H_{11}(t,s)=\frac{4}{3\sqrt{\pi}}\begin{cases}(\ln t)^{\frac{3}{2}}-(\ln t-\ln s)^{\frac{3}{2}}, & 1\le s\le t\le e;\\ (\ln t)^{\frac{3}{2}}, & 1\le t\le s\le e.\end{cases}$$

It is easy to see $N_1=N_2=1$ and $N_3=1$. In addition, for each $R_1>r_1>0$,

$$\begin{aligned}f(t,u,v)&=\frac{v}{10\sqrt{\ln t(1-\ln t)}}\left(u^{\frac{1}{2}}+\frac{1}{u}\right)\\&\le\frac{1}{10\sqrt{\ln t(1-\ln t)}}\left(R_1^{\frac{1}{2}}+1\right).\end{aligned}$$

Take $\Psi_{r_1,R_1}(t)=\frac{1}{10\sqrt{\ln t(1-\ln t)}}\left(R_1^{\frac{1}{2}}+1\right)$, and this means that ($C_2$) holds. Moreover, choose

$$r=\frac{1}{100},\ \Phi_r(t)=\frac{1}{20(r+1)\sqrt{\ln t(1-\ln t)}},$$

and

$$R=100,\ \Psi_{R,R}(t)=\frac{1}{10\sqrt{\ln t(1-\ln t)}}\left(R^{\frac{1}{2}}+1\right).$$

By careful calculations, we can obtain

$$\begin{aligned}\Theta(\Psi_{R,R})&=\max_{t\in[1,e]}\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)\Psi_{R,R}(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\&\le\int_1^e G_1(e,s)\phi_q\left(\int_1^e H_1(e,\tau)\Psi_{R,R}(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\&\le 99.5722\\&<100,\end{aligned}$$

and

$$\begin{aligned}\Theta(\Phi_r) &= \max_{t\in[1,e]}\int_1^e G_1(t,s)\phi_q\left(\int_1^e H_1(s,\tau)\Phi_r(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\ &\ge \max_{t\in[1,e]}(\ln t)^{\frac{3}{2}}\int_1^e G_1(e,s)\phi_q\left((\ln s)^{\frac{3}{2}}\int_1^e H_1(e,\tau)\Phi_r(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\ &\ge \int_1^e G_1(e,s)\phi_q\left((\ln s)^{\frac{3}{2}}\int_1^e H_1(e,\tau)\Phi_r(\tau)\frac{\mathrm{d}\tau}{\tau}\right)\frac{\mathrm{d}s}{s}\\ &\ge 0.0557\\ &> \frac{1}{100}.\end{aligned}$$

That is to say, (C_3) holds. Furthermore, since

$$\lim_{\substack{|v|<Nu\\ u\to+\infty}}\min_{t\in[a,b]}\frac{f(t,u,v)}{u^{p-1}} = \lim_{\substack{|v|<Nu\\ u\to+\infty}}\min_{t\in[a,b]}\frac{\frac{v}{10\sqrt{\ln t(1-\ln t)}}\left(u^{\frac{1}{2}}+\frac{1}{u}\right)}{u^{p-1}} = +\infty,$$

this implies that (C_4) holds. Hence, according to *Theorem 3.1*, SBVP (4.1) has at least two nontrivial solutions. □

5. Conclusion

This study investigates a category of singular boundary value problems of Hadamard fractional differential systems involving p-Laplacian operator. Firstly, the sequences of approximate solutions to SBVP (1.1) are obtained by the fixed point index theory to overcome the singularity. Secondly, the existence and multiplicity of positive solutions are established. Finally, an illustrative example is provided to validate our main findings. Additionally, the introduction of Hadamard fractional derivative and the p-Laplacian operator deepens our comprehension of singular boundary value problems. This research also extends the current body of literature and highlights the potential to be applied in various fields.

Acknowledgements

The authors would like to thank the referee for his/her valuable suggestions and comments.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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