

A Hybrid ESA-CCD Method for Variable-Order Time-Fractional Diffusion Equations

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Abstract

In this paper, we study the solutions for variable-order time-fractional diffusion equations. A three-point combined compact difference (CCD) method is used to discretize the spatial variables to achieve sixth-order accuracy, while the exponential-sum-approximation (ESA) is used to approximate the variable-order Caputo fractional derivative in the temporal direction, and a novel spatial sixth-order hybrid ESA-CCD method is implemented successfully. Finally, the accuracy of the proposed method is verified by numerical experiments.

Keywords

Variable-Order Caputo Fractional Derivative, Combined Compact Difference Method, Exponential-Sum-Approximation

1. Introduction

In the last few decades, based on the rapid development of fractional calculus theory, fractional derivative has been proved to be able to more accurately describe physical processes with memory, heredity and path dependence properties [1], so fractional diffusion equations have become powerful tool for modeling abnormal diffusion phenomena and physical mechanics. However, many important dynamical problems exhibit the order of the fractional operator varying with time, space, or some other factors in [2] [3]. Therefore, the variable-order (VO) fractional derivatives played a more and more important role in the related applications of fields for science and engineering.

In a VO fractional diffusion equation, fractional differential operators are non-local operators, which means that when solving fractional diffusion equations, it is necessary to consider the information of the whole time or space domain, not

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just the local information. This non-locality makes the process of solving the equation more complicated and time-consuming. Fractional Caputo derivatives introduce new numerical difficulties, including the weak singular integral kernel. The existence of such an integral kernel makes the numerical integration process more complicated.

In 1993, Samko and Ross [4] formulated the concept of VO operators. Later, Lorenzo and Hartley [2] first introduced the concept of VO fractional derivative which is a function that can change with the change of time or space variables. In order to better describe some complex physical phenomena, some VO fractional variations have also been proposed by [5] [6]. The fractional derivative in a constant fractional diffusion equation remains constant throughout the solution domain, while the fractional derivative in a VO fractional diffusion equation changes with time, space, or other variables. This change makes the mathematical form of the equation more complicated and difficult to solve directly. In order to accurately solve the VO fractional diffusion equation, more intensive meshing and longer computation time are usually required, which increases the computation cost and resource consumption.

Recently, some scholars have proposed many effective methods to solve the numerical solutions of the corresponding equations with VO fractional derivatives. Currently, finite difference methods, spectral methods, matrix methods and spline interpolation methods are used to approximate VO fractional derivatives, where the finite difference methods include the explicit scheme, the implicit scheme and the Crank-Nicholson scheme. In 2009, Zhuang *et al.* [7] considered a VO fractional advection-diffusion equation with a nonlinear source term on a finite domain. Explicit and implicit Euler approximations for the equation were proposed. In 2014, Sierociuk *et al.* [8] introduced a numerical scheme based on matrix approach. In 2015 [9], second-order approximation formulas for the VO fractional time derivatives were derived. Bhrawy and Zaky [10] provided an accurate spectral collocation method for solving VO fractional nonlinear cable equations. In 2017, Moghaddam and Machado [11] put forward the linear B-spline approximation and the Du Fort-Frankel algorithm for a class of nonlinear time VO fractional partial differential equations. In 2020, Zheng *et al.* [12] developed a numerical approximation to a hidden-memory VO space-time fractional diffusion equation, which provided a physically more relevant VO fractional diffusion equation modeling. In 2021, Liao *et al.* [13] proposed a Crank-Nicolson type scheme with variable steps for the time fractional Allen-Cahn equation. In 2022, Du *et al.* [14] developed a temporal second-order finite difference scheme for a VO time-fractional wave partial differential equation in multiple space dimensions via the order reduction. Furthermore, finite difference methods provide the solution of the problem on mesh points only and accuracy of the techniques is reduced in non-smooth and non-regular domains, see [3] [15]-[18]. For more information on algorithms for VO partial differential equations, please refer to [19]-[29]. However, exponential-sum-approximation (ESA) techniques allow for more flexibility in

dealing with derivatives of different orders. The finite difference method is effective when dealing with fractional derivatives of fixed order, but may require complex mesh adjustment and difference scheme design when dealing with VO derivatives. In contrast, the approximation method based on exponential function may be easier to adapt to changes in order. Spectral methods may have high accuracy in solving fractional differential equations in the frequency domain, but they may require complex transformations when dealing with VO derivatives. The approximate method based on exponential function is processed directly in time domain, avoiding the complexity of frequency domain transformation.

Although spatially second-order or fourth-order methods were investigated in multiple publications for VO time-fractional partial differential equations, the study of higher order numerical method has been in its infancy. In addition, we note that the combined compact difference (CCD) method only requires the information of three points to obtain the numerical solutions of the partial differential equations, and can achieve sixth order accuracy, which is higher than many proposed methods. Due to these advantages of the CCD method and the fact that it has never been combined with exponential-sum-approximation (ESA) to solve the variable-order time-fractional diffusion equations (VO-TFDEs), we propose the hybrid ESA-CCD method to solve the VO-TFDEs. More precisely, the ESA strategy is employed for approximating the VO Caputo fractional derivative in the temporal direction, while the CCD method is incorporated in the space to achieve sixth-order accuracy. Numerical results are shown to demonstrate the effectiveness of the novel method.

Combining the two high-precision methods is expected to significantly improve the accuracy of numerical solutions, especially when dealing with complex or higher-order differential equations. Because of the need to deal with high-precision approximations in both space and time, need to store more data, so memory requirements can be higher. Due to the combination of the advantages of the two methods, the hybrid method may have a wider range of applications, such as financial engineering, materials science, biomedical engineering, etc. Combining two high-precision methods can result in a significant increase in algorithm complexity, requiring more computing resources and time. Stability analysis with hybrid methods can be more complex, as both CCD and ESA contributions to stability and their interactions need to be considered. While some accuracy can also be achieved with CCD or ESA alone, the hybrid method may further improve accuracy through complementary advantages. Although hybrid methods may increase computational complexity, in some cases, by optimizing the algorithm and parameter settings, it is possible to achieve higher computational efficiency than either method alone.

The remainder of this paper is organized as follows. In Section 2, a novel hybrid ESA-CCD method is proposed. For convenience, the hybrid ESA-CCD method is applied to solve the VO time-fractional diffusion equations (VO-TFDEs) in Section 3. In Section 4, numerical results are reported to demonstrate the efficiency

of the proposed method. Concluding remarks are given in Section 5.

2. ESA-CCD Method

Generally, when numerically solving time VO fractional differential equations, the approximation method of the time fractional derivative term is considered, then the approximation method of spatial variables is combined to solve the differential equation. Next, we will introduce the ESA method for approximating Caputo time variant fractional derivatives, as well as the CCD method for dealing with spatial derivative terms in equations. Unlike general methods, we do not directly discretize spatial variables, but instead treat them as unknowns to the equations. In the following, we will consider the ESA method approximation of variable order Caputo derivatives first.

2.1. ESA Method for VO Caputo Fractional Derivative

The VO Caputo fractional derivative is defined by [30]

$${}_0^c D_t^{\alpha(t)} u(t) \equiv \frac{1}{\Gamma(1-\alpha(t))} \int_0^t \frac{u'(\tau)}{(t-\tau)^{\alpha(t)}} d\tau,$$

where $\alpha(t) \in [\underline{\alpha}, \bar{\alpha}] \subset (0,1)$ is the VO function, $\Gamma(\cdot)$ is the Gamma function.

In order to approximate the VO Caputo fractional derivative in the time direction. Firstly, we have a lemma about the ESA.

Lemma 1. (see [31]) For arbitrary constant

- 1) $c > 0$;
- 2) $0 < \delta \leq t \leq 1$;
- 3) $0 < \varepsilon \leq \frac{1}{e}$.

Then there is a constant d , integers M and N satisfy

$$\begin{aligned} d &\leq \frac{2\pi}{\log 3 + c \log(\cos 1)^{-1} + \log \varepsilon^{-1}}, \\ N &\geq \frac{1}{d} \left(\frac{1}{c} \log \varepsilon + \frac{1}{c} \log \Gamma(1+c) \right), \\ M &\leq \frac{1}{d} \left(\log \frac{1}{\delta} + \log \log \frac{1}{\varepsilon} + \log c + \frac{1}{2} \right), \end{aligned} \tag{2.1}$$

such that

$$\left| t^{-c} - \sum_{j=N+1}^M \theta_{c,j} e^{-\lambda_j t} \right| \leq t^{-c} \varepsilon,$$

where $\theta_{c,j} = \frac{de^{cjd}}{\Gamma(c)}$, and $\lambda_j = e^{jd}$. In addition, the total number of terms can be defined as

$$Z_\varepsilon = M - N \leq \frac{1}{10} \left(2 \log \frac{1}{\varepsilon} + \log c + 2 \right) \left(\log \frac{1}{\delta} + \frac{1}{c} \log \frac{1}{\varepsilon} + \log \log \frac{1}{\varepsilon} + \frac{3}{2} \right).$$

Then we discuss how to approach VO Caputo fractional derivative. Generally,

the $L1$ formula is used to approximate the Caputo derivative, which is based on the interpolation polynomials to $u'(\tau)$, the expression is as follows:

$${}^L_0 D_t^{\alpha_k} u(t_k) = \frac{1}{\Gamma(1-\alpha_k)} \int_0^{t_k} \frac{\Lambda u'(\tau)}{(t_k - \tau)^{\alpha_k}} d\tau, \tag{2.2}$$

where $\Lambda u'(\tau)$ is the piecewise approximation function:

$$\Lambda u'(\tau) = \{ \Lambda^k u'(\tau) \mid \tau \in [t_{k-1}, t_k], k = 1, 2, \dots, n \},$$

with $\Lambda^k u'(\tau) = \frac{u(t_k) - u(t_{k-1})}{\Delta t}$ in the time interval $[t_{k-1}, t_k]$.

In fact, the integral in Formula (2.2) can divide into two parts

$$\begin{aligned} {}^L_0 D_t^{\alpha_k} u(t_k) &= \frac{1}{\Gamma(1-\alpha_k)} \left[\int_0^{t_{k-1}} \frac{\Lambda u'(\tau)}{(t_k - \tau)^{\alpha_k}} d\tau + \int_{t_{k-1}}^{t_k} \frac{\Lambda u'(\tau)}{(t_k - \tau)^{\alpha_k}} d\tau \right] \\ &\equiv \frac{1}{\Gamma(1-\alpha_k)} \left[\Phi_{\Delta t}^{0,k-1}(t_k) + \Phi_{\Delta t}^{k-1,k}(t_k) \right]. \end{aligned} \tag{2.3}$$

We observed that the second term of the above Formula (2.3) can be directly calculated using integration by parts:

$$\begin{aligned} \Phi_{\Delta t}^{k-1,k}(t_k) &= \int_{t_{k-1}}^{t_k} \frac{\Lambda u'(\tau)}{(t_k - \tau)^{\alpha_k}} d\tau = \Lambda^k u'(\tau) \frac{-1}{1-\alpha_k} (t_k - \tau)^{1-\alpha_k} \Big|_{t_{k-1}}^{t_k} \\ &= \frac{u(t_k) - u(t_{k-1})}{\Delta t^{\alpha_k} (1-\alpha_k)}, \end{aligned}$$

namely

$$\frac{1}{\Gamma(1-\alpha_k)} \Phi_{\Delta t}^{k-1,k}(t_k) = \frac{u(t_k) - u(t_{k-1})}{\Delta t^{\alpha_k} \Gamma(2-\alpha_k)}.$$

For the first term of the Formula (2.3), we obtain

$$\Phi_{\Delta t}^{0,k-1}(t_k) = \int_0^{t_{k-1}} \frac{\Lambda u'(\tau)}{(t_k - \tau)^{\alpha_k}} d\tau = T^{-\alpha_k} \int_0^{t_{k-1}} \Lambda u'(\tau) \left(\frac{t_k - \tau}{T} \right)^{-\alpha_k} d\tau. \tag{2.4}$$

By Lemma 1 we have

$$\alpha_k > 0, 0 < \frac{\Delta t}{T} \leq \frac{t_k - \tau}{T} \leq 1, \tau \in [0, t_{k-1}].$$

Hence, the kernel $\left(\frac{t_k - \tau}{T} \right)^{-\alpha_k}$ in Formula (2.4) can be approximated by

$$\Phi_{\Delta t}^{0,k-1}(t_k) \approx T^{-\alpha_k} \sum_{j=N+1}^M \theta_{k,j} \int_0^{t_{k-1}} \Lambda u'(\tau) e^{\frac{-\lambda_j(t_k - \tau)}{T}} d\tau,$$

where $\theta_{k,j} = \frac{de^{\alpha_k jd}}{\Gamma(\alpha_k)}$, $\lambda_j = e^{jd}$, d , M and N are defined in Formula (2.1).

Remark: From Lemma 1, the expected accuracy ε is given once, the selected parameters d , M and N will be determined. To guarantee better approximation in the actual calculation, for each time level, we always choose $\varepsilon = \left(\frac{\Delta t}{T} \right)^2$, and

$$d = \frac{2\pi}{\log 3 + \bar{\alpha} \log (\cos 1)^{-1} + \log \varepsilon^{-1}},$$

$$N = \left\lceil \frac{1}{d\bar{\alpha}} \left[\log \varepsilon + \log \Gamma(1 + \bar{\alpha}) \right] \right\rceil,$$

$$M = \left\lceil \frac{1}{d} \left(\log \frac{T}{\Delta t} + \log \log \varepsilon^{-1} + \log \bar{\alpha} + 2^{-1} \right) \right\rceil.$$

Finally, we obtain the ESA approximation formula for the VO Caputo fractional derivative:

$$\begin{aligned} {}_0^{ESA}D_t^{\alpha_k} u(t_k) &\equiv \frac{1}{\Gamma(1-\alpha_k)} \left[\Phi_{\Delta t, \varepsilon}^{0, k-1}(t_k) + \Phi_{\Delta t}^{k-1, k}(t_k) \right] \\ &= \frac{T^{-\alpha_k}}{\Gamma(1-\alpha_k)} \sum_{j=N+1}^M \theta_{k,j} \int_0^{t_{k-1}} \Lambda u'(\tau) e^{-\frac{\lambda_j(t_k-\tau)}{T}} d\tau \quad (2.5) \\ &\quad + \frac{u(t_k) - u(t_{k-1})}{\Delta t^{\alpha_k} \Gamma(2-\alpha_k)}, \quad k = 1, 2, 3, \dots, n. \end{aligned}$$

Let

$$\phi_{k,j} = \int_0^{t_{k-1}} \Lambda u'(\tau) e^{-\frac{\lambda_j(t_k-\tau)}{T}} d\tau,$$

then we have

$$\begin{aligned} {}_0^{ESA}D_t^{\alpha_k} u(t_k) &\equiv \frac{1}{\Gamma(1-\alpha_k)} \left[\Phi_{\Delta t, \varepsilon}^{0, k-1}(t_k) + \Phi_{\Delta t}^{k-1, k}(t_k) \right] \\ &= \frac{T^{-\alpha_k}}{\Gamma(1-\alpha_k)} \sum_{j=N+1}^M \theta_{k,j} \phi_{k,j} + \frac{u(t_k) - u(t_{k-1})}{\Delta t^{\alpha_k} \Gamma(2-\alpha_k)}, \quad k = 1, 2, 3, \dots, n. \end{aligned}$$

When $k = 1$, then $\phi_{1,j} = 0$,

$${}_0^{ESA}D_t^{\alpha_1} u(t_1) \equiv \frac{1}{\Gamma(1-\alpha_1)} \Phi_{\Delta t}^{0,1}(t_1) = \frac{u(t_1) - u(t_0)}{\Delta t^{\alpha_1} \Gamma(2-\alpha_1)}.$$

When $2 \leq k \leq n$, $\phi_{k,j}$ have the following recursive formula:

$$\begin{aligned} \phi_{k,j} &= \int_0^{t_{k-2}} \Lambda^{k-1} u'(\tau) e^{-\frac{\lambda_j(t_k-\tau)}{T}} d\tau + \int_{t_{k-2}}^{t_{k-1}} \Lambda^{k-1} u'(\tau) e^{-\frac{\lambda_j(t_k-\tau)}{T}} d\tau \\ &= \int_0^{t_{k-2}} \Lambda^{k-1} u'(\tau) e^{-\frac{\lambda_j(t_{k-1}-\tau)}{T}} \cdot e^{-\frac{\lambda_j(t_k-t_{k-1})}{T}} d\tau + \int_{t_{k-2}}^{t_{k-1}} \Lambda^{k-1} u'(\tau) e^{-\frac{\lambda_j(t_k-\tau)}{T}} d\tau \\ &= e^{-\frac{\lambda_j \Delta t}{T}} \phi_{k-1,j} + T \frac{e^{-\frac{\lambda_j \Delta t}{T}} - e^{-\frac{2\lambda_j \Delta t}{T}}}{\lambda_j \Delta t} \left[u(t_{k-1}) - u(t_{k-2}) \right]. \end{aligned}$$

2.2. CCD Method

In this section, we briefly review the CCD method, if $u(x, t) \in C^{8,2}(\Omega)$, defining a grid function

$$u_j^k = u(x_j, t_k), \quad (u_x)_j^k = u_x(x_j, t_k),$$

$$(u_{xx})_j^k = u_{xx}(x_j, t_k), \quad f_j^k = f(x_j, t_k), \quad 0 \leq j \leq m+1, \quad 0 \leq k \leq n.$$

The CCD method in [32] is established by local hermitian polynomial approximation function $u(x)$ between each cell $[x_{j-1}, x_{j+1}]$. In fact, the derivation of these relations can be easily obtained by Taylor expansion, which is expressed as

$$\begin{aligned} & \frac{7}{16}[(u_x)_{j+1}^k + (u_x)_{j-1}^k] + (u_x)_j^k - \frac{h}{16}[(u_{xx})_{j+1}^k - (u_{xx})_{j-1}^k] \\ &= \frac{15}{16h}(u_{j+1}^k - u_{j-1}^k) + O(h^6), \end{aligned} \quad (2.6)$$

$$\begin{aligned} & \frac{9}{8h}[(u_x)_{j+1}^k - (u_x)_{j-1}^k] - \frac{1}{8}[(u_{xx})_{j+1}^k + (u_{xx})_{j-1}^k] + (u_{xx})_j^k \\ &= \frac{3}{h^2}(u_{j+1}^k - 2u_j^k + u_{j-1}^k) + O(h^6). \end{aligned} \quad (2.7)$$

The CCD boundary formulas are expressed as follows:

$$\begin{aligned} & 14(u_x)_0^k + 16(u_x)_1^k + 2h(u_{xx})_0^k - 4h(u_{xx})_1^k \\ &= -\frac{1}{h}(31u_0^k - 32u_1^k + u_2^k) + O(h^5), \end{aligned} \quad (2.8)$$

$$\begin{aligned} & 14(u_x)_m^k + 16(u_x)_{m-1}^k - 2h(u_{xx})_m^k + 4h(u_{xx})_{m-1}^k \\ &= \frac{1}{h}(31u_m^k - 32u_{m-1}^k + u_{m-2}^k) + O(h^5). \end{aligned} \quad (2.9)$$

When solving partial differential equations, the ESA method is used to approximate fractional order time derivatives, while the spatial derivatives are processed using the CCD method to obtain the ESA-CCD method proposed above.

3. ESA-CCD Scheme for VO-TFDEs

For a class of partial differential equations containing Caputo temporal and spatial derivatives, we can apply the ESA-CCD scheme proposed in Section 2 for solving them. For simplicity, we study the following VO-TFDEs:

$${}^c D_t^{\alpha(t)} u(x, t) = \frac{\partial^2 u(x, t)}{\partial x^2} + f(x, t), \quad x \in (x_L, x_R), \quad t \in (0, T], \quad (3.1)$$

$$u(x, 0) = \varphi(x), \quad x \in [x_L, x_R], \quad (3.2)$$

$$u(x_L, t) = u(x_R, t) = 0, \quad t \in (0, T], \quad (3.3)$$

where $[x_L, x_R]$ is the spatial interval, $[0, T]$ is the time interval, $f(x, t)$ and $\varphi(x)$ are smooth functions.

To solve the VO-TFDEs, we uniformly divide the spatial interval $[x_L, x_R]$ into m parts and the time interval $[0, T]$ into n parts. We define $x_j = jh, (0 \leq j \leq m)$, $\varphi_j = \varphi(x_j), h = \frac{x_R - x_L}{m}$ and $t_k = k\tau$, where $\tau = \frac{T}{n}$. In order to obtain a high spatial accuracy finite difference method for solving time-fractional diffusion equations. By ESA (2.5) and CCD method (2.6) and (2.7) with boundary Formulas (2.8) and (2.9). We can obtain the following formulas:

$$\left\{ \begin{aligned} & {}^{ESA}_0 D_t^{\alpha_k} u_j^k = (u_{xx})_j^k + f_j^k + P_j^k, 0 \leq j \leq m, 1 \leq k \leq n, \\ & \frac{7}{16} [(u_x)_{j+1}^k + (u_x)_{j-1}^k] + (u_x)_j^k - \frac{h}{16} [(u_{xx})_{j+1}^k - (u_{xx})_{j-1}^k] \\ & = \frac{15}{16h} (u_{j+1}^k - u_{j-1}^k) + Q_j^k, 1 \leq j \leq m-1, 1 \leq k \leq n, \\ & \frac{9}{8h} [(u_x)_{j+1}^k - (u_x)_{j-1}^k] - \frac{1}{8} [(u_{xx})_{j+1}^k + (u_{xx})_{j-1}^k] + (u_{xx})_j^k \\ & = \frac{3}{h^2} (u_{j+1}^k - 2u_j^k + u_{j-1}^k) + R_j^k, 1 \leq j \leq m-1, 1 \leq k \leq n, \\ & 14(u_x)_0^k + 16(u_x)_1^k + 2h(u_{xx})_0^k - 4h(u_{xx})_1^k = -\frac{1}{h} (31u_0^k - 32u_1^k + u_2^k) + S_j^k, \\ & 14(u_x)_m^k + 16(u_x)_{m-1}^k - 2h(u_{xx})_m^k + 4h(u_{xx})_{m-1}^k = \frac{1}{h} (31u_m^k - 32u_{m-1}^k + u_{m-2}^k) + T_j^k, \\ & u_j^k = 0, j = 0, m, 1 \leq k \leq n, \\ & u_j^0 = \varphi_j, 0 \leq j \leq m. \end{aligned} \right.$$

Omitting the small term $P_j^k, Q_j^k, R_j^k, S_j^k$ and T_j^k , the ESA-CCD finite difference scheme is obtained as follows:

$$\left\{ \begin{aligned} & {}^{ESA}_0 D_t^{\alpha_k} u_j^k = (u_{xx})_j^k + f_j^k, 0 \leq j \leq m, 1 \leq k \leq n, \\ & \frac{7}{16} [(u_x)_{j+1}^k + (u_x)_{j-1}^k] + (u_x)_j^k - \frac{h}{16} [(u_{xx})_{j+1}^k - (u_{xx})_{j-1}^k] \\ & = \frac{15}{16h} (u_{j+1}^k - u_{j-1}^k), 1 \leq j \leq m-1, 1 \leq k \leq n, \\ & \frac{9}{8h} [(u_x)_{j+1}^k - (u_x)_{j-1}^k] - \frac{1}{8} [(u_{xx})_{j+1}^k + (u_{xx})_{j-1}^k] + (u_{xx})_j^k \\ & = \frac{3}{h^2} (u_{j+1}^k - 2u_j^k + u_{j-1}^k), 1 \leq j \leq m-1, 1 \leq k \leq n, \\ & 14(u_x)_0^k + 16(u_x)_1^k + 2h(u_{xx})_0^k - 4h(u_{xx})_1^k + \frac{1}{h} (31u_0^k - 32u_1^k + u_2^k) = 0, \\ & 14(u_x)_m^k + 16(u_x)_{m-1}^k - 2h(u_{xx})_m^k + 4h(u_{xx})_{m-1}^k - \frac{1}{h} (31u_m^k - 32u_{m-1}^k + u_{m-2}^k) = 0, \\ & u_j^k = 0, j = 0, m, 1 \leq k \leq n, \\ & u_j^0 = \varphi_j, 0 \leq j \leq m. \end{aligned} \right.$$

Next, we will use numerical examples to verify the effectiveness of the proposed method in solving a class of partial differential equations.

4. Numerical Experiments

In this section, we will verify the effectiveness and accuracy of the proposed method. To express the results, we denote the errors in the infinite norm with spatial accuracy and temporal accuracy is defined as

$$Err(h, \tau) = \max_{0 \leq j \leq m} |U_j - u_j|,$$

where U_j and u_j denote the exact and numerical solutions, respectively.

$$Order_t = \log 2 \frac{Err(h, \tau)}{Err\left(\frac{h}{2}, \tau\right)}, Order_x = \log 2 \frac{Err(h, \tau)}{Err\left(h, \frac{\tau}{2}\right)}.$$

Example 1. Let $[x_L, x_R] = [0, 1]$, $T = 1$, we consider the initial-boundary value problems of VO-TFDEs (3.1) - (3.3) with source term, where

$$f(x, t) = 20x^8(1-x) \left(\frac{t^{2-\alpha(t)}}{\Gamma(3-\alpha(t))} + \frac{t^{1-\alpha(t)}}{\Gamma(2-\alpha(t))} \right) - 80x^6(7-9x)(t+1)^2,$$

initial value is $\varphi(x) = 10x^8(1-x)$ and the exact solution is $u = 10x^8(1-x)(t+1)^2$.

Firstly, according to **Figure 1**, it clearly shows that numerical solutions fit well with the exact solutions of Example 1 by the proposed method.

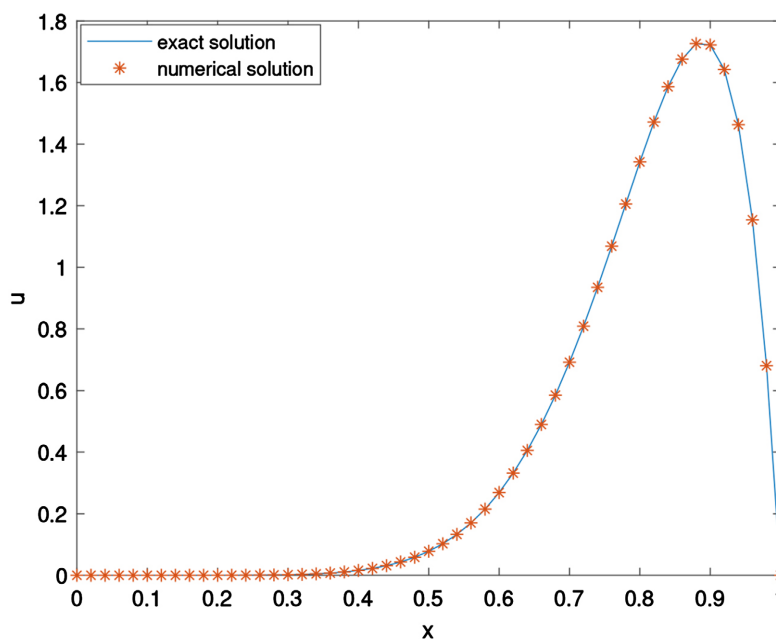


Figure 1. Taking $m = 50$, $n = 100$, $\alpha(t) = \frac{2 + \sin(t)}{4}$, exact and numerical solutions for Example 1.

Secondly, from **Table 1**, the $(2 - \bar{\alpha})$ th-order accuracy can be achieved in time. Similarly, by **Table 2**, the spatial accuracy is verified.

Table 1. The error and temporal accuracy of the proposed method at a fixed $h = 1/5000$, $\alpha(t) = \frac{2 + \sin(t)}{4}$ for Example 1.

$m = 5000$	ESA-CCD		
	n	$Err(h, \tau)$	$Order_t$
$T = 1$	10	4.0568e-04	-
	20	1.6638e-04	1.2859
	40	6.8049e-05	1.2898
	80	2.7792e-05	1.2919
	160	1.1346e-05	1.2926

Table 2. The error and spatial accuracy of the proposed method at a fixed $\tau = \frac{1}{100000}$, $\alpha(t) = \frac{2 + \sin(t)}{4}$ for Example 1.

$n = 100,000$	ESA-CCD		
	m	$Err(h, \tau)$	$Order_x$
$T = 1$	3	1.4153e-00	-
	6	4.9800e-02	4.8289
	12	1.2000e-03	5.4086
	24	2.2186e-05	5.7235
	48	3.8198e-07	5.8600

In summary, numerical experiments verify the effectiveness of the proposed method.

Example 2. Let $[x_L, x_R] = [0, 1]$, $T = 1$, we consider the initial-boundary value problems of VO-TFDEs (3.1) - (3.3) with source term, where

$$f(x, t) = \frac{2t^{(2-\alpha(t))}}{\Gamma(3-\alpha(t))} \sin(2\pi x) + 4\pi^2 t^2 \sin(2\pi x),$$

initial value is $\varphi(x) = 0$ and the exact solution is $u = t^2 \sin(2\pi x)$.

From this example, it is easy to know that the $(2 - \bar{\alpha}) = 1.05$, which is exactly consistent with the numerical results in **Table 3**. From the numerical results in **Table 4**, it can be seen that the proposed method achieves high order accuracy in the spatial direction, verifying the accuracy of the proposed method once again.

Table 3. The error and temporal accuracy of the proposed method at a fixed $h = 1/5000$, $\alpha(t) = 0.8 + 0.15t$ for Example 2.

$m = 5000$	ESA-CCD		
	n	$Err(h, \tau)$	$Order_x$
$T = 1$	10	2.1000e-03	-
	20	1.0000e-03	1.0528
	40	4.8420e-04	1.0532
	80	2.3331e-04	1.0533
	160	1.1242e-04	1.0534

Table 4. The error and spatial accuracy of the proposed method at a fixed $\tau = \frac{1}{100000}$, $\alpha(t) = 0.8 + 0.15t$ for Example 2.

$n = 100,000$	ESA-CCD		
	m	$Err(h, \tau)$	$Order_x$
$T = 1$	3	7.9300e-02	-

Continued

6	2.4000e-03	5.0639
12	2.5571e-05	6.5342
24	2.5749e-07	6.6338

5. Concluding Remarks

In this paper, we developed a novel spatial sixth-order hybrid ESA-CCD method for a class of VO-TFDEs. The proposed method not only achieves high-order convergence rate but also requires information from three points in the computational region only. Finally, numerical examples demonstrate the effectiveness and accuracy of the method. Although the ESA-CCD method for solving the VO-TFDEs is proposed only in this paper, in fact, it is suitable for solving a class of equations containing fractional order operators.

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Data Availability Statement

Data or code will be made available on request.

Conflicts of Interest

The authors declare no competing interests.

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