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Inverse Electromagnetic Scattering by a Penetrable Chiral Object

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Abstract

We consider the inverse electromagnetic scattering problem of determining the shape of a perfectly conducting core inside a penetrable chiral body. We prove the well-posedness of the corresponding direct scattering problem by the variational method. We focus on a uniqueness result for the inverse scattering problem that is under what conditions an obstacle can be identified by the knowledge of the electric far-field pattern corresponding to all time-harmonic incident planes waves with a fixed wave number. To this end, we establish a chiral mixed reciprocity relation that connects the electric far-field pattern of a spherical wave with the scattered field of a plane wave.

Keywords

Chiral Media, Inverse Scattering Problem, Chiral Mixed Reciprocity Relation

1. Introduction

In this paper, we consider a time-harmonic electromagnetic scattering problem by a chiral dielectric with a perfectly conducting core. Chiral media exhibit optical activity that a chiral body cannot be brought into congruence with its mirror image by translation and rotation. They are examples of media that are characterized by two generalized constitutive relations in which the electric and magnetic fields are coupled by a material parameter, chirality. In this work, chirality is introduced via the Drude-Born-Fedorov equations which are symmetric under time-reversality and duality transformation [1]. An electromagnetic field, that is incident upon a chiral obstacle, is composed of left-circularly polarized and right-circularly polarized components which are propagated in an isotropic and homogeneous medium independently with different phase speeds. In applications, we use Bohren transformation where the electric and magnetic fields are expressed in terms of Beltrami fields that solve first order differential equations [2]. For details of the physical properties of electromagnetic scattering in chiral media we refer to the books [1] and [2]. In recent years, direct and inverse scattering problems in chiral media have been studied in many scientific publications. In [3] the authors proved that the classical Silver-Müller radiation condition remains valid in chiral media while in [4] the same authors proved existence and uniqueness of solution to a diffraction problem of a plane electromagnetic field by a chiral thin layer covering a perfectly conducting object. We also refer to [5] and [6] where the well posedness of the direct electromagnetic scattering problem has been studied for a chiral dielectric and by an impedance screen in chiral media respectively. Uniqueness of the inverse obstacle scattering problem in chiral media for the case of a two-dimensional chiral scatterer using boundary integral equation approach has been studied in [7] and for a chiral dielectric by establishing an orthogonality type result in [8].

The inverse scattering problem has been studied extensively and there are numerous research papers concerning both uniqueness theorems and determining physical and geometrical properties of the obstacle. In [9] the author proved a uniqueness theorem for a transmission problem by means of boundary integral equation methods. Uniqueness theorems for scattering from a piecewise homogeneous medium were studied considering either the same wave number in [10] or different wave numbers between the layers of the scatterer in [11]. Applications of the inverse scattering problem can be found in radar, geophysical exploration and medical imaging. In particular, chiral media falls within the research area of detecting an obstacle that may be (partially) coated with an unknown material in order to avoid its identification. Concerning uniqueness theorems for electromagnetic scattering by a perfect conductor we refer to the books [12] [13], whereas by homogeneous anisotropic media in [14].

For uniqueness results to the inverse obstacle scattering problem, we indicatively refer to [15] for acoustic scattering and [16] for elastic waves. In [17] uniqueness issues for scattering by an obstacle with an impedance boundary condition using the mixed reciprocity relation were proved.

In Section 2, we study the well-posedness of the direct scattering problem by the variational method using Calderon operator. In Section 3, we concentrate on the unique determination of the obstacle from a knowledge of the far-field pattern that corresponds to incident plane waves with arbitrary directions of propagation and polarizations and a fixed real wave number. Moreover, a mixed reciprocity relation for chiral media is proved which will be used in the proof of the uniqueness theorem.

Formulation of the problem

In what follows we give a precise formulation of the dimensionless scattering problem. Let D be a bounded domain with a known C^2-boundary $\partial D = S_0$. The domain D, which will be called the scatterer, is divided into two homogeneous, non intersecting layers D_1 and D_2 by a C^2-surface S_1 as shown in **Figure 1**. The domain D_1 is filled with a homogeneous and isotropic chiral material with chirality measure β, electric permittivity ε, and magnetic permeability μ while the domain D_2 is a perfectly conducting core. The exterior region $D_0 = \mathbb{R}^3 \setminus \bar{D}$ of the scatterer is an isotropic, infinite, homogeneous achiral medium with electric permittivity ε_0 and magnetic permeability μ_0. In domain D_1 chirality is introduced via the Drude-Born-Fedorov constitutive relations [1] where the electric displacement D and the magnetic induction B are connected with the total electric and magnetic fields E and H,

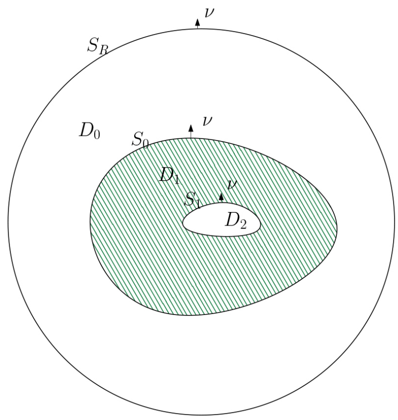

Figure 1. Chiral dielectric with perfectly conducting core.

$$D = \varepsilon\left(E + \beta\nabla\times E\right),\ B = \mu\left(H + \beta\nabla\times H\right). \tag{1}$$

In a source free region we have

$$\nabla\times E - i\omega B = 0,\ \nabla\times H + i\omega D = 0, \tag{2}$$

where we have suppressed a time dependence of $\mathrm{e}^{-i\omega t}$, $\omega > 0$ being the angular frequency. From above equations, for the electric and magnetic fields in a chiral medium we take

$$\nabla\times E = \beta\gamma^2 E + i\omega\mu\left(\frac{\gamma}{k}\right)^2 H, \tag{3}$$

$$\nabla\times H = \beta\gamma^2 H - i\omega\varepsilon\left(\frac{\gamma}{k}\right)^2 E, \tag{4}$$

where $k^2 = \omega^2\varepsilon\mu$ and $\gamma^2 = k^2\left(1-\beta^2k^2\right)^{-1}$ with $|k\beta| < 1$ [1].

Let $\left(E^i, H^i\right)$ be a time harmonic plane electromagnetic wave incident upon the scatterer D and let $\left(E^s, H^s\right)$ be the corresponding scattered field. Then the total electromagnetic field $\left(E^0, H^0\right)$ in D_0 is given by

$$E^0 = E^i + E^s,\quad H^0 = H^i + H^s. \tag{5}$$

It is convenient to consider a dimensionless version of the problem. Therefore, we replace the fields $\left(E^0, H^0\right)$, $\left(E, H\right)$ with $\left(\sqrt{\mu_0}E^0, \sqrt{\varepsilon_0}H^0\right)$ and

$\left(\sqrt{\mu}E,\sqrt{\varepsilon}H\right)$ respectively. Similar scaling holds for the scattered and incident fields (see [12] for achiral and [18] for chiral case). Then, the transmission problem we study is described by the following equations

$$\left.\begin{aligned}\nabla\times E^0-ik_0H^0&=0\\ \nabla\times H^0+ik_0E^0&=0\end{aligned}\right| \text{ in } D_0, \tag{6}$$

$$\left.\begin{aligned}\nabla\times E-i\frac{\gamma^2}{k}H-\beta\gamma^2E&=0\\ \nabla\times H+i\frac{\gamma^2}{k}E-\beta\gamma^2H&=0\end{aligned}\right| \text{ in } D_1, \tag{7}$$

where $k_0^2=\omega^2\varepsilon_0\mu_0$. We note that k_0 is the wave number in D_0 while k is not a wave number but a short hand notation [1]. The transmission conditions we impose on the boundary S_0 are given by [18]

$$\left.\begin{aligned}\nu\times E^0&=\delta^{-1}\nu\times E\\ \nu\times H^0&=\zeta^{-1}\nu\times H\end{aligned}\right| \text{ on } S_0, \tag{8}$$

where ν is the unit outward normal to S_0, $\delta=\sqrt{\frac{\mu_0}{\mu}}$ and $\zeta=\sqrt{\frac{\varepsilon_0}{\varepsilon}}$. For the boundary of the perfectly conducting core D_2, we have

$$\nu\times E=0 \text{ on } S_1. \tag{9}$$

We assume that all physical parameters are real positive constants. The scattered field satisfies the Silver-Müller radiation condition

$$\hat{x}\times H^s+E^s=o\left(\frac{1}{|x|}\right),\ |x|\to\infty, \tag{10}$$

uniformly in all directions $\hat{x}=\frac{x}{|x|}$.

2. Fundamental Principles

In this section, we study the well-posedness of the scattering problem (5)-(10). We will first prove that the corresponding homogeneous scattering problem has only the trivial solution.

Theorem 1. The scattering problem (5)-(10) admits at most one solution.

Proof. We set

$$\mathcal{E}=\begin{cases}E^s & \text{in } D_0\\ E & \text{in } D_1\end{cases},\quad \mathcal{H}=\begin{cases}H^s & \text{in } D_0\\ H & \text{in } D_1\end{cases},$$

then

$$\nabla\times\mathcal{E}-i\frac{\gamma^2}{k}\mathcal{H}-\beta\gamma^2\mathcal{E}=0 \text{ in } D_0\cup D_1, \tag{11}$$

$$\nabla\times\mathcal{H}+i\frac{\gamma^2}{k}\mathcal{E}-\beta\gamma^2\mathcal{H}=0 \text{ in } D_0\cup D_1, \tag{12}$$

where $\beta=0$, $\gamma=k=k_0$ in $\mathbb{R}^3\setminus D_1$. We consider the corresponding homogeneous scattering problem ($E^i = 0$, $H^i = 0$) and we multiply the complex conjugate of (11) by the magnetic field $\mathcal{H}$ and (12) by $\bar{\mathcal{E}}$ (complex conjugate of $\mathcal{E}$), we apply the divergence theorem in D_1

$$\int_{D_1} \left[\mathcal{H} \cdot \nabla \times \bar{\mathcal{E}} - \bar{\mathcal{E}} \cdot \nabla \times \mathcal{H} \right] \mathrm{d}x = (\delta\zeta)^{-1} \int_{S_0} \nu \cdot \left(\bar{\mathcal{E}} \times \mathcal{H} \right) \mathrm{d}s - \int_{S_1} \nu \cdot \left(\bar{\mathcal{E}} \times \mathcal{H} \right) \mathrm{d}s. \tag{13}$$

Taking into account (11), (12) and the boundary condition on S_1, we get

$$\int_{D_1} \frac{i\gamma^2}{k} \left(|\mathcal{H}|^2 - |\mathcal{E}|^2 \right) \mathrm{d}x = (\delta\zeta)^{-1} \int_{S_0} \nu \cdot \left(\bar{\mathcal{E}} \times \mathcal{H} \right) \mathrm{d}s. \tag{14}$$

The real part of (14) gives

$$Re\left\{ \int_{S_0} \nu \cdot \left(\bar{E}^s \times H^s \right) \mathrm{d}s \right\} = 0,$$

where the transmission conditions (8) has been used. By Theorem 6.10 of [12], (Rellich's lemma), we deduce that $E^s = 0$ in $\mathbb{R}^3 \setminus \bar{D}$. Then, $H^s = 0$ from (6). The transmission conditions (14) on S_0 and the continuity of the fields give

$$\nu \times E = \nu \times H = 0 \text{ on } S_0. \tag{15}$$

Finally, by Holmgren's uniqueness theorem [19] we conclude that $E = H = 0$ in D_1.

In order to study a general transmission problem we need to define the following Sobolev spaces:

$$L_t^2(S_j) = \left\{ u \in \left(L^2(S_j) \right)^3 : \nu \cdot u = 0 \text{ on } S_j, j = 0,1 \right\},$$

$$H(\mathrm{curl}, B_\rho \cap D_0) = \left\{ u \in \left(L^2(B_\rho \cap D_0) \right)^3 : \nabla \times u \in \left(L^2(B_\rho \cap D_0) \right)^3 \right\},$$

$$X_{\mathrm{loc}}(D_0, S_0) = \left\{ u \in H(\mathrm{curl}, B_\rho \cap D_0) : \nu \times u|_{S_0} \in L_t^2(S_0) \right\},$$

$$H^{-1/2}(\mathrm{Div}, S_\rho) = \left\{ u \in \left(H^{-1/2}(S_\rho) \right)^3 : u \in \left(L^2(S_\rho) \right)^3 \ \nabla_{S_\rho} \cdot u \in H^{-1/2}(S_\rho) \right\},$$

where B_ρ is a ball of radius $\rho > 0$ that contains the scatterer D with $S_\rho = \partial B_\rho$ and $H^{-1/2}(S_\rho)$ is the completion of $L_t^2(S_\rho)$. In order to study the existence of solution to this problem we need to define the following space for the domain D_1 with $S = S_0 \cup S_1$:

$$X(D_1, S) = \left\{ u \in H(\mathrm{curl}, D_1) : \nu \times u|_S \in L_t^2(S) \right\},$$

equipped with the norm

$$\|u\|_{X(D_1,S)}^2 = \|u\|_{(L^2(D_1))^3}^2 + \|\nabla \times u\|_{(L^2(D_1))^3}^2 + \|\nu \times u\|_{L^2(S)}^2.$$

We set

$$X(B_\rho, S) = X_{\mathrm{loc}}(D_0, S_0) \cup X(D_1, S)$$

and we define the space of the require solution

$$X := \left\{ u \in X\left(B_\rho, S\right) : u\big|_{D_0} \in X_{\text{loc}}\left(D_0, S_0\right), u\big|_{D_1} \in X\left(D_1, S\right), \nu \times u\big|_S \in L_t^2\left(S\right) \right\}.$$

We restate the problem (5)-(10) for the scattered electric field by eliminating the magnetic field and taking into account that the incident field is a solution of Equation (6) we have the following mixed boundary value problem:

Given $F \in \left(L_t^2\left(D_0\right)\right)^3$, $J, K \in L_t^2\left(S_0\right)$ and $U \in L_t^2\left(S_1\right)$, find $E \in X$ such that:

$$\nabla \times \nabla \times E - k_0^2 E = 0 \text{ in } D_0, \tag{16}$$

$$\nabla \times \nabla \times E - 2\beta\gamma^2 \nabla \times E - \gamma^2 E = F \text{ in } D_1 \tag{17}$$

$$\nu \times E_+ - \frac{1}{\delta} \nu \times E_- = K \text{ on } S_0, \tag{18}$$

$$\nu \times \nabla \times E_+ + \frac{k_0 k \beta\gamma^2}{\zeta} \nu \times E_- - \frac{k_0 k}{\zeta} \nu \times \nabla \times E_- = J \text{ on } S_0, \tag{19}$$

$$\nu \times E = U \text{ on } S_1, \tag{20}$$

$$\hat{x} \times \nabla \times E + i k_0 E = o\left(\frac{1}{|x|}\right), \ |x| \to \infty, \tag{21}$$

where $K = \left(\frac{1}{\delta} - 1\right) \nu \times E^i$, $F = 2\beta\gamma^2 \nabla \times E^i + \left(\gamma^2 - k_0^2\right) E^i$, $U = -\nu \times E^i$ and $J = -\nu \times \nabla \times E^i + \frac{k_0 k}{\zeta} \nu \times \nabla \times E^i - \frac{k_0 k \beta\gamma^2}{\zeta} \nu \times E^i$ and the field E is the scattered field of the problem (5)-(10). Also, E_+, E_- denote the limit values of E when the variable tends to the surface S_0 from outside and inside, respectively. We have the following result.

Theorem 2. Let $F \in \left(L^2\left(D_1\right)\right)^3$, $J, K \in L_t^2\left(S_0\right)$, $U \in L_t^2\left(S_0\right)$ then the problem (16)-(21) has a unique solution $E \in X$ satisfying

$$\|E\|_X \le C \left\{ \|F\|_{\left(L^2\left(D_1\right)\right)^3} + \|J\|_{L_t^2\left(S_0\right)} + \|K\|_{L_t^2\left(S_0\right)} + \|U\|_{L_t^2\left(S_1\right)} \right\}, \tag{22}$$

where $C > 0$ is independent of F, J, K, U.

Proof. Let $\left\{ \tilde{X} = x \in X, \nu \times u\big|_{S_1} = 0 \right\}$. We multiply (16) with a vector test function $\phi \in \tilde{X}$, apply successively the Gauss theorem for E and $\overline{\phi}$ on $B_\rho \cap D_0$ and taking into account the transmission boundary conditions we get the variational formulation for the problem (16)-(21)

$$\begin{aligned} &\int_{B_\rho \cap D_0} \left(\nabla \times E\right) \cdot \left(\nabla \times \overline{\phi}\right) \mathrm{d}x - k_0^2 \int_{B_\rho \cap D_0} E \cdot \overline{\phi} \, \mathrm{d}x + i k_0 \int_{S_\rho} G_e\left(\nu \times E\right) \cdot \overline{\phi}_T \mathrm{d}s \\ &+ \frac{k_0 k}{\zeta} \int_{D_1} \left[\nabla \times E \cdot \nabla \times \overline{\phi} - \beta\gamma^2 E \cdot \nabla \overline{\phi} - \gamma^2 E \cdot \overline{\phi} \right] \mathrm{d}x \\ &= \int_{S_0} J \cdot \overline{\phi}_T \mathrm{d}s + \frac{k_0 k}{\zeta} \int_{D_1} F \cdot \overline{\phi} \, \mathrm{d}x, \end{aligned} \tag{23}$$

where G_e is the electric to magnetic Calderon operator defined in [13], $\phi_T = \left(\nu \times \phi\right) \times \nu$ chosen such that $\phi_T = 0$ on the boundary S_1. Next, we define the following bilinear form $\mathcal{A} : X \times X \to \mathbb{C}$

$$\mathcal{A}(E,\phi) := \int_{B_\rho \cap D_0} (\nabla \times E)\cdot(\nabla \times \overline{\phi})\,\mathrm{d}x - k_0^2 \int_{B_\rho \cap D_0} E \cdot \overline{\phi}\,\mathrm{d}x + ik_0 \int_{S_\rho} G_e(\nu \times E)\cdot \overline{\phi}_T\,\mathrm{d}s + \frac{k_0 k}{\zeta}\int_{D_1}\left[\nabla \times E \cdot \nabla \times \overline{\phi} - \beta\gamma^2 E \cdot \nabla \overline{\phi} - \gamma^2 E \cdot \overline{\phi}\right]\mathrm{d}x, \tag{24}$$

for all $\phi \in X$. We also set

$$\mathcal{B}(\phi) = \int_{S_0} J \cdot \overline{\phi}_T\,\mathrm{d}s + \frac{k_0 k}{\zeta}\int_{D_1} F \cdot \overline{\phi}\,\mathrm{d}x, \tag{25}$$

therefore we deduce that (23) via (24)-(25) can be written

$$\mathcal{A}(E,\phi) = \mathcal{B}(\phi). \tag{26}$$

We will present a brief proof for the existence of the solution to (26) following the book [13] and the paper [6]. Due to Theorem 1 and the equivalence between the scattering problem (16)-(21) and (26) the latter has at most one solution [13].

By the definition of $Y(S)$, there exists a function $\tilde{U} \in H_0(\mathrm{curl}; B_\rho)$ such that $\nu \times \tilde{U}\big|_{S_1} = U$. If we set $E = W + \tilde{U}$, then $\nu \times W = 0$ on S_1 and substituting in (26) we get

$$\mathcal{A}(W,\phi) = \mathcal{B}(\phi) - \mathcal{A}(\tilde{U},\phi). \tag{27}$$

If $W \in X$ is a solution of (27) then $E = W + \tilde{U} \in X(B_\rho, S)$ is a solution to the equivalent problem (26). We follow ([13], Theorem 10.2) and [6] and by setting the space of functions $\tilde{\Sigma} = \left\{p \in H^1(B_\rho \setminus D_2) : p\big|_{S_1} = 0\right\}$ we prove that (27) has a unique solution in $\nabla\tilde{\Sigma}$ for every $\xi \in \tilde{\Sigma}$

$$\mathcal{A}(\nabla p, \nabla \xi) = \mathcal{B}(\nabla \xi) - \mathcal{A}(\tilde{U}, \nabla \xi). \tag{28}$$

Next, we use the Helmholtz decomposition as in ([13], Section 10.3.1). To this end, we define the space $\tilde{X}_0 = \left\{u \in X : \mathcal{A}(u, \nabla \xi) = 0, \forall \xi \in \tilde{\Sigma}\right\}$ and we prove that X can be written as the direct sum of $\tilde{X}_0$ and $\nabla\tilde{\Sigma}$

$$X = \tilde{X}_0 \oplus \nabla\tilde{\Sigma}.$$

It can be shown that $\tilde{X}_0$ is compactly embedded in $\left(L^2(B_\rho \setminus D_2)\right)^3$ following the proof of Lemma 10.4 page 268 in [13]. We seek solution $W = W_0 + \nabla p$ where $W_0 \in \tilde{X}_0$ and $\nabla p \in \nabla\tilde{\Sigma}$. Substituting in (27) we take,

$$\mathcal{A}(W_0 + \nabla p, \psi + \nabla \xi) = B(\psi + \nabla \xi) - \mathcal{A}(\tilde{U}, \psi + \nabla \xi)\ \forall \psi \in \tilde{X}_0,\ \nabla \xi \in \nabla\tilde{\Sigma}.$$

Via (62) and the definition of space $\tilde{X}_0$ we get

$$\mathcal{A}(W_0, \psi) = B(\psi) - \mathcal{A}(\tilde{U}, \psi) - \mathcal{A}(\nabla p, \psi). \tag{29}$$

We can show that (29) has a unique solution in $\tilde{X}_0$ following a similar proof to theorem 10.6 page 271 [13]. The estimate (22) follows from Equations (62), (29) and a duality argument ([13], p: 272).

3. The Inverse Scattering Problem

In order to establish uniqueness for the inverse scattering problem, we will prove a mixed reciprocity relation. We assume that the scatterer D is excited by a plane and a spherical electromagnetic wave. The mixed reciprocity theorem relates the scattered field due to the incident plane electric wave and the far-field pattern due to the incident spherical wave. Such reciprocity relations are used in the point-source method for solving inverse scattering problems [20]. A mixed reciprocity theorem is contained in [21] for chiral media.

The plane electromagnetic wave for achiral case (in normal form) is

$$E^i(x;d,q)=\frac{i}{k_0}\nabla\times\nabla\times\left(q\mathrm{e}^{ik_0 r\cdot d}\right)=ik_0 q\mathrm{e}^{ik_0 x\cdot d}, \tag{30}$$

$$H^i(x;d,q)=d\times E^i(x;d,q)=ik_0 d\times q\mathrm{e}^{ik_0 x\cdot d}, \tag{31}$$

where the unit vector d is the direction of propagation and the constant vector q is the polarization. By writing $\left(E^0(x;d,q),H^0(x;d,q)\right)$, $\left(E(x;d,q),H(x;d,q)\right)$, $\left(E^s(x;d,q),H^s(x;d,q)\right)$, $\left(E^\infty(\hat{x};d,q),H^\infty(\hat{x};d,q)\right)$ we denote the dependence on the direction of propagation d and polarization q for the total fields in D_0 and D_1, the scattered fields and the far-field patterns, respectively. The incident spherical electromagnetic wave due to a point source with position vector $a\in D_0$, with respect to the origin, is given by

$$E_a^i(x;p)=\frac{i}{k_0}\nabla\times\nabla\times\left(p\Phi(x,a;k_0)\right), \tag{32}$$

$$H_a^i(x;p)=\nabla\times\left(p\Phi(x,a;k_0)\right), \tag{33}$$

where $\Phi(x,a;k_0)=\dfrac{\mathrm{e}^{ik_0|x-a|}}{4\pi|x-a|}$ and $x\neq a$. The wave field $\left(E_a^i(x;p),H_a^i(x;p)\right)$ represents the field generated by an electric dipole with polarization p. We shall denote the total fields in D_0 and D_1, the far-field patterns and the scattered fields by writing $\left(E_a^0(x;p),H_a^0(x;p)\right)$, $\left(E_a(x;p),H_a(x;p)\right)$, $\left(E_a^\infty(x;p),H_a^\infty(x;p)\right)$, and $\left(E_a^s(x;p),H_a^s(x;p)\right)$, respectively.

As it is well-known, [1], in a homogeneous isotropic chiral medium, the electromagnetic field is composed of Left Circularly Polarized (LCP) and Right Circularly Polarized (RCP) components which are propagated with different face speeds. In the chiral domain D_1 the electromagnetic field has to be expressed in terms of LCP and RCP Beltrami fields Q_L and Q_R respectively via the Bohren decomposition (see [1] [2]), $E=Q_L+Q_R$, $H=\dfrac{\sqrt{\varepsilon}}{i\sqrt{\mu}}(Q_L-Q_R)$. Hence, using the similar scaling as before we get

$$E=Q_L+Q_R,\ H=-i(Q_L-Q_R). \tag{34}$$

These fields satisfy the equations

$$\nabla\times Q_L=\gamma_L Q_L,\ \nabla\times Q_R=-\gamma_R Q_R, \tag{35}$$

where γ_L and γ_R given by

$$\gamma_L=\frac{k}{1-\beta k},\ \gamma_R=\frac{k}{1+\beta k}$$

are wave numbers. From (35) by applying the curl operator we get

$$\nabla\times\nabla\times Q_L=\gamma_L^2 Q_L,\quad \nabla\times\nabla\times Q_R=\gamma_R^2 Q_R.$$

Also Q_L and Q_R are divergence free and satisfy the vector Helmholtz equation

$$\Delta Q_L+\gamma_L^2 Q_L=0,\ \Delta Q_R+\gamma_R^2 Q_R=0.$$

Using Beltrami fields, we consider an incident plane electromagnetic wave in a chiral medium of the form [1]

$$E^i\left(x\mid d,p_L,p_R\right)=Q_L^i\left(x;d,p_L\right)+Q_R^i\left(x;d,p_R\right), \tag{36}$$

$$H^i\left(x\mid d,p_L,p_R\right)=-i\left(Q_L^i\left(x;d,p_L\right)-Q_R^i\left(x;d,p_R\right)\right), \tag{37}$$

where the LCP component is

$$Q_L^i\left(x;d,p_L\right)=p_L \mathrm{e}^{i\gamma_L d\cdot x} \tag{38}$$

and the RCP component is

$$Q_R^i\left(x;d,p_R\right)=p_R \mathrm{e}^{i\gamma_R d\cdot x}, \tag{39}$$

with corresponding polarizations p_L and p_R. We also assume that the LCP and RCP waves have the same direction of propagation d and it holds $p_L\cdot d=p_R\cdot d=0$ ([1], p: 476). We denote by $\left(E^0\left(x\mid d,p_L,p_R\right),H^0\left(x\mid d,p_L,p_R\right)\right)$, $\left(E\left(x\mid d,p_L,p_R\right),H\left(x\mid d,p_L,p_R\right)\right)$, $\left(E^s\left(x\mid d,p_L,p_R\right),H^s\left(x\mid d,p_L,p_R\right)\right)$ and $\left(E^\infty\left(x\mid d,p_L,p_R\right),H^\infty\left(x\mid d,p_L,p_R\right)\right)$ the dependence on the direction of propagation and the polarizations p_L, p_R of the total fields in D_0 and D_1, the scattered fields and the far-field patterns, respectively. Next, we consider the Bohren decomposition for the chiral spherical waves due to a source located at a point with vector position $a\in D_1$

$$E_a^i\left(x\mid p_L,p_R\right)=Q_{aL}^i\left(x;p_L\right)+Q_{aR}^i\left(x;p_R\right), \tag{40}$$

$$H_a^i\left(x\mid p_L,p_R\right)=-i\left(Q_{aL}^i\left(x;p_L\right)-Q_{aR}^i\left(x;p_R\right)\right), \tag{41}$$

where the LCP point source Q_{aL}^i is given by

$$Q_{aL}^i\left(x;p_L\right)=\tilde{T}_L\Phi\left(x,a;\gamma_L\right)\cdot p_L, \tag{42}$$

where $\tilde{T}_L=\frac{k}{2\gamma^2}\left(\gamma_L\tilde{I}+\frac{1}{\gamma_L}\nabla\oplus\nabla+\nabla\times\tilde{I}\right)$ with $\tilde{I}=\hat{x}\oplus\hat{x}+\hat{y}\oplus\hat{y}+\hat{z}\oplus\hat{z}$ be the identity dyadic and

$$\Phi\left(x,a;\gamma_L\right)=\frac{\mathrm{e}^{i\gamma_L|x-a|}}{4\pi|x-a|}.$$

The RCP point source Q_{aR}^i is given by

$$Q_{aR}^i\left(x;p_R\right)=\tilde{T}_R\Phi\left(x,a;\gamma_R\right)\cdot p_R, \tag{43}$$

where

$$\tilde{T}_R=\frac{k}{2\gamma^2}\left(\gamma_R\tilde{I}+\frac{1}{\gamma_R}\nabla\oplus\nabla-\nabla\times\tilde{I}\right)$$

and

$$\Phi(x,a;\gamma_R)=\frac{e^{i\gamma_R|x-a|}}{4\pi|x-a|}.$$

More details for spherical waves in a chiral medium can be found in [21]. The spherical chiral [21] and achiral [18] fields satisfy the Silver-Müller radiation condition [3]. We also state that when the point source tends to infinity the spherical waves become plane with direction of propagation $-\hat{a}$ [18] [21]. For two electromagnetic fields (E_1,H_1) and (E_2,H_2) on a surface S we introduce the Twersky notation

$$\{E_1,E_2\}_S=\int_S\left[\nu\cdot(E_1\times H_2-E_2\times H_1)\right]ds. \tag{44}$$

In particular, for the chiral case by applying the Bohren decomposition $E_j=Q_{jL}+Q_{jR}$, $H_j=-i(Q_{jL}-Q_{jR})$, $j=1,2$, relation (44) becomes

$$\{E_1,E_2\}_S=\frac{2}{i}\int_S\left[\nu\cdot(Q_{1L}\times Q_{2L}-Q_{1R}\times Q_{2R})\right]ds. \tag{45}$$

For the spherical waves we prove the following properties.

Lemma 1. Let $(E_a^i(x;p),H_a^i(x;p))$ be an incident spherical electromagnetic wave due to a point source $a\in D_0$ and let $(E^i(x;-d,q),H^i(x;-d,q))$ be an incident plane wave with direction of propagation -*d*. Then, we have

$$\lim_{r\to\infty}\{E_a^i(x;p),E^s(x;-d,q)\}_{S_r}=0, \tag{46}$$

$$\lim_{\epsilon\to 0}\{E_a^i(x;p),E^s(x;-d,q)\}_{S_{a,\epsilon}}=-\frac{i}{k}p\cdot E^s(x;-d,q), \tag{47}$$

where S_r is a large sphere of radius r enclosing the scatterer D and the small sphere $S_{a,\epsilon}$ which is of radius ϵ centered at *a*.

Proof. The incident spherical and the scattered waves satisfy the Silver-Müller radiation condition (10). Hence, we can make use of the radiation conditions (59) and (62) of the paper [22] and the proof of (46) is immediate. The proof of (47) is based on Lemma 8 of [22] taking into account the normal form of the spherical incident wave (32), (33).

Lemma 2. Let $(E_a^i(x|p_L,p_R),H_a^i(x|p_L,p_R))$ be an incident spherical chiral electromagnetic wave due to a point source $a\in D_1$ and let $(E^i(x;-d,q),H^i(x;-d,q))$ be an incident plane electromagnetic wave with direction of propagation $-d$. Then, we have

$$\lim_{r\to\infty}\{E_a^i(x|p_L,p_R),E^s(x;-d|q)\}_{S_r}=0, \tag{48}$$

$$\begin{aligned}&\lim_{\epsilon\to 0}\{E_a^i(x|p_L,p_R),E(x;-d|q)\}_{S_{a,\epsilon}}\\&=-\frac{ik}{\gamma^2}\left(p_L\cdot Q_L(x;-d,q)+p_R\cdot Q_R(x;-d,q)\right),\end{aligned} \tag{49}$$

where $S_{a,\epsilon}$ is a small sphere of radius ϵ centered at *a* and S_r is a large sphere of radius r enclosing the scatterer D.

Proof. For the proof of (48), taking into account that $E_a^i(x \mid p_L, p_R)$, $E^s(x;-d \mid q)\}_{S_r}$ satisfy the Silver-Müller radiation condition (10), we apply the same procedure as in Lemma 1 for (46). For Equation (49) we use (45) to obtain the relation

$$\left\{E_a^i(x \mid p_L, p_R), E(x;-d \mid q)\right\}_{S_{a,\epsilon}} = -2i\int_{S_{a,\epsilon}}\left[\nu\cdot\left(Q_{aL}^i \times Q_L - Q_{aR}^i \times Q_R\right)\right]\mathrm{ds}.$$

We substitute Q_{aL}^i, Q_{aR}^i by (42) and (43) respectively and after some calculations we obtain for the LCP part

$$\begin{aligned}\int_{S_{a,\epsilon}}\left[\nu\cdot\left(Q_{aL}^i \times Q_L\right)\right]\mathrm{ds} = &\frac{k}{2\gamma^2\gamma_L}\int_{S_{a,\epsilon}}\nu\cdot(\Phi p_L \times Q_L)\mathrm{ds}\\ &+\frac{k}{2\gamma^2\gamma_L}\int_{S_{a,\epsilon}}\nu\cdot\left[(\nabla\oplus\nabla\Phi\cdot p_L)\times Q_L\right]\mathrm{ds}\\ &+\frac{k}{2\gamma^2}\int_{S_{a,\epsilon}}\nu\cdot\left[\nabla\times(\nabla\Phi\cdot p_L)Q_L\right]\mathrm{ds}.\end{aligned}$$

The third integral on the right-hand side vanishes by the Stoke's theorem. Applying the mean value theorem on the remaining two integrals and letting $\epsilon\to 0$ we obtain

$$-2i\int_{S_{a,\epsilon}}\left[\nu\cdot\left(Q_{aL}^i \times Q_L\right)\right]\mathrm{ds} = -\frac{ik}{\gamma^2}p_L\cdot Q_L. \tag{50}$$

We follow the same procedure for the RCP part

$$2i\int_{S_{a,\epsilon}}\left[\nu\cdot\left(Q_{aR}^i \times Q_R\right)\right]\mathrm{ds} = -\frac{ik}{\gamma^2}p_R\cdot Q_R. \tag{51}$$

The proof is complete.

Remark 1. We note that relations (47) of Lemma 1 and (49) of Lemma 2 are in general valid for bounded functions.

Now we will prove a mixed reciprocity theorem that connects the far-field pattern of a spherical wave with the scattered field of a plane wave.

Theorem 3. Let $E_a^i(x;p)$ be an incident achiral electric spherical wave due to a point source $a\in D_0$ with polarization p and $E^i(x;-d,q)$ be an incident electric plane wave with polarization q and direction of propagation $-d$. We also consider a chiral incident electric wave $E_a^i(x \mid p_L, p_R)$ due to a point source located at $a\in D_1$. Then, we have

$$4\pi q\cdot E_a^\infty(d;p) = p\cdot E^s(a;-d,q),\ a\in D_0, \tag{52}$$

$$\begin{aligned}&4\pi q\cdot E_a^\infty(d \mid p_L, p_R)\\ &=\frac{k^2}{\gamma^2\delta\zeta}\left(p_L\cdot Q_L^s(a;-d,q) + p_R\cdot Q_R^s(a;-d,q)\right)\\ &\quad+\frac{k^2}{\gamma^2}\left(\frac{1}{\delta\zeta}-1\right)\left(p_L\cdot Q_L^i(a;-d,q)+p_R\cdot Q_R^i(a;-d,q)\right),\ a\in D_1.\end{aligned} \tag{53}$$

Proof

For $a\in D_0$, see **Figure 2**, in view of (5) and due to the bilinearity of (44), we have

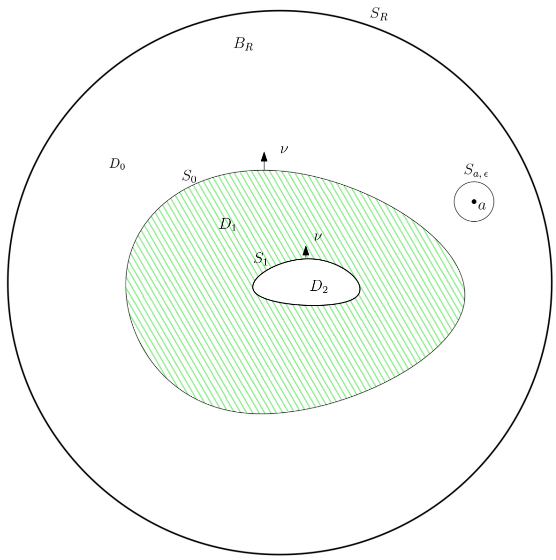

Figure 2. Point source in D_0.

$$\begin{aligned}&\left\{E_a^0(x;p),E^0(x;-d,q)\right\}_{S_0}\\&=\left\{E_a^i(x;p),E^i(x;-d,q)\right\}_{S_0}+\left\{E_a^i(x;p),E^s(x;-d,q)\right\}_{S_0}\\&\quad+\left\{E_a^s(x;p),E^i(x;-d,q)\right\}_{S_0}+\left\{E_a^s(x;p),E^s(x;-d,q)\right\}_{S_0}.\end{aligned} \tag{54}$$

The incident fields are regular solutions of (16) in D. Applying the Gauss theorem, we take

$$\left\{E_a^i(x;p),E^i(x;-d,q)\right\}_{S_0}=0. \tag{55}$$

For the scattered fields, we consider a sphere S_r of radius r large enough to include the scatterer. We apply the Gauss' theorem to transform (44) from the boundary S_0 to the boundary of the sphere S_r. Letting $r\to\infty$ we pass to the radiation zone and via the radiation condition we conclude

$$\left\{E_a^s(x;p),E^s(x;-d,q)\right\}_{S_0}=0. \tag{56}$$

For the integral of the total fields, we use the transmission conditions (8) on S_0 and the boundary condition (9) on S_1 and applying Gauss' theorem in D_1 we conclude to

$$\left\{E_a^0(x;p),E^0(x;-d,q)\right\}_{S_0}=0. \tag{57}$$

For the term $\left\{E_a^i(x;p),E^s(x;-d,q)\right\}_{S_0}$ we consider again the sphere S_r and a small sphere $S_{a,\epsilon}$ of radius ϵ centered at a which is inside the S_r. We apply Gauss' theorem in exterior to S_0 and $S_{a,\epsilon}$ and interior to S_r, we take

$$\begin{aligned}&\left\{E_a^i(x;p),E^s(x;-d,q)\right\}_{S_r}-\left\{E_a^i(x;p),E^s(x;-d,q)\right\}_{S_0}\\&-\left\{E_a^i(x;p),E^s(x;-d,q)\right\}_{S_{a,\varepsilon}}=0.\end{aligned}$$

Next by applying Lemma 1 we get

$$\left\{E_a^i(x;p), E^s(x;-d,q)\right\}_{S_0} = \frac{i}{k} E^s(a;-d,q)\cdot p. \quad (58)$$

Finally, taking into account the integral representation of the far-field pattern relation (6.24) of [12] we have

$$\left\{E_a^s(x;p), E^i(x;-d,q)\right\}_{S_0} = -\frac{4\pi i}{k} q\cdot E_a^\infty(d;p). \quad (59)$$

Substituting (55)-(59) into (54) we get (52).

For $a\in D_1$, see **Figure 3**, taking into account (5) and due to the bilinearity of (44), we have

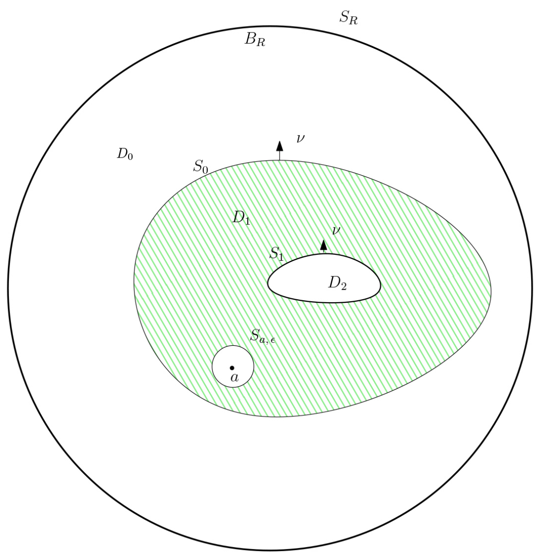

Figure 3. Point source in D_1.

$$\begin{aligned}&\left\{E_a^0(x\mid p_L,p_R), E^0(x;-d,q)\right\}_{S_0}\\ &=\left\{E_a^i(x\mid p_L,p_R), E^i(x;-d,q)\right\}_{S_0}+\left\{E_a^i(x\mid p_L,p_R), E^s(x;-d,q)\right\}_{S_0}\\ &\quad+\left\{E_a^s(x\mid p_L,p_R), E^i(x;-d,q)\right\}_{S_0}+\left\{E_a^s(x\mid p_L,p_R), E^s(x;-d,q)\right\}_{S_0}.\end{aligned} \quad (60)$$

We consider again the sphere S_r. Applying the divergence theorem in the region exterior to S_0 and interior to S_r and letting $r\to\infty$ we pass to the radiation zone. Taking into account that the scattered and spherical waves satisfy the Silver-Müller radiation condition and by using Lemma 2 we get

$$\left\{E_a^i(x\mid p_L,p_R), E^s(x;-d,q)\right\}_{S_0} = \left\{E_a^s(x\mid p_L,p_R), E^s(x;-d,q)\right\}_{S_0} = 0. \quad (61)$$

As in the relation (59), for the chiral case we have

$$\left\{E_a^s(x\mid p_L,p_R), E^i(x;-d,q)\right\}_{S_0} = -\frac{4\pi i}{k} q\cdot E_a^\infty(d\mid p_L,p_R). \quad (62)$$

Next, we apply the Gauss' theorem in D_1 apart from the sphere $S_{a,\epsilon}$

$$\begin{aligned}&\left\{E_a^i(x\mid p_L,p_R), E^i(x;-d,q)\right\}_{S_0}\\ &=\left\{E_a^i(x\mid p_L,p_R), E^i(x;-d,q)\right\}_{S_1}+\left\{E_a^i(x\mid p_L,p_R), E^i(x;-d,q)\right\}_{S_{a,\epsilon}}.\end{aligned}$$

The incident fields are regular solutions of (16) in D_2 and hence

$$\left\{E_a^i\left(x \mid p_L, p_R\right), E^i\left(x;-d,q\right)\right\}_{S_1}=0.$$

For the sphere $S_{a,\epsilon}$, we obtain

$$\left\{E_a^i\left(x \mid p_L, p_R\right), E^i\left(x;-d,q\right)\right\}_{S_{a,\epsilon}}$$
$$=-2i\int_{S_{a,\epsilon}}\left[v\cdot\left(Q_{aL}^i\left(x;p_L\right)\times Q_L^i\left(x;-d,q\right)-Q_{aR}^i\left(x;p_R\right)\times Q_R^i\left(x;-d,q\right)\right)\right]\mathrm{d}s.$$

Letting $\epsilon\to 0$ we apply Lemma 2 (where the total plane wave has been replaced by the incident plane wave) we get

$$\lim_{\epsilon\to 0}\left\{E_a^i\left(x \mid p_L, p_R\right), E^i\left(x;-d,q\right)\right\}_{S_{a,\epsilon}}$$
$$=-\frac{ik}{\gamma^2}\left(p_L\cdot Q_L^i\left(x;-d,q\right)+p_R\cdot Q_R^i\left(x;-d,q\right)\right).$$

Therefore

$$\left\{E_a^i\left(x \mid p_L, p_R\right), E^i\left(x;-d,q\right)\right\}_{S_0}$$
$$=-\frac{ik}{\gamma^2}\left(p_L\cdot Q_L^i\left(r;-d,q\right)+p_R\cdot Q_R^i\left(x;-d,q\right)\right). \tag{63}$$

For the integral of the total exterior fields, we apply the conditions (8) and (9) and using Gauss' theorem in D_1 apart from the sphere $S_{a,\epsilon}$ we have that

$$\left\{E_a^0\left(x \mid p_L, p_R\right), E^0\left(x;-d,q\right)\right\}_{S_0}=\frac{1}{\delta\zeta}\left\{E_a\left(x \mid p_L, p_R\right), E\left(x;-d,q\right)\right\}_{S_{a,\epsilon}}$$
$$=\frac{1}{\delta\zeta}\left\{E_a^i\left(x \mid p_L, p_R\right), E\left(x;-d,q\right)\right\}_{S_{a,\epsilon}}+\frac{1}{\delta\zeta}\left\{E_a^s\left(x \mid p_L, p_R\right), E\left(x;-d,q\right)\right\}_{S_{a,\epsilon}}.$$

The second integral of the right-hand side is equal to zero since $E_a^s\left(x \mid p_L, p_R\right)$ and $E\left(x;-d,q\right)$ are regular solutions of (7) in the sphere $S_{a,\epsilon}$ in D_1. For the first integral of the right-hand side, we apply Formula (49) and we get

$$\left\{E_a^0\left(x \mid p_L, p_R\right), E^0\left(x;-d,q\right)\right\}_{S_0}$$
$$=\frac{-ik}{\gamma^2\delta\zeta}\left(p_L\cdot Q_L\left(a;-d,q\right)+p_R\cdot Q_R\left(a;-d,q\right)\right). \tag{64}$$

Taking into account that $Q_A=Q_A^i+Q_A^s,\ A=L,R$ and substituting (61)-(63) into (60) we get (53).

Lemma 3. Let $D_2,\tilde{D}_2$ be subsets of D and let G be the unbounded component of $\mathbb{R}^3\setminus\left(\overline{D_2\cup\tilde{D}_2}\right)$. Assume that $E^\infty\left(\hat{x};d,q\right)=\tilde{E}^\infty\left(\hat{x};d,q\right)$ for all $\hat{x},d\in S^2$, $q\in\mathbb{R}^3$, where $\tilde{E}^\infty\left(\hat{x};d,q\right)$ is the electric far field pattern of the scattered field $\tilde{E}^s\left(x;d,q\right)$ that corresponds to the same plane wave $E^i\left(x;d,q\right)$ incident to $\tilde{D}_2$. Let $E_z^i\left(x \mid p_L, p_R\right)$ be the chiral spherical electric wave due to a source located at a point with position vector $z\in D_1\cap G$ and let $E_z^s=E_z^s\left(x \mid p_L, p_R\right)$ be the unique solution of the problem:

$$\nabla\times\nabla\times E_z^s-k_0^2E_z^s=0\ \text{in}\ D_0, \tag{65}$$

$$\nabla\times\nabla\times E_z^s-2\beta\gamma^2\nabla\times E_z^s-\gamma^2E_z^s=\left(\gamma^2-k_0^2\right)E_z^i+2\beta\gamma^2\nabla\times E_z^i\ \text{in}\ D_1, \tag{66}$$

$$\nu \times E_{z+}^{s} = \delta^{-1} \nu \times E_{z-}^{s} + \left(\delta^{-1} - 1\right) \nu \times E_{z}^{i} \text{ on } S_0, \tag{67}$$

$$\begin{aligned} &\nu \times \nabla \times E_{z+}^{s} - \frac{k_0 k}{\zeta \gamma^2} \nu \times \nabla \times E_{z-}^{s} + \frac{k_0 k \beta}{\zeta} \nu \times E_{z-}^{s} \\ &= \left(\frac{k_0 k}{\zeta \gamma_2} - 1\right) \nu \times \nabla \times E_{z}^{i} - \frac{k_0 k \beta}{\zeta} \nu \times E_{z}^{i} \text{ on } S_0, \end{aligned} \tag{68}$$

$$\nu \times E_{z-}^{s} = -\nu \times E_{z}^{i} \text{ on } S_1, \tag{69}$$

$$\hat{x} \times \nabla \times E_{z}^{s} + i k_0 E_{z}^{s} = o\left(\frac{1}{|x|}\right), \ |x| \to \infty. \tag{70}$$

Assume that $\tilde{E}_z^s = \tilde{E}_z^s\left(x \mid p_L, p_R\right)$ is the unique solution of the problem (65)-(70) with D_2 replaced by $\tilde{D}_2$ and D_1 replaced by $\tilde{D}_1 := D \setminus \overline{\tilde{D}_2}$. Then we have

$$E_z^s\left(x \mid p_L, p_R\right) = \tilde{E}_z^s\left(x \mid p_L, p_R\right), \quad x \in D_1 \cap G. \tag{71}$$

Remark 2. We note that the problem (65)-(70) has a unique solution due to Theorem 2 (**Figure 4**).

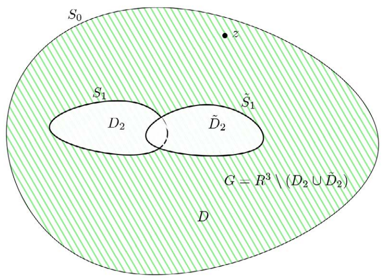

Figure 4. Uniqueness of D_2.

Proof. Since $E^\infty\left(\hat{x}; d, q\right) = \tilde{E}^\infty\left(\hat{x}; d, q\right)$ for all $\hat{x}, d \in S^2$, $q \in \mathbb{R}^3$ Rellich's lemma [12] (Theorem 6.9, p.164) gives

$$E^s\left(x; d, q\right) = \tilde{E}^s\left(x; d, q\right) \text{ in } D_0 = \mathbb{R}^3 \setminus \bar{D}.$$

By continuity on the boundary S_0 and by using Holmgren's uniqueness theorem ([19] p: 194) we get that

$$E^s\left(z; d, q\right) = \tilde{E}^s\left(z; d, q\right) \text{ for } z \in D_1 \cap G.$$

Taking into account that $H^s\left(z; d, q\right) = \tilde{H}^s\left(z; d, q\right)$, the corresponding Beltrami fields in the chiral domain $D_1 \cap G$ satisfy

$$Q_L^s\left(z; d, q\right) + Q_R^s\left(z; d, q\right) = \tilde{Q}_L^s\left(z; d, q\right) + \tilde{Q}_R^s\left(z; d, q\right), \tag{72}$$

$$Q_L^s\left(z; d, q\right) - Q_R^s\left(z; d, q\right) = \tilde{Q}_L^s\left(z; d, q\right) - \tilde{Q}_R^s\left(z; d, q\right), \tag{73}$$

from which we conclude $Q_L^s\left(z; d, q\right) = \tilde{Q}_L^s\left(z; d, q\right)$ and $Q_R^s\left(z; d, q\right) = \tilde{Q}_R^s\left(z; d, q\right)$. Therefore we obtain $p_L \cdot Q_L^s\left(z; d, q\right) = p_L \cdot \tilde{Q}_L^s\left(z; d, q\right)$, $p_R \cdot Q_R^s\left(z; d, q\right) = p_R \cdot \tilde{Q}_R^s\left(z; d, q\right)$ for every $p_L, p_R \in \mathbb{R}^3$. We make use of the mixed reciprocity Theorem 3 to conclude to

$$E_z^\infty\left(d \mid p_L, p_R\right) = \tilde{E}_z^\infty\left(d \mid p_L, p_R\right), \ z \in D_1 \cap G, \ d \in S^2.$$

By Rellich's lemma again we have

$$E_z^s(x \mid p_L, p_R) = \tilde{E}_z^s(x \mid p_L, p_R),\ x \in S_0,\ z \in D_1 \cap G,\ p \in \mathbb{R}^3.$$

By continuity on the boundary S_0 and by using Holmgren's theorem we derive that

$$E_z^s(x \mid p_L, p_R) = \tilde{E}_z^s(x \mid p_L, p_R),\ x \in D_1 \cap G,$$

which proves the lemma.

We consider a scatterer $\tilde{D}_0$ with boundary $\tilde{S}_0$ with a perfectly conducting obstacle $\tilde{D}_2$ inside $\tilde{D}_0$. Then, we can formulate the following theorem.

Theorem 4. Let D_2 and $\tilde{D}_2$ be two perfectly conducting obstacles such that $E^\infty(\hat{x}; d, q) = \tilde{E}^\infty(\hat{x}; d, q)$, for all $\hat{x}, d \in S^2$ and $q \in \mathbb{R}^3$, for the same incident electric field $E^i(x; d, q)$, then $D_2 = \tilde{D}_2$ ($S_1 = \tilde{S}_1$).

Proof. Let $D_2 \neq \tilde{D}_2$ ($S_1 \neq \tilde{S}_1$). Then without loss of generality we may choose $z_0 \in S_1$ and $z_0 \notin \overline{\tilde{D}_2}$ and $h > 0$ such that the sequence

$$z_j = z_0 + \frac{h}{j}\nu(z_0),\ \ j = 1, 2, \cdots,$$

is contained in $D \cap G$, where G is the unbounded component of $\mathbb{R}^3 \setminus \overline{(D_2 \cup \tilde{D}_2)}$ and $\nu(z_0)$ the outward normal to ∂D_2 at z_0. Taking into account that $z_j \notin \overline{\tilde{D}_2}$ and in view of the well posedness of the direct problem (65)-(70) we have that $\left|\nu \times \tilde{E}_{z_j}^s(z_0 \mid p_L, p_R)\right| \le c$ for appropriate positive constant c uniformly for $j \ge 1$ and for all polarizations $p_L, p_R \in \mathbb{R}^3$. We also note that on the boundary S_1 it holds,

$$\begin{aligned}\lim_{j\to\infty}\left|\nu \times \tilde{E}_{z_j}^s(z_0 \mid p_L, p_R)\right| &= \lim_{j\to\infty}\left|\nu \times E_{z_j}^s(z_0 \mid p_L, p_R)\right| \\ &= \lim_{j\to\infty}\left|-\nu \times E_{z_j}^i(z_0 \mid p_L, p_R)\right| \to \infty,\end{aligned}$$

for all $p_L, p_R \perp \nu(z_0)$. This is a contradiction, which implies $S_1 = \tilde{S}_1$.

4. Conclusion

In this work, we are concerned with the scattering model for a perfectly conducting core inside a chiral layer. First, we proved the well-posedness of the direct problem. In the sequel, we focused on the determination of the shape of the core. For this purpose, we proved a mixed reciprocity theorem which was used for the proof of the solution's uniqueness of an inverse scattering problem (*i.e.* the determination of the core's shape). In the same manner, we can work when the scatterer has more layers. A generalization of this work in case that the scatterer is embedded in a chiral environment is under consideration.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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