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Natural Numbers and the Strong Goldbach Conjecture

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Abstract

This study introduces the representation of natural number sets as row vectors and pretends to offer a new perspective on the strong Goldbach conjecture. The natural numbers are restructured and expanded with the inclusion of the zero element as the source of a strong Goldbach conjecture reformulation. A prime Boolean vector is defined, pinpointing the positions of prime numbers within the odd number sequence. The natural unit primality is discussed in this context and transformed into a source of quantum-like indetermination. This approach allows for rephrasing the strong Goldbach conjecture, framed within a Boolean scalar product between the prime Boolean vector and its reverse. Throughout the discussion, other intriguing topics emerge and are thoroughly analyzed. A final description of two empirical algorithms is provided to prove the strong Goldbach conjecture.

Keywords

Natural Numbers, Prime Numbers, Vector Description of Natural Numbers, Prime Boolean Vectors, Primality of the Natural Unit, Strong Goldbach's Conjecture, Vector Reversal, Pairing Conjecture, Natural Matrix Squeezing

1. Introduction

This study focuses on the strong Goldbach conjecture, a prominent unsolved problem in mathematics. Much of the literature on this subject deals with attempts to construct a conjecture's solution; some authors even claim it has been found.

Despite its seemingly simple nature, this problem's enduring form underscores its significance. Owing to their recent publication, several references dealing with the strong Goldbach conjecture can be chosen. References [1]-[15] deal with the main subject of this paper, but readers can also find information about a related item, prime numbers, and their distribution among natural numbers.

In the present study, the strong Goldbach conjecture is defined from a slightly different point of view, considering that any natural number in the set: $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$ can be written as the sum of two prime number generators, which are both elements contained in $\mathbb{N}_0$. That is:

$$\forall n \in \mathbb{N}_0 : \exists \{p, q\} \subset \mathbb{N}_0 \wedge \{P = 2p+1, Q = 2q+1\} \subset \mathbf{P} \Rightarrow n = p + q,$$

where $\mathbf{P}$ is the set of prime numbers.

At the same time, one will describe the set of natural numbers in a new way, developed in this laboratory, associating the odd natural numbers ordered structure with two vertices of a Boolean hypercube [16]-[18] acting as Boolean tags. For example, see references [19]-[21] for more information on Boolean tagged sets.

The present research is based on the author's experience in several aspects of natural number theory, as seen from the various problems studied in references [18] [22]-[37].

The new approach presents odd natural numbers as a Boolean tagged set constructed through prime and composite numbers distribution. This straightforward method allows for a redefinition of strong Goldbach's conjecture from a Boolean perspective (see references [18] [22] [26]), offering a simple path toward proof if it exists. The potential of this approach further reinforces this optimistic outlook.

A significant discovery was made while exploring the rules of the Boolean description of the odd natural number set. An unexpected quantum-mechanical-like superposition situation emerged, providing a unique starting point insight into the relationship between prime and composite elements. The implications of this finding are detailed in the text unfolding, stressing its importance.

One discusses the Goldbach conjecture in light of the Boolean description of the odd-number prime-composite distribution. Vector reversal, a newly described operation over vectors [38], is also used to study this subject further. Thus, based on a Boolean tagged set formalism, the present paper proposes an alternative equivalent to the strong Goldbach conjecture that one might phrase: "Any prime Boolean vector tag cannot be orthogonal to its reverse vector".

To arrive there, one will start a section defining two vectors that permit constructing the natural number set as a vector. The description of Boolean tags attached to natural numbers follows in another section. Having defined these two previous steps, this study will redefine the strong Goldbach conjecture and construct an algorithm to test it. Section 6, the largest of them all, will analyze the product of the prime Boolean vector and its reverse. Section 7 provides two algorithms to empirically test the strong Goldbach conjecture and contains the concluding remarks.

2. Two Vectors Redefine the Natural Number Set

One usually defines the natural number set $\mathbb{N} = \{1, 2, 3, \cdots, N, \cdots\}$ without zero as the first element. However, the present study also employs the zero-extended natural number set $\mathbb{N}_0 = \{0\} \cup \mathbb{N} = \{0,1,2,3,\cdots,N,\cdots\}$. Such extension is needed because natural numbers can be connected with the vertices of arbitrary-dimensional Boolean hypercubes; for example, the reference [24] [29] [30] deals with this construction.

Also, the set $\mathbb{N}_0$ plays a leading role in ordering sequences of natural numbers, and its possible use as a generator of the natural number set has determined its presence and use here.

2.1. Vector Form of the Natural Number Sets

In addition, one can consider the set $\mathbb{N}_0$ as a row[1] vector of arbitrary dimension: $\langle \mathbf{N}_0 |$. That is, one can write:

$$\mathbb{N}_0 \Leftrightarrow \langle \mathbf{N}_0 | = (0,1,2,3,\cdots,I,\cdots). \tag{1}$$

In the same way that one can accept the existence of a zero and unity (or Mersenne) vectors of the same dimension, for instance:

$$\langle \mathbf{0} | = (0,0,0,\cdots,0,\cdots) \tag{2}$$

and

$$\langle \mathbf{1} | = (1,1,1,\cdots,1,\cdots), \tag{3}$$

respectively.

Then, keeping in mind these previous vectors, one can construct the vector of even natural numbers $\langle \boldsymbol{\varepsilon} |$ as a homothecy of the vector $\langle \mathbf{N}_0 |$ when defined as in the Equation (1); that is:

$$\langle \boldsymbol{\varepsilon} | = 2\langle \mathbf{N}_0 | = (0,2,4,\cdots,2I,\cdots). \tag{4}$$

So, one can consider the even natural numbers vector constructed with the origin reference at the zero vector: $\langle \mathbf{0} |$. Moreover, one can construct the odd natural number vector $\langle \boldsymbol{\omega} |$ similarly, but in this case writing:

$$\langle \boldsymbol{\omega} | = 2\langle \mathbf{N}_0 | + \langle \mathbf{1} | = \langle \boldsymbol{\varepsilon} | + \langle \mathbf{1} | = (1,3,5,7,\cdots,2I+1,\cdots), \tag{5}$$

thus, the Equation (5) allows one to consider the vector of odd natural numbers as the even natural number vector but origin-shifted or translated employing the unity vector: $\langle \mathbf{1} |$.

Resuming, the even natural number vector is defined as having zero origin. At the same time, one can suppose the odd natural number vector is constructed as the even vector but origin-shifted by the unity vector.

One can conclude that even and odd natural numbers, constructed as elements of two vectors of the same dimension, are both homothecies and translations of the vector: $\langle \mathbf{N}_0 |$, associated with the zero-augmented natural set $\mathbb{N}_0$.

Such circumstances, involving a vector representation based on the extended natural number set [24] [25], compelled the present author to previously underline its

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[1] Used in this way for simple typographic reasons, because the transpose column representation in dual natural space is equally valid.

importance in the development of the present paper.

Hence, two vectors: $\{\langle\boldsymbol{\varepsilon}|;\langle\boldsymbol{\omega}|\}$ redefine the natural number set within a row vector space framework. Thus, using vector operations within the vector representation of natural number sets permits mathematical developments that could be difficult to describe while keeping the reasoning within the set theory. For instance, the possibility of obtaining the odd vector natural powers:

$$\forall p \in \mathbb{N} : \langle\boldsymbol{\omega}^p| = \left(\omega_1^p, \omega_2^p, \omega_3^p, \cdots, \omega_I^p, \cdots\right), \tag{6}$$

with a similar definition for the even vector natural powers $\langle\boldsymbol{\varepsilon}^p|$. Also, the previous power definition permits to write the unity vector as the zeroth power of any odd vector:

$$\langle\boldsymbol{\omega}^0| = \langle\mathbf{1}| \tag{7}$$

however, to obtain the same on even vectors, one must take care of the zero as the first element according to the Equation (4), and modify the vectors accordingly.

Other manipulations, which are irrelevant to the context of the present paper, can be performed but will be described elsewhere.

As a consequence of these previous vectorial constructs, one must consider the vectors' dimensions: $\{\langle\boldsymbol{\varepsilon}|;\langle\boldsymbol{\omega}|\}$ equal to provide a computationally and geometrically coherent point of view.

However, this dual vector possibility needs also a correspondence with the even and odd natural number sets, then one can write:

$$\langle\boldsymbol{\varepsilon}| \Leftrightarrow \mathbf{E} = \{0, 2, 4, \cdots, 2I, \cdots\} \tag{8}$$

and

$$\langle\boldsymbol{\omega}| \Leftrightarrow \boldsymbol{\Omega} = \{1, 3, 5, \cdots, 2I+1, \cdots\}. \tag{9}$$

2.2. Finite-Dimensional Natural Vectors

One could also consider finite the associated natural number vector dimensions to reflect the possibility of working with natural number ordered sets. In this case, one can indicate the dimension N as a vector subindex, such as: $\{\langle\boldsymbol{\varepsilon}_N|;\langle\boldsymbol{\omega}_N|\}$. Then one can proceed similarly to describe the two associated ordered natural number sets with finite cardinality so that one can write: $\{\mathbf{E}_N;\boldsymbol{\Omega}_N\}$.

Admitting the infinite nature of the set $\mathbb{N}_0$, therefore considering the infinite natural even-odd numbers whole sets, the associated vector dimension, when needed, might be accepted as infinite for all of them. Equations (4), (5) and (6) are valid for any dimension, though.

3. The Boolean Tags Connected to the Odd Natural Number Vector

One can circumscribe the arguments to the odd natural number vector and the connected set when discussing the strong Goldbach conjecture and other natural number properties.

The present study might initially be set up using the following description, linked with constructing the odd natural number set by the union of prime and composite odd numbers sets: $\mathbf{P}$ and $\mathbf{K}$, respectively. In doing this, the even prime number 2 does not appear as such. Here, one considers the prime set $\mathbf{P}$ only to contain the odd primes.

That is, one can write:

$$\mathbf{\Omega} = \mathbf{P} \cup \mathbf{K}, \tag{10}$$

this partition and posterior union are undoubtedly feasible. However, proceeding in this way can cause one to lose the ordering positions of the elements of both sets, as one can suppose that every separate set is reordered on its terms and elements.

Such original ordering loss appears because the distribution of primes within the odd natural numbers is not smooth but presents a chaotic look. However, there are indications that some structure is hidden within the odd natural number set.

Because the primes appear within the odd number set with a characteristic (pseudo-)randomness that one cannot foresee or even describe without some difficult, if not impossible, mathematical setup. A recent study explains [39] such prime number set feature, where the authors describe some algorithms allowing one to know the position of the infinite natural prime elements in sequential finite recursive partitions. But even with these recently described prime location recipes, there does not appear to be a clear, simple algorithm for prime numbers' positions within the odd natural number set.

Yet, one can describe the distribution of primes' positions within the ordering of odd numbers by defining two more natural vectors, respectively orthogonal, in their scalar product sense. They must present the adequate ordering position of elements: one vector holding primes and the other linked similarly to composites. Both vectors bear additional zeroes in the corresponding places associated with the prime-composite nature of each vector element.

But this is a clumsy procedure, which can be substituted by an elegant extended definition of the odd natural number set as a Boolean tagged set. For more information about the mathematical basis of this simple construct, peruse references [19]-[21].

Indeed, one can preserve the odd vector $\langle \boldsymbol{\omega} |$, as the Equation (5) defines. While undisturbing the natural ordering of the whole odd vector elements, one can construct two complementary Boolean vectors of the same dimension, that is:

$$\langle \boldsymbol{\pi} | = \left(\pi_0, \pi_1, \pi_2, \cdots, \pi_I, \cdots \right) \leftarrow \forall I \in \mathbb{N}_0 : \pi_I \in \{0,1\} \tag{11}$$

and

$$\langle \boldsymbol{\kappa} | = \left(\kappa_0, \kappa_1, \kappa_2, \cdots, \kappa_I, \cdots \right) \leftarrow \forall I \in \mathbb{N}_0 : \kappa_I \in \{0,1\}. \tag{12}$$

One can choose both vectors complementary from the Boolean point of view. Therefore, one can write:

$$\langle \mathbf{1}| = \langle \boldsymbol{\pi}| + \langle \boldsymbol{\kappa}|. \tag{13}$$

Using Boolean algebra, one can also compute the scalar product of this vector couple:

$$\langle \boldsymbol{\pi} | \boldsymbol{\kappa} \rangle = \sum_{I \in \mathbb{N}_0} \pi_I \kappa_I = 0. \tag{14}$$

Accordingly, one associates an orthogonal Boolean vector pair: $\{\langle \boldsymbol{\pi}|;\langle \boldsymbol{\kappa}|\}$, bearing the ordering of primes and composites within the odd natural number vector. Such a Boolean vector pair corresponds to two unique vertices of a Boolean hypercube of the appropriate dimension.

Choosing both vector dimensions, finite or infinite, this construct matches the particular distribution of primes and composites within the sets of odd natural numbers with finite or infinite cardinality.

However, one must be aware that the vector pair is unique. It appears in any chosen dimension as a well-defined characteristic of the natural odd-number ordered subsets.

As the involved vectors are Boolean, one can translate them into a unique natural number decimal pair, which one can consider as a couple of constants associated with the chosen working dimension, bearing the information on the prime and composite odd numbers ordering compactly.

3.1. A Finite-Dimensional Example of the Boolean Prime-Composite Vector Tags

An example will illustrate these described Boolean constructs for a dimension $N = 16$, containing the odd numbers vector up to the Mersenne number $M(5) = 2^5 - 1 = 31$:

- Odd Number Vector:

$$\langle \boldsymbol{\omega}_{16}| = (1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,31) \tag{15}$$

- Prime Boolean Tag Vector:

$$\langle \boldsymbol{\pi}_{16}| = (0,1,1,1,0,1,1,0,1,1,0,1,0,0,1,1) \tag{16}$$

- Composite Boolean Tag Vector:

$$\langle \boldsymbol{\kappa}_{16}| = (1,0,0,0,1,0,0,1,0,0,1,0,1,1,0,0). \tag{17}$$

Note that the Boolean vectors shown in the Equations (16) and (17) have been written with the leftmost bit as the less significant one, contrary to the convention in the decimal representation of natural numbers. Such ordering means that the row vectors used here have the leftmost element as the initial one, labeled with a 0 subindex, as one can observe in the Equations (1), (4) and (5).

Equations (16) and (17) have the initial elements written as: $\pi_0 = 0$ and $\kappa_0 = 1$, respectively. That is, accepting the natural unit as *not* prime but composite. Such choices, for the moment, must be considered as not definitive, as other related possibilities are later discussed in the next paragraphs.

3.2. On the Superposition of Two Boolean Descriptions of the Natural Unit 1

Also, in the above and following examples, for any prime Boolean vector of arbitrary dimension, the first odd number, the natural unit 1, can be chosen with a bit $\{0\}$, and also as a bit $\{1\}$ within the zeroth element of the prime Boolean tag vector.

Goldbach seems to have considered the natural unit 1 a prime, while Euler's opinion was the contrary. However, the natural unit is neither prime nor composite in modern acceptance.

This study has chosen the prime and composite vector tags as redundant, complementary, and compliant with the Equation (13). However, such a choice compels us to consider the unit as a composite if one discards its primality, as adopted here in the above example. However, the alternative also appears to be another not-so-weird choice.

One can check that the Boolean vector tags are also orthogonal under Boolean, even within decimal algebra. As the Equation (14) shows, such a property appears when admitting the complementarity relation: $\pi_0 + \kappa_0 = 1$.

In the former particular example in Section 3.1, using the Equations (16) and (17), one can write:

$$\langle \mathbf{1}_{16} | = \langle \boldsymbol{\pi}_{16} | + \langle \boldsymbol{\kappa}_{16} | \wedge \langle \boldsymbol{\pi}_{16} | \boldsymbol{\kappa}_{16} \rangle = 0 . \tag{18}$$

However, the Boolean vector construct could also be solved in an alternative way, considering the unit as a prime, as accepting Goldbach's opinion, that is: $\pi_0 = 1$, and not composite, thus: $\kappa_0 = 0$, to preserve the complementarity of both tag vectors.

Nothing will substantially change if they are chosen in this manner or within the alternative option, more nearly Euler's thought, considering the natural unit not as prime but composite, as adopted here in the Equations (16) and (17).

Yet, in the case of zeroing the zeroth position of both Boolean vectors, that is, admitting: $\pi_0 = \kappa_0 = 0$, and thus adopting the current nature of the natural unit as being neither prime nor composite, then this fact creates the problem of transforming the Boolean prime-composite tag vector pair into a non-complementary, although orthogonal pair.

On the other hand, choosing the natural unit as simultaneously prime and composite: $\pi_0 = \kappa_0 = 1$, will make the Boolean pair complementary but not orthogonal. Still, one could consider the Equation (13) true, but the vector pair will appear correlated, not orthogonal.

Perhaps this condition of the natural unit might indicate that its primality or compositionality depends on the context in which it is discussed. The situation is such that the actual definition of primes and composites, inducing the natural unit to be neither prime nor composite, creates a problem of complementarity or correlation of the tag sets attached to the odd number set.

Consequently, in the present paper, the natural unit will be chosen as a composite in some cases and a prime in others. One has adopted this ambiguous choice because, interestingly enough, perhaps paradoxically, the complementary choice of the natural unit as prime or composite is equally plausible from the Boolean tag description of the present study, as will be discussed below in due time.

One might modify the definition of prime and composite natural numbers to make this Boolean indetermination of the natural unit conceivable. The natural numbers computation framework easily accommodates such ambiguous possibilities.

3.3. Quantum-Mechanical-Like Description of the Natural Unit 1: The Goldbach and Euler States

Perhaps one can use the attached quantum mechanical undefinition concept in this case. Thus, one can assume that the prime and composite nature of the natural unit is unknown until one constructs a Boolean representation tag of the prime-composite odd number sequence or, in quantum mechanical parlance, observes the nature of natural unit 1.

Then, to preserve the complementarity of the vector pair $\left\{\left\langle \boldsymbol{\pi}_N \right|;\left\langle \boldsymbol{\kappa}_N \right|\right\}$, one faces *two* possible obvious choices: either the natural unit is prime and not composite, or else this unit is not prime but composite.

One can explain the meaning of this dual choice as follows. Until one seeks a complementary Boolean description of the primality or the compositality of odd numbers, one can consider the natural unit in a superposition of two Boolean descriptions or states: $\{0,1\} \Leftrightarrow \{1,0\}$.

This dual, quantum mechanically looking property of the natural unit induces the existence of two distinct but equally sound, Boolean complementary descriptions, representations, of the prime-composite odd number sets. One can call the $\{0,1\}$ representation the Euler state, as one has adopted here in the previous example, and a chosen alternative representation $\{1,0\}$ can be named the Goldbach state.

However, such a quantum mechanical-like structure of the simple natural unit, when adopting the natural number set as a Boolean tagged set, might imply the possibility of at least two diverse ways of studying the odd number set properties.

3.4. An Alternative Description

Moreover, even if the Goldbach and Euler states of the Boolean representation of the natural unit are plausible representations, two other possibilities can be similarly defined like in the Euler and Goldbach states, but being their bits now written as: $\{0,0\} \Leftrightarrow \{1,1\}$.

Thus, one can propose two new states as these previously discussed Boolean definitions while considering the description of the natural unit as being associated with a dual formalism.

It is neither prime nor composite as in the $\{0,0\}$ state or prime and composite simultaneously as in the state $\{1,1\}$.

One can distinguish these new states as S_{00} and S_{11}, and while referring to the Goldbach and Euler states as S_{10} and S_{01} respectively.

The state S_{00} cannot generate a prime Boolean vector complementary to the corresponding composite Boolean vector, but both are orthogonal. On the other hand, the state S_{11} corresponds to the two complementary vectors, but they are correlated because, in this case: $\pi_0 = \kappa_0 = 1 \to \pi_0 \kappa_0 = 1$, while: $\forall I > 0 : \pi_I \kappa_I = 0$.

Despite these characteristics, which differ from the states S_{10} and S_{01}, one can computationally adopt both states S_{00} and S_{11}, but considering their particular appearances. In this paper, only the states S_{10} and S_{01} will be taken into account.

4. A Reformulation of the Strong Goldbach Conjecture

Section 2. of this study already discusses several aspects of the natural number set. However, more elements must be presented before rephrasing the strong Goldbach conjecture.

4.1. The Construction of the Natural Number Set and Its Extension

One can describe the natural number set as:

$$\mathbb{N} = \{1, 2, 3, \cdots, I, \cdots\}; \tag{19}$$

so, it might have a vector representation, and similarly to the Equation (1), one can write:

$$\mathbb{N} \Leftrightarrow \langle \mathbf{N} | = (1, 2, 3, \cdots, I, \cdots). \tag{20}$$

Then, one can express the connection of the natural number set $\mathbb{N}$ and the extended with an additional zero natural set $\mathbb{N}_0$ as:

$$\mathbb{N}_0 \Leftrightarrow \langle \mathbf{N}_0 | = \langle \mathbf{N} | - \langle \mathbf{1} | = (0, 1, 2, \cdots, I-1, \cdots), \tag{21}$$

the result of this simple vector operation yields the same vector as the one represented in the Equation (1). Alternatively, one can also write:

$$\mathbb{N} \Leftrightarrow \langle \mathbf{N} | = \langle \mathbf{N}_0 | + \langle \mathbf{1} |. \tag{22}$$

Therefore, the vector representation of both sets is related by an origin shift. Considering this, the Equation (22) appears more adequate to the natural number algebra.

4.2. Vector Reversal Operator

In a previous study, see reference [38], a vector reversal operator was described in depth to give readers a general definition, detailed properties, and extension to matrix algebra applications.

Here, one will describe the necessary elements to follow the understanding of the reversal operation, particularly as applied to discussing the strong Goldbach conjecture.

Given any *N*-dimensional natural vector, like the already defined one:

$$\langle \mathbf{N}_{0,N} | = (0, 1, 2, \cdots, I, \cdots, N-1), \tag{23}$$

then, one outlines, with the help of the supraindex R noting the reversal operator, the reverse vector through:

$$\left\langle \mathbf{N}_{0,N}^{R} \right| = \left\langle \mathbf{N}_{0,N} \right|^{R} = \left(N-1, N-2, \cdots, N-(I+1), \cdots, 2,1,0 \right). \qquad (24)$$

One can consider, in general, that the reversal operator produces an order reversal in the elements of any vector, such that one can write:

$$\left\langle \mathbf{A}_{N} \right| = \left(a_0, a_1, a_2, \cdots, a_I, \cdots, a_{N-1} \right) \to \left\langle \mathbf{A}_{N}^{R} \right| = \left(a_{N-1}, a_{N-2}, \ldots, a_{N-(I+1)}, \cdots, a_1, a_0 \right), \quad (25)$$

and one can be aware that: $\left\langle \mathbf{A}_{N}^{R} \right|^{R} = \left\langle \mathbf{A}_{N} \right|$, thus, one can also write:

$$\forall I = 0, N-1: a_I^R = a_{N-(I+1)}. \qquad (26)$$

4.3. An Equivalent Formulation of Strong Goldbach Conjecture

One can transcribe the strong Goldbach conjecture in the following way:

$$\forall n \in \mathbb{N} \to \exists \{P,Q\} \subset \mathbf{P}: 2n = P+Q. \qquad (27)$$

However, being the pair of natural numbers $\{P,Q\}$ prime, thus odd as adopted here, one can also write them with the aid of two prime generators $\{p,q\}$, like:

$$\exists \{p,q\} \subset \mathbb{N}_0 \Rightarrow P = 2p+1 \wedge Q = 2q+1, \qquad (28)$$

therefore, one can rewrite the strong Goldbach conjecture as:

$$\forall n \in \mathbb{N} \to \exists \{p,q\} \subset \mathbb{N}_0: 2n = 2p+2q+2, \qquad (29)$$

which one can further transform by simplifying the common factor 2, like:

$$\forall n \in \mathbb{N} \to \exists \{p,q\} \subset \mathbb{N}_0: n = p+q+1. \qquad (30)$$

Furthermore, with a simple rearrangement, one can write an equivalent proposition:

$$\forall n \in \mathbb{N} \to \exists \{p,q\} \subset \mathbb{N}_0: n-1 = p+q, \qquad (31)$$

considering: $m = n-1 \in \mathbb{N}_0$, one can finally enunciate the strong Goldbach conjecture as:

$$\forall m \in \mathbb{N}_0 \to \exists \{p,q\} \subset \mathbb{N}_0: m = p+q. \qquad (32)$$

Thus, the strong Goldbach conjecture is equivalent to saying: "One can express any element of the extended natural set $\mathbb{N}_0$ as the sum of two prime number generators, belonging to $\mathbb{N}_0$".

This reformulation of the strong Goldbach conjecture constitutes a simpler definition involving elements of the set $\mathbb{N}_0$ only.

Alternatively, one can say: "All the elements of the set $\mathbb{N}_0$ can be generated as sums of two prime generators found within $\mathbb{N}_0$".

5. An Algorithm to Probe Strong Goldbach Conjecture: Role of the Primality Boolean Vectors

Given the extended natural set $\mathbb{N}_0$, one can use the already described tensor sum of two vectors to observe how the elements of the resultant square matrix can be written as a sequence of sums involving two elements of the same set, as explained in reference [38].

5.1. Vector Reversal and the Strong Goldbach Conjecture

However, one can use the vector reversal operator described earlier in Section 4., and with more detail in reference [38], for the same purpose. This operator lets one know if some of these sums result from a generator pair of prime numbers. The prime Boolean vectors of the adequate dimension provide the possibility of knowing the primality of the summand pair.

Suppose that one wants to generate all the sums of pairs of elements of the set $\mathbb{N}_0$ yielding precisely some number $N-1 \in \mathbb{N}_0$. One can perform such an algorithm taking into account that one can write:

$$\forall I = 0, N-1: N-1 = I + (N-1-I). \tag{33}$$

Then, defining over the set $\mathbb{N}_0$ an *N*-dimensional vector and its reverse, like:

$$\begin{gathered} \langle \mathbf{N}_{0,N} | = (0,1,2,\cdots,I-1,\cdots,N-1) \wedge \\ \langle \mathbf{N}_{0,N}^{R} | = (N-1,N-2,\cdots,N-1-I,\cdots,2,1,0) \end{gathered} \tag{34}$$

one can write the sum of both vectors as:

$$\langle \mathbf{N}_{0,N} | + \langle \mathbf{N}_{0,N}^{R} | = (N-1) \langle \mathbf{1}_N | . \tag{35}$$

Such a result is the same as constructing the anti-diagonal elements in the tensor sum of the two vectors, which is easily defined and was already used in previous research on the strong Goldbach conjecture [22]. The Equation (35) corresponds to the elements equal to $N-1$, belonging to the principal anti-diagonal of an $(N \times N)$-dimensional tensor sum matrix $\mathbf{M}$ of two vectors:

$$\begin{gathered} \mathbf{M} = \langle \mathbf{N}_{0,N} | \oplus \langle \mathbf{N}_{0,N}^{R} | \rightarrow \\ \forall I,J = 0,N-1: M_{IJ} = N_{0,N;I} + N_{0,N;J}^{R} \equiv I + N - (J+1). \end{gathered} \tag{36}$$

The elementary algorithm yielding this result for the number $N-1$ is sketched in the Equation (33) above, as the sum of each vector $\langle \mathbf{N}_{0,N} |$ element, with each element of its reversed vector $\langle \mathbf{N}_{0,N}^{R} |$, is the sum of natural pairs yielding the number $N-1$.

These summations of element pairs correspond to sums of index pairs, each summand being a generator of the sequence of odd numbers via the construction depicted by the Equation (5). One must know that the primality Boolean vector $\langle \boldsymbol{\pi}_N |$ and the vector $\langle \mathbf{N}_{0,N} |$ elements are ordered similarly in a one-to-one correspondence.

This fact means that if the values of some of the Boolean elements: $\pi_I = 1$ and $\pi_I^R = \pi_{N-1-I} = 1$ hold, then the modified strong Goldbach conjecture is true for the number $N-1$, and thus, the original strong Goldbach conjecture is true, too.

So, there is no problem with considering the construction of the principal diagonal of the tensor sum of a vector and its reverse, as shown in the Equation (35).

Still, the presence of Boolean elements equal to the bit $\{1\}$, intervening in the sums and appearing in the same element index order indicates that the natural number $N-1$ complies with the strong Goldbach conjecture.

To determine if the number $N-1$ is strong Goldbach conjecture compliant, one must consider the primality distribution, or location, included within the Boolean vector $\langle \boldsymbol{\pi}_N|$ and its reverse $\langle \boldsymbol{\pi}_N^R|$.

Also, if the Boolean scalar product yields 1, that is:

$$\langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle = \langle \boldsymbol{\pi}_N^R | \boldsymbol{\pi}_N \rangle = 1,$$

the vectors $\{\langle \boldsymbol{\pi}_N|;\langle \boldsymbol{\pi}_N^R|\}$ are correlated, which means that at least a pair of bits: $\{\pi_I;\pi_I^R\}$ possess the values $\{1,1\}$. Such a result will have the same meaning as the strong Goldbach conjecture compliance of the natural number $N-1$.

Proving that the Boolean vector pair $\{\langle \boldsymbol{\pi}_N|;\langle \boldsymbol{\pi}_N^R|\}$ is orthogonal, that is: $\langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle = \langle \boldsymbol{\pi}_N^R | \boldsymbol{\pi}_N \rangle = 0$ will be equivalent, saying that the natural number $N-1$ is *not* compliant with the Goldbach conjecture.

Now, the question is whether any vector pair $\{\langle \boldsymbol{\pi}_N|;\langle \boldsymbol{\pi}_N^R|\}$ is orthogonal or correlated. Remembering the uniqueness of the vector $\langle \boldsymbol{\pi}_N|$, as earlier commented, its reverse is also unique for a given dimension. Moreover, both vectors remain a unique sequence of bits in every dimension as the working dimension grows to infinity. This can be interpreted when the dimension of the odd natural vectors grows that the primality vector takes the form of a *unique* Boolean hypercube vertex of the same dimension. And the same applies to its reverse.

At every number to be described this way, a specific *N*-dimensional Boolean hypercube vertex is taken among the available 2^N ones. There is no arbitrary choice but a compulsive result due to the positions of primes in the odd numbers vector sequence. Such a compulsive structure applies to the reverse vector.

If one can consider both vectors correlated in the Boolean scalar product sense (and in the decimal, too, if necessary), then using the Euler description for any dimension greater than 2, one could write:

$$\forall N > 2 : \langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle = \langle \boldsymbol{\pi}_N^R | \boldsymbol{\pi}_N \rangle = 1, \tag{37}$$

a result meaning that the strong Goldbach conjecture is true. Therefore, the Equation (37) acts as a probe to test if the conjecture is true at any *N*-dimensional level.

Thus, the pair of prime Boolean vectors $\{\langle \boldsymbol{\pi}_N|;\langle \boldsymbol{\pi}_N^R|\}$, belongs to some subset of the Boolean Hypercube vertices, which are related by the reversal operator and correlated.

5.2. The Prime Boolean Vector, the Unity Vector, and the Prime Counting Function

As one has mentioned when describing the Boolean scalar products involving the prime Boolean vector and its reverse: $\{\langle \boldsymbol{\pi}_N|;\langle \boldsymbol{\pi}_N^R|\}$, it is interesting to present some simple arguments for using the Boolean bits as decimal zeros and units. If it is so, one can write the scalar product between the two vectors in question as:

$$\mathrm{D}\left\langle \boldsymbol{\pi}_N \middle| \boldsymbol{\pi}_N^R \right\rangle = M_N , \tag{38}$$

D means the scalar product is performed considering decimal elements in both vectors and M_N denotes the number of coincident 1's on the same positions of both *N*-dimensional vectors. In the example of the Equation (16), the decimal scalar product becomes:

$$\mathrm{D}\left\langle \boldsymbol{\pi}_{16} \middle| \boldsymbol{\pi}_{16}^R \right\rangle = 4_{16} , \tag{39}$$

denoting that the number 31 has 4 times coincident prime generators in its anti-diagonal development. Thus, 31 complies with the strong Goldbach conjecture.

There is an interesting but trivial application of the decimal calculation of the scalar products of the prime Boolean vector. One has previously been developed in this section. Another remarkable point consists of an equivalent formulation of the so-called prime-counting function. One can define the prime-counting or prime-cardinality function as the number of prime numbers included in a sequential interval in the set $\mathbb{N}_0$. Still will be the same to define it within the odd sequential sets $\boldsymbol{\Omega}_N$, indicating the prime cardinality at each element of the set. Adopting such a definition, one can write:

$$\Pi\left(\boldsymbol{\Omega}_N\right) = P_N , \tag{40}$$

where P_N is the number of primes contained in the set $\boldsymbol{\Omega}_N$. One can make equivalent this definition to the decimal scalar products:

$$\Pi\left(\boldsymbol{\Omega}_N\right) = \mathrm{D}\left\langle \boldsymbol{\pi}_N \middle| \boldsymbol{\pi}_N \right\rangle = \mathrm{D}\left\langle \boldsymbol{\pi}_N^R \middle| \boldsymbol{\pi}_N^R \right\rangle = \mathrm{D}\left\langle \boldsymbol{\pi}_N \middle| \mathbf{1} \right\rangle = \mathrm{D}\left\langle \boldsymbol{\pi}_N^R \middle| \mathbf{1} \right\rangle = P_N . \tag{41}$$

In the example of the equation (16), one can write:

$$\Pi\left(\boldsymbol{\Omega}_{16}\right) = \mathrm{D}\left\langle \boldsymbol{\pi}_{16} \middle| \boldsymbol{\pi}_{16} \right\rangle = \mathrm{D}\left\langle \boldsymbol{\pi}_{16}^R \middle| \boldsymbol{\pi}_{16}^R \right\rangle = \mathrm{D}\left\langle \boldsymbol{\pi}_{16} \middle| \mathbf{1} \right\rangle = \mathrm{D}\left\langle \boldsymbol{\pi}_{16}^R \middle| \mathbf{1} \right\rangle = 10_{16} \tag{42}$$

Thus, one counts 10 prime numbers in the odd interval vector of dimension 16. Note that the prime counting here systematically discards the even prime 2.

Also, it is the moment to indicate that one can find a sound relation between the number of primes in sequential intervals involving Mersenne numbers, which can be used to recursively construct the natural number set $\mathbb{N}_0$, see references [26] [29]-[32] for more details. Such a relationship might indicate that the number of primes grows steadily as the Mersenne interval grows. This previous result would indicate that one can expect the number of bits $\{1\}$ to grow steadily when looking at large prime Boolean vector intervals. Moreover, this could indicate that the decimal scalar products of the vector pair $\left\{\left\langle \boldsymbol{\pi}_N \right| ; \left\langle \boldsymbol{\pi}_N^R \right|\right\}$ also augment smoothly when observed between large dimension intervals. The number of primes in a given interval has been a well-studied natural number problem; for example, see references [1] [30] [32].

This increase in the number of bits $\{1\}$ of the prime Boolean vectors indicates that the Boolean scalar products steadily correspond to a correlation between the two vectors, backing up the strong Goldbach conjecture. But, alas! it does not prove it.

5.3. A Simple Example of Probing Strong Goldbach Conjecture Compliance

An example can illustrate that, in practice, one does not need to know the whole prime Boolean vector. Suppose one wants to know if some prime number, $M = 313$[2], say, appears to be strong Goldbach conjecture compliant. This number, among many other two summands, can be expressed as the sum: $134 + 179 = 313$. Both summands generate two prime numbers; as one can write: $2 \times 134 + 1 = 269$ and $2 \times 179 + 1 = 359$, and the pair of elements: $\{269, 359\}$ is a prime subset. Therefore, the number 313 can be decomposed into the sum of two natural numbers $\{134, 179\}$, generating two prime numbers $\{269, 359\}$. This arrangement means that the prime Boolean vector of dimension $N = 314$, has in the positions $\{134, 179\}$ two bits $\{1, 1\}$; indicating that the number 313 will be strong Goldbach conjecture compliant.

Such a procedure can be applied to any natural number to prove its Goldbach conjecture compliance.

6. The Product of the Prime Boolean Vector by Its Reverse

This section will discuss several aspects of the strong Goldbach conjecture problem transformed in the guise of the previous paragraphs.

Possible simplifications and particular aspects of the Boolean mathematical structure developed so far will be presented.

Essentially, one will use the prime Boolean vector as the basic tool to develop theoretical reasoning for the strong Goldbach conjecture.

6.1. Preliminary Considerations

The previous discussion has led one to study the scalar product of the prime Boolean vector pair: $\left\{\left\langle \boldsymbol{\pi}_N \right|; \left\langle \boldsymbol{\pi}_N^R \right|\right\}$. One has concluded that proving in the Boolean sense that these vectors are correlated:

$$\forall N > 2 : \left\langle \boldsymbol{\pi}_N \middle| \boldsymbol{\pi}_N^R \right\rangle = 1, \tag{43}$$

becomes the same as considering the strong Goldbach conjecture true for the number $N - 1$.

This section will study several aspects of the Equation (43) to ensure its validity.

6.2. Using the Euler's State

Then, remembering the possibility of considering two states of the vector pair $\left\{\left\langle \boldsymbol{\pi}_N \right|; \left\langle \boldsymbol{\pi}_N^R \right|\right\}$ and choosing Euler's as the working state, the first element of the vector $\left\langle \boldsymbol{\pi}_N \right|$ and, thus, the last of the vector $\left\langle \boldsymbol{\pi}_N^R \right|$ are composed by the bit $\{0\}$ zero.

Such a situation might be expressed in the following way:

$$\left\langle \boldsymbol{\pi}_N \right| = \left(0; \left\langle \mathbf{p}_{N-1} \right|\right) \wedge \left\langle \boldsymbol{\pi}_N^R \right| = \left(\left\langle \mathbf{p}_{N-1}^R \right|; 0\right) \tag{44}$$

---

[2] Chosen in this example as a prime number, but any number to be tested can be chosen as a composite at will.

with the definitions of the pair of reduced prime Boolean vectors:

$$\langle \mathbf{p}_{N-1}| = (\pi_1, \pi_2, \pi_3, \cdots, \pi_{N-2}, \pi_{N-1}) \wedge \langle \mathbf{p}_{N-1}^R| = (\pi_{N-1}, \pi_{N-2}, \pi_{N-3}, \cdots, \pi_2, \pi_1). \quad (45)$$

So, one can express the scalar product of the vector pair: $\{\langle \boldsymbol{\pi}_N|; \langle \boldsymbol{\pi}_N^R|\}$, reducing it still in one dimension, like:

$$\langle \boldsymbol{\pi}_N| = (0; \langle \mathbf{p}_{N-2}|; \pi_{N-1}) \to \langle \boldsymbol{\pi}_N^R| = (\pi_{N-1}; \langle \mathbf{p}_{N-2}^R|; 0). \quad (46)$$

Consequently, one can write the scalar product in terms of the reduced pair: $\{\langle \mathbf{p}_{N-2}|; \langle \mathbf{p}_{N-2}^R|\}$; like:

$$\langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle = \langle \mathbf{p}_{N-2} | \mathbf{p}_{N-2}^R \rangle = \sum_{I=1}^{N-2} \pi_I \pi_{N-(I+1)} \quad (47)$$

and in the next step of dimension $N+1$, one can write the bit summation with one additional term.

6.3. Using the Goldbach State

However, a different landscape will develop if the Goldbach state is chosen for the vector $\langle \boldsymbol{\pi}_N|$, in this case, the Equation (44) has to be rewritten as:

$$\langle \boldsymbol{\pi}_N| = (1; \langle \mathbf{p}_{N-1}|) \wedge \langle \boldsymbol{\pi}_N^R| = (\langle \mathbf{p}_{N-1}^R|; 1), \quad (48)$$

and the Equation (46) shall be modified accordingly. But now, within the Goldbach state, the Equation (47) might be written with an extra term, owing to the Boolean algebra written as:

$$\langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle = \pi_{N-1} + \langle \mathbf{p}_{N-2} | \mathbf{p}_{N-2}^R \rangle = \pi_{N-1} + \sum_{I=1}^{N-2} \pi_I \pi_{N-(I+1)}. \quad (49)$$

This means that if the last bit of the prime Boolean vector is $\{1\}$, then the product is correlated, and the associated natural number will be strongly Goldbach compliant.

As a consequence of this property of the Goldbach state of the natural unit, one could perhaps be tempted to consider the possibility of admitting the natural number 1 as being a prime. If this is the case, when the dimensions of the considered vectors increase steadily, one can be sure that at a dimension value *N*, the prime Boolean vector element has the value: $\pi_{N-1} = 1$ and thus $N-1$ is a prime generator. Then, necessarily due to that, the first prime Boolean vector element is now written: $\pi_0 = 1$, because of the chosen Goldbach state, one can admit that $N-1$ can be considered strongly Goldbach compliant.

One can finally resume the previous finding: "In the *N*-dimensional cases, where $N-1$ is a prime generator, then $N-1$ is strongly Goldbach compliant within the unit prime assumption of the Goldbach state".

6.4. One Step Ahead in Euler's State of the Natural Unit

Now, one can consider the structure of the next $(N+1)$-dimensional prime Boolean vector in the Euler state, that is:

$$\left\langle \boldsymbol{\pi}_{N+1}\right| = \left(0;\left\langle \mathbf{p}_{N-1}\right|;\pi_N\right) \rightarrow \left\langle \boldsymbol{\pi}_{N+1}^{R}\right| = \left(\pi_N;\left\langle \mathbf{p}_{N-1}^{R}\right|;0\right), \tag{50}$$

in this form, their scalar product can be written in terms of the reduced prime Boolean vectors:

$$\left\langle \boldsymbol{\pi}_{N+1} \middle| \boldsymbol{\pi}_{N+1}^{R}\right\rangle = \left\langle \mathbf{p}_{N-1} \middle| \mathbf{p}_{N-1}^{R}\right\rangle = \sum_{I=1}^{N-1} \pi_I \pi_{N-I}\ . \tag{51}$$

This result means that in the Euler state description of odd natural numbers, knowing the pair: $\left\{\left\langle \boldsymbol{\pi}_N\right|;\left\langle \boldsymbol{\pi}_N^{R}\right|\right\}$ of prime Boolean *N*-dimensional vector elements, one can know the result of the scalar product of the pair $\left\{\left\langle \boldsymbol{\pi}_{N+1}\right|;\left\langle \boldsymbol{\pi}_{N+1}^{R}\right|\right\}$ of $(N+1)$-dimensional vectors, which is independent of the value of the last bit added on the right of the previous vector $\left\langle \boldsymbol{\pi}_N\right|$ to form the vector $\left\langle \boldsymbol{\pi}_{N+1}\right|$.

The problem resides in the Equations (47) or (50) and the Equation (51), which deals with different dimensions now. Therefore, the pairings of possible products of bits $\{1\}$ are not the same because one bit is added on the right, and the possible pairing is shifted by one unit.

6.5. One Step Ahead in the Goldbach State of the Natural Unit

However, in the Goldbach unit state, the bit $\{0\}$ in the Equation (50) becomes a unit bit $\{1\}$, and the following Equation (51) shall be written with an additional term π_N, stating that as in the previous *N*-dimensional case, the number *N* will be strongly Goldbach compliant whenever $\pi_N = 1$ holds.

Moreover, the Goldbach state of the primality of the natural unit makes relevant the last bit of the prime Boolean vector, which, if it is $\{1\}$, then the Boolean scalar product of the vector pair: $\left\{\left\langle \boldsymbol{\pi}_N\right|;\left\langle \boldsymbol{\pi}_N^{R}\right|\right\}$ becomes correlated. The Goldbach state choice, $\pi_0 = 1$, leads to a trivial situation, as in the increasing path of every Boolean vector dimension from $N \rightarrow N+1$, and corresponding to a prime connection then $\pi_N = 1$, therefore, the scalar product yields: $\left\langle \boldsymbol{\pi}_{N+1} \middle| \boldsymbol{\pi}_{N+1}^{R}\right\rangle = 1$, therefore becoming correlated. This will happen for all the prime Boolean vectors with a bit $\{1\}$ at the last working position.

This means that the Boolean scalar product, when the choice is the Goldbach state of the unit, is always correlated in all the dimensions corresponding to generators of prime numbers.

However, one can obtain more information from the prime Boolean vectors than in the previous sections, keeping the Euler representation of the natural unit as not prime, thus making the observation of the strong Goldbach compliance more difficult.

6.6. Odd-Dimensional Case

The vector elements in the pair: $\left\{\left\langle \boldsymbol{\pi}_N\right|;\left\langle \boldsymbol{\pi}_N^{R}\right|\right\}$ can be symmetrically situated, so only one-half of the bit pairing of the elements of both vectors is needed.

One can start studying the odd-dimensional case, where one can suppose that: $N = 2n+1$, then the vectors can be separated in two halves with a common bit: π_n located at the central vector position. Thus, one can write:

$$\langle \boldsymbol{\pi}_{2n+1} | = \left(\langle \boldsymbol{\lambda}_n |; \pi_n; \langle \boldsymbol{\delta}_n | \right), \tag{52}$$

and therefore, following the vector reversal rule, one can also write:

$$\langle \boldsymbol{\pi}^R_{2n+1} | = \left(\langle \boldsymbol{\delta}^R_n |; \pi_n; \langle \boldsymbol{\lambda}^R_n | \right). \tag{53}$$

From the Equations (52) and (53), one sees that the bit π_n located at the center of both vectors is unmoved by the reversal operation. Therefore, using Boolean algebra, the central term corresponding to the scalar product of both vectors can be expressed by: $\pi_n^2 \equiv \pi_n$. That is, by the bit content of π_n itself.

Thus, the central term can only contribute with a bit value $\{0\}$ or $\{1\}$ to the sum of products forming the scalar product. It is important to note that if the contribution is $\{1\}$ it will specify that necessarily: $\pi_n = 1$, holds. This indicates that, besides other possible bit $\{1\}$ coincidences, if this is the case, the Boolean scalar product will be correlated: $\langle \boldsymbol{\pi}_{2n+1} | \boldsymbol{\pi}^R_{2n+1} \rangle = 1$.

A result signifying that the dimension is expressible as the sum of one repeated generator of a possible prime number. Thus, twice the natural value of the dimension will be Goldbach compliant.

Therefore, one can consider all prime numbers multiplied by 2 as Goldbach compliant. Constituting a trivial result:

$$\langle \mathbf{P} | + \langle \mathbf{P} | = 2 \langle \mathbf{P} | = \langle \mathbf{D} | \rightarrow \langle \mathbf{D} | \Leftrightarrow \mathbf{D} \rightarrow \mathbf{D} \subset \mathbb{N}_0 . \tag{54}$$

This obvious result corresponds to the diagonal of the tensor sum of the vector of prime numbers. One can also write:

$$Diag \left| \langle \mathbf{P} | \oplus \langle \mathbf{P} | \right| = 2 Diag |\mathbf{P}| \leftarrow Diag |\mathbf{P}| = Diag \left(p_I | I = 1, 2, 3, \cdots \right). \tag{55}$$

However, if the odd composite dimension is not prime then: $\pi_n = 0$. One can handle this case like in the even dimension, where no unique central bit in the vector pair: $\left\{ \langle \boldsymbol{\pi}_N |; \langle \boldsymbol{\pi}^R_N | \right\}$ exists. The next section will deal with this situation.

6.7. Even-Dimensional Case

The case of even dimensions, that is: $N = 2n$, will use a pattern associated with the two vectors: $\left\{ \langle \boldsymbol{\pi}_N |; \langle \boldsymbol{\pi}^R_N | \right\}$, which can be described as:

$$\langle \boldsymbol{\pi}_{2n} | = \left(\langle \boldsymbol{\lambda}_n |; \langle \boldsymbol{\delta}_n | \right) \wedge \langle \boldsymbol{\pi}^R_{2n} | = \left(\langle \boldsymbol{\delta}^R_n |; \langle \boldsymbol{\lambda}^R_n | \right), \tag{56}$$

where the vector pair: $\left\{ \langle \boldsymbol{\lambda}_n |; \langle \boldsymbol{\delta}_n | \right\}$ has the same meaning as in the previous section, representing the left and right halves of the prime Boolean vector.

The decomposition of the Equation (56) is symmetric, and the Boolean scalar product can be expressed in a simplified form owing to the rules of Boolean algebra:

$$\langle \boldsymbol{\pi}_{2n} | \boldsymbol{\pi}^R_{2n} \rangle = \langle \boldsymbol{\lambda}_n | \boldsymbol{\delta}^R_n \rangle + \langle \boldsymbol{\delta}_n | \boldsymbol{\lambda}^R_n \rangle = \langle \boldsymbol{\lambda}_n | \boldsymbol{\delta}^R_n \rangle + \langle \boldsymbol{\lambda}_n | \boldsymbol{\delta}^R_n \rangle \equiv \langle \boldsymbol{\lambda}_n | \boldsymbol{\delta}^R_n \rangle = \sum_{I=0}^{n-1} \pi_I \pi_{2n-(I+1)} . \tag{57}$$

This algorithm can be considered equivalent to multiplying the first element of the prime Boolean vector with the last one and continuing to sum the products of the elements sequentially upwards and downwards, with indices augmenting and diminishing by one at each step until the upper-lower bound is reached.

6.8. Remark about the Correlated Product and an Example

According to the previous considerations, the product $\langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle$ can be quite simplified, and in fact, it is only necessary to find a product within the scalar product summation terms like:

$$\exists I \in \{0, N-1\} : \pi_I \pi_{N-(I+1)} = 1, \tag{58}$$

somewhere within the elements of the Equation (57), to consider correlated the involved vector pair.

Section 5.3 gives an example to illustrate the possible finding of a pair of bits fulfilling the Equation (58). However, this example could be solved even more simply than previously described.

This is because the chosen dimension: $N = 313$ is prime. Thus, the reduced dimension might be written: $n = 156 \leftarrow 2n + 1 = 313 = N$. Therefore, one can write that: $\pi_{156} = 1$, so the scalar product implying strong Goldbach compliance will certainly be correlated. Of course, using the Goldbach unit state will also automatically fulfill the compliance.

6.9. Orthogonality and Boolean Hypercube Vertices

When trying to find whether the strong Goldbach conjecture regarding the correlated Boolean scalar product: $\langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle$ is untrue, one can suppose that at some dimension step, the prime Boolean vector and its reverse are orthogonal.

However, one can remember that if the Goldbach or Euler states of the prime Boolean vector have been chosen, they are complementary in the Boolean sense to a composite Boolean vector of the same dimension $\langle \boldsymbol{\kappa}_N |$, both being orthogonal, that is: $\langle \boldsymbol{\pi}_N | \boldsymbol{\kappa}_N \rangle = 0$.

Therefore, an orthogonal vector to the prime Boolean vector is always present and known at every dimension step. If the reverse of the prime Boolean vector is orthogonal to the original one, this could mean that, most directly, the complementary Boolean vector and the reversed prime possess the same property; thus, the prime original and its reverse will have no coincidental pair of bits equal to $\{1\}$.

Hence, strong Goldbach's conjecture is equivalent to affirming that: "at any dimension step, the reverse of the prime Boolean vector *cannot* coincide with the complementary composite Boolean vector".

However, apart from containing the prime Boolean vector $\langle \boldsymbol{\pi}_N |$, any *N*-dimensional Boolean hypercube has a Mersenne number of: $2^N - 1$ available vertices, including the composite Boolean vector $\langle \boldsymbol{\kappa}_N |$, complementary to $\langle \boldsymbol{\pi}_N |$. The reverse vector $\langle \boldsymbol{\pi}_N^R |$ corresponds to another Boolean hypercube vertex, which could be studied whether it coincides with $\langle \boldsymbol{\kappa}_N |$ or not. If this is the case, one can write: $\langle \boldsymbol{\pi}_N^R | = \langle \boldsymbol{\kappa}_N |$.

This will also indicate that one can write using the reversal operator: $\langle \boldsymbol{\pi}_N | = \langle \boldsymbol{\kappa}_N^R |$.

Thus, the reverse of the vector $\langle \boldsymbol{\kappa}_N |$ will be coincident with the original prime Boolean vector.

If this is the case, one can use the reverse of the complementarity of $\langle \boldsymbol{\pi}_N |$, which yields:

$$\langle \boldsymbol{\pi}_N^R | + \langle \boldsymbol{\kappa}_N^R | = \langle \mathbf{1} |, \tag{59}$$

if one sums up this last equation with the previously supposed equalities, it will appear that the vectors $\langle \boldsymbol{\pi}_N^R |$ and $\langle \boldsymbol{\pi}_N |$ are complementary, that is:

$$\langle \boldsymbol{\pi}_N^R | + \langle \boldsymbol{\pi}_N | = \langle \mathbf{1} |, \tag{60}$$

therefore, both vectors will be orthogonal, resulting in a false strong Goldberg conjecture.

Such reasoning provides a sound basis for proving the strong Goldberg conjecture. The prime vector pair $\{ \langle \boldsymbol{\pi}_N | ; \langle \boldsymbol{\pi}_N^R | \}$ cannot be complementary in any dimension if the strong Goldberg conjecture has to be true.

Moreover, $\langle \boldsymbol{\pi}_N^R |$ and $\langle \boldsymbol{\pi}_N |$ possess the same number of bits $\{1\}$, $\nu_\pi[1]$ say, and $\{0\}$, $\nu_\pi[0]$, which becomes a constant pair at every dimension. The complementary composite Boolean vector and its reverse have these numbers reversed, that is: $\nu_\kappa[1] = \nu_\pi[0]$ and $\nu_\kappa[0] = \nu_\pi[1]$.

Then, one can be sure that the prime Boolean reverse vector and the composite Boolean vectors are not equal at every dimension step, that is:

$$\forall N : \langle \boldsymbol{\pi}_N^R | \neq \langle \boldsymbol{\kappa}_N |. \tag{61}$$

This indicates that the Boolean scalar product will comply at every dimension: $\forall N > 2 : \langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle = 1$, and consequently, the strong Goldberg conjecture appears true.

Although these characteristics of the involved Boolean vectors seem to indicate the natural number compliance of the strong Goldberg conjecture, it is needed to prove that, apart from the orthogonal complementary vector existence, there cannot exist any Boolean hypercube vertex coincident with the reverse of the prime Boolean vector, which can be orthogonal to the prime Boolean vector. If such a vertex exists, it belongs to the class of vertices with the number of bits: $N = \nu_\pi[1] + \nu_\pi[0]$. The collection $\mathbf{C}_N$ of vertices, possible candidates to be orthogonal to $\langle \boldsymbol{\pi}_N |$, has a cardinality corresponding to the combinations:

$$Card(\mathbf{C}_N) = \binom{N}{\nu_\pi[1]} = \frac{N!}{\nu_\pi[1]!(N - \nu_\pi[1])!}. \tag{62}$$

Concerning the prime Boolean vector $\langle \boldsymbol{\pi}_N |$, one of these possible candidates corresponds to the reverse $\langle \boldsymbol{\pi}_N^R |$.

To ensure that no Boolean hypercube vertex: $\langle \mathbf{h}_{\nu_\pi[1]} |$, lying within the collection related to the prime Boolean vector $\langle \boldsymbol{\pi}_N |$ is correlated, one needs to see that: $N[\langle \mathbf{h}_{\nu_\pi[1]} | : \langle \pi_N |]$, the number of bits $\{1\}$ coincident between $\langle \mathbf{h}_{\nu_\pi[1]} |$ and $\langle \boldsymbol{\pi}_N |$ is positive definite.

However, in case that: $\nu_\pi[1] > \nu_\pi[0]$ holds, one can ensure that no matter the location of bits $\{1\}$ within all the vertices $\langle \mathbf{h}_{\nu_\pi[1]} |$, the scalar products will be correlated, that is: $\langle \mathbf{h}_{\nu_\pi[1]} | \boldsymbol{\pi}_N \rangle = 1$.

Therefore, one can write:

$$\forall \left(N = \nu_\pi[1] + \nu_\pi[0] \wedge \nu_\pi[1] > \nu_\pi[0] \right) : \langle \boldsymbol{\pi}_N | \boldsymbol{\pi}_N^R \rangle = 1, \tag{63}$$

This result corresponds to many situations along the path of prime Boolean vectors with small dimensions. Large dimensions include fewer primes relative to composite cardinality, therefore producing a reversal of the bit numbers: $\nu_\pi[1] < \nu_\pi[0]$, and the correlation cannot be systematically assured.

Looking at the collection $\mathbf{C}_N$, one can easily see that only half of the vertices are relevant because the other half correspond to their reverses. If the dimension is odd, some vertices are invariant upon reversal; the rest have a different reverse companion.

6.10. Prime Numbers Pairing

In several previous papers, one has discussed some aspects of the prime numbers pairing [30]-[32]. Thus, it is interesting from the point of view of the strong Goldbach conjecture to introduce prime pairing to obtain a possible connection between the two mathematical concepts.

If two prime numbers $\{P < Q\} \subset \mathbf{P}$ are *paired*, then one can write the following relation:

$$\exists \nu \in \mathbb{N} \to Q = P + 2^\nu \tag{64}$$

and if this previous equality is true, then using their prime generators, one can also write:

$$\exists \{p, q\} \subset \mathbb{N}_0 : (2q+1) = (2p+1) + 2^\nu, \tag{65}$$

that one can simplify to:

$$\exists \{p < q\} \subset \mathbb{N}_0 : q = p + 2^{\nu-1}. \tag{66}$$

Therefore, this results in a possible property of the elements of the prime Boolean vector $\langle \boldsymbol{\pi}_N |$, which one can write as: $\pi_p = \pi_q = 1 \equiv \pi_p = \pi_{p+2^{\nu-1}} = 1$.

These coincident non-zero elements of the prime Boolean vector permit the assurance that, in some cases, the correlation of the Boolean scalar product is connected with the strong Goldbach conjecture.

Indeed, as the dimension of the prime Boolean vector increases, and remembering the partition into a left and right part as discussed in Sections 6.6 and 6.7, an element π_p located on the left side of the vector can be coincident with an element π_q situated on the right part of the vector when performing the reversal to obtain the Boolean scalar product, then the scalar product will be equal to the unit.

6.10.1. Some Nuances of the Pairing Conjecture

That the whole set of prime numbers is paired can be expressed through the set of all natural powers of 2:

$$\mathbf{2}^{\mathbb{N}} \Leftrightarrow \left\langle \mathbf{2}^{\mathbb{N}} \right| = \left(2, 2^2, 2^3, \cdots, 2^J, \cdots\right) \tag{67}$$

and the set of prime numbers $\mathbf{P}$ when adopting the Goldbach state for the natural unit:

$$\mathbf{P} \Leftrightarrow \left\langle \mathbf{P} \right| = (1, 3, 5, \cdots). \tag{68}$$

Then, constructing the tensor sum of both vectors, expressible as the algorithm:

$$\mathbf{Q} = \left\langle \mathbf{P} \right| \oplus \left\langle \mathbf{2}^{\mathbb{N}} \right| = \left\{ Q_{IJ} = \delta\left[\left(P_I + 2^J\right) \in \mathbf{P}\right]\left(P_I + 2^J\right) \right\}, \tag{69}$$

where the symbol δ (Logical Expression) is a logical Kronecker's delta, equal to 1 if true and 0 if false, see reference [40]. Not all the above tensor sum elements are prime, but some will appear to be paired primes. Thus, one can construct the matrix $\mathbf{Q}$ with the resulting paired primes as non-null elements while the resulting composite elements are zero. In this case, one can suppose:

$$[\![\mathbf{Q}]\!] \subseteq \mathbf{P}, \tag{70}$$

where $[\![\mathbf{Q}]\!]$ means the set of non-zero elements of the matrix $\mathbf{Q}$.

However, if one accepts the conjecture that all the prime numbers are paired, then both sets might coincide, that is: $[\![\mathbf{Q}]\!] = \mathbf{P}$. Therefore, the strong Goldbach conjecture can be considered equivalent to the pairing conjecture of the prime numbers.

There is empirical evidence that pairing can be found in large prime numbers; for instance, one can write:

$$10000019 + 2^{11} = 10002067, \tag{71}$$

which corresponds to the pairing of two large prime numbers[3].

6.10.2. Origin Shift and Prime Numbers Pairing

The Equation (69) can also be written as an origin shift of the prime numbers vector. Indeed, pairing of prime numbers can be written as follows:

$$\forall N \in \mathbb{N}: \left\langle \mathbf{P}_{[N]} \right| = \left\langle \mathbf{P} \right| + 2^N \left\langle \mathbf{1} \right|. \tag{72}$$

The above Equation (72) can be used to write: $\mathbf{P}_{[N]} = \left[\!\!\left[\left\langle \mathbf{P}_{[N]} \right| \right]\!\!\right] \rightarrow \mathbf{P}_{[N]} \subseteq \mathbf{P}$, the symbol $\left[\!\!\left[\left\langle \mathbf{P}_{[N]} \right| \right]\!\!\right]$ describes the prime numbers set contained in the origin-shifted vector (72).

The infinite origin shifts of the prime vector described by the above Equation (72), corresponds to the infinite set of vectors containing the paired primes with the pairing shift 2^N. That is when $N = 1$, the vector $\left\langle \mathbf{P}_{[1]} \right|$ contains some elements, which are twin primes. Larger origin shifts behave in the same manner. Thus, whenever there are infinite prime elements in the vector $\left\langle \mathbf{P}_{[1]} \right|$ nothing in principle opposes the idea that one can consider them the same in any vector $\left\langle \mathbf{P}_{[N]} \right|$.

---

[3] Here, one has used information and computation obtained from the references [1] and [12] respectively.

6.10.3. Pairing of Mersenne Numbers

Another interesting point is related to the pairing of prime Mersenne numbers, which are defined as:

$$M(N)=2^N-1. \tag{73}$$

If the Mersenne primes are paired, the structure of the resultant primes could be written as the symmetric expression:

$$P_P(M(N))=M(N)+2^P=2^N-1+2^P. \tag{74}$$

As an example, one can consider the pairing:

$$P_8(M(13))=2^{13}-1+2^8\rightarrow 8191+256=8447, \tag{75}$$

Between the prime Mersenne number $M(13)$ with the 8th power of 2, yielding a prime number.

7. Final Considerations and Conclusions

Several points should be noted and highlighted after what has been said beforehand.

7.1. Natural Sequences Attached to the Prime and Composite Boolean Vectors

The decimal correspondence of the vectors in Equations (16) and (17), can be easily computed.

The following results are to be considered within the Euler representation of the natural unit.

Taking the rightmost bit as less significant yields the numbers 52078 and 13457, respectively. Those numbers are characteristic of the ordered odd number set associated with the dimension $N=16$. One can assign them: $\{52078;13457\}_{16}$ as a pair of scalar labels to prime and composite number distributions for the corresponding dimension appearing as a subindex. On the contrary, taking the leftmost bit as less significant, one gets the pair: $\{30419,35116\}_{16}$.

One must be aware that these sequences are different when considered within the Goldbach representation of the natural unit, so one could construct two kinds of these, which will also differ whenever one takes the less significant bit as the leftmost or rightmost one.

Moreover, these pairs of numbers can be associated with a natural sequence evolving with the considered dimension.

Therefore, eight natural sequences can be attached to the prime and composite Boolean vector dimension evolution. They correspond to unique natural number sequences associated with the distribution of primes among the odd natural numbers.

7.2. The Sets of Natural Numbers Set and the Tensor Sum of $\langle \mathbf{N}_0 |$

First, one can generate a set of sets with natural numbers as basic ordering indices.

To describe this situation, one must recall the possibility of writing the natural set, with zero added: $\mathbb{N}_0 = \{0,1,2,\cdots,N,\cdots\}$, as a row (or column) vector, that is: $\mathbb{N}_0 \Leftrightarrow \langle \mathbf{N}_0| = (0,1,2,\cdots,N,\cdots)$.

Note that the dimension of the vector, which becomes equivalent to the associated set cardinality, is left indefinite. In this way, the expressions and mathematical elements used from now on are valid for any cardinality and dimension, except if explicitly provided as a subindex.

Then, one can construct an indefinite set of natural sets and vectors, using an origin shift, based on the unity vector: $\langle \mathbf{1}| = (1,1,1,\cdots,1,\cdots)$ homotheties, or:

$$\forall I \in \mathbb{N}_0 : \mathbb{N}_I \Leftrightarrow \langle \mathbf{N}_I| = \langle \mathbf{N}_0| + I\langle \mathbf{1}| = (I, I+1, I+2,\cdots, I+N,\cdots), \tag{76}$$

the vectors defined above, as commented before, are of indefinite dimension, and one can write for each attached shifted set:

$$\forall I \in \mathbb{N}_0 : \mathbb{N}_I = \{I, I+1, I+2,\cdots, I+N,\cdots\}. \tag{77}$$

Therefore, an ordered set of sets associated with a set of vectors is easily defined in this manner. When considering all implied sets, the cardinality of each set is $card(\mathbb{N}_0)$, and the cardinality of the whole natural set of sets or vectors is the same number. In this framework, the total cardinality of the sets of natural numbers is $[card(\mathbb{N}_0)]^2 = card(\mathbb{N}_0)$.

The linearly dependent vector set:

$$\mathrm{V} = \{\langle \mathbf{N}_I| \,|\, \forall I \in \mathbb{N}_0\}, \tag{78}$$

can be alternatively seen as the rows of the tensor sum $\mathbf{S}$:

$$\mathbf{S} = \langle \mathbf{N}_0| \oplus \langle \mathbf{N}_0| = \{S_{IJ} = I + J \,|\, \forall I, J \in \mathbb{N}_0\}. \tag{79}$$

One can easily realize that the algorithm gives the rows of the matrix $\mathbf{S}$:

$$\forall I \in \mathbb{N}_0 : \langle \mathbf{s}_I| = (I, I+1, I+2,\cdots, I+J,\cdots) \equiv \langle \mathbf{N}_I|. \tag{80}$$

Therefore, the tensor sum $\mathbf{S}$ collects all the elements of the vector set V.

7.3. Squeezing a Naturally-Defined Matrix

Let us define the *squeezing* of a matrix as an operation over natural number-defined matrices of arbitrary dimension. Suppose a natural matrix of any sort: $\mathbf{M}$. By squeezing this matrix, one can understand the construction of a row (or column) vector $\langle \mathbf{z}|$ where the dimension will be given by:

$$\dim(\langle \mathbf{z}|) = \max_{\forall(I,J)}(M_{IJ}) = D_{\mathbf{z}}.$$

and the structure of the vector squeezed matrix is obtained with the following algorithm:

$$\langle \mathbf{z}_{D_\mathbf{z}}| = \langle \mathbf{0}|;\ \forall(I,J) : z_{M_{IJ}} = 1.$$

Moreover, any squeezed matrix vector can be written as:

$$\mathbf{M}^Z = \langle \mathbf{z}|.$$

When the squeezed matrix vector yields the unity vector: $\langle \mathbf{z}| = \langle \mathbf{1}|$, then the squeezed matrix might be called *complete*. Any squeezed matrix vector holding zero(s) is produced by an *incomplete* matrix.

For example, the following squeezed matrix:

$$\begin{pmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 2 & 3 & 4\end{pmatrix}^{Z}=\left(1,1,1,1,1\right)$$

is complete, while the squeezed matrix:

$$\begin{pmatrix}0 & 2\\ 2 & 5\\ 4 & 2\end{pmatrix}^{Z}=\left(1,0,1,0,1,1\right)$$

is incomplete.

7.4. Squeezing the Tensor Sum of $\langle \mathbf{N}_0 |$

Using the matrix squeezing definition, the tensor sum of the vector $\langle \mathbf{N}_0 |$, is inwardly multiplied, see reference [16] [17] [21] for more details of the inward product, by the digital pro-prime matrix:

$$\boldsymbol{\Pi}=\left\{\forall I,J\in\mathbb{N}_0:\Pi_{IJ}=\delta\left(\left\{I,J\right\}\subset\mathbf{p}\right)\right\},$$

must be a complete matrix whenever the natural set $\mathbb{N}$ is strongly Goldbach compliant.

The expression of this property can be written as:

$$\mathbf{M}=\left(\langle\mathbf{N}_0|\oplus\langle\mathbf{N}_0|\right)*\boldsymbol{\Pi}\rightarrow\mathbf{M}^{Z}=\langle\mathbf{1}|.$$

The pro-prime matrix $\boldsymbol{\Pi}$, which can also be transformed into a Boolean matrix, might also be constructed by the decimal version of the Boolean pro-prime vector $\langle\boldsymbol{\pi}|$, holding in the $\langle\mathbf{N}_0|$ appropriate position element a 1 if such an element corresponds to a prime generator number. In contrast, the composite natural number positions hold a 0. Then it is trivial to write: $\boldsymbol{\Pi}=|\boldsymbol{\pi}\rangle\langle\boldsymbol{\pi}|$, that is, as the tensor product of the prime Boolean vector $\langle\boldsymbol{\pi}|$.

This is because the symmetric matrix $\mathbf{M}$ contains only nonzero elements sums of the set $\mathbf{p}$ of prime generator numbers. Therefore, the following algorithm, valid for any dimension, applies to the matrix $\mathbf{M}$ building structure:

$$\begin{aligned}&\forall I,J\in\mathbb{N}_0:M_{IJ}=0;\\&\text{iff } I,J\in\mathbf{p}\rightarrow P=2I+1,Q=2J+1:P,Q\in\mathbf{P}\\&\Rightarrow M_{IJ}=I+J\end{aligned}$$

If the squeezed matrix is complete for a given dimension *D*: $\mathbf{M}_D^{Z}=\langle\mathbf{1}|$, then the squeezed matrix at some higher dimension $D+M$ might also be complete, and consequently, the natural number set N will comply with the strong Goldbach conjecture.

When setting the dimensions, one must keep in mind that the numbers $D-1$ and $D+M-1$ both might be prime generator numbers, that is:

$$\{(D-1),(D+M-1)\}\subset\mathbf{p}\rightarrow\{(2D-1),(2D+2M-1)\}\subset\mathbf{P}\ .$$

One can choose the squeezed matrix $\mathbf{M}_D^Z$ complete. Next, one can choose the dimension increment M so that in the next working dimension, when squeezed, the matrix $\mathbf{M}_{(D+M)}^Z$, becomes complete too.

In the Goldbach representation, the sequence of matrices is such that one gets complete matrices up to dimension $D=4$. From here, not until $D=7$, one gets a new complete matrix, even if the dimension $D=6$ produces the last element of the first row or column as a prime generator number. The procedure can be repeated indefinitely to empirically test the Goldbach conjecture.

For this purpose, one can write the following algorithm:

```
Start : $\mathbf{M}_{D_0}^Z = \langle\mathbf{1}|$;
    do: Choose $M$;
        $D = D_0 + M \wedge (2D+1) \in \mathbf{P}$;
        if : $\mathbf{M}_D^Z \neq \langle\mathbf{1}|$ then $D_0 = D$;
```

7.5. Testing the Goldbach Conjecture in the Anti-Diagonals of the Matrix M

As discussed earlier, the already defined matrix $\mathbf{M}$ can be observed from the point of view of its anti-diagonals; see references [18] [22]. The matrix $\mathbf{M}$ possesses a very interesting structure, where the anti-diagonals hold the elements of the set $\mathbb{N}_{0D}$, increasing in dimension as they approach the main anti-diagonal and decreasing after this situation up to the last anti-diagonal.

When considering a dimension *D*, the matrix $\mathbf{M}_D$ possesses a main anti-diagonal dimension *D*, which decreases going to the past and future anti-triangle vertices. In naming the anti-diagonal sets, one can use the *past* anti-triangle holding the whole anti-diagonals from 0 to $D-2$, the *present* anti-diagonal being the main anti-diagonal, and the *future* anti-triangle holding the anti-diagonals from D to $2(D-1)$.

Each anti-diagonal holds the elements of $\mathbb{N}_{0D}$ in sequence, starting in the past and ending in the future. Like for $D=3$:

$$\mathbf{M}_3=\begin{pmatrix}0&1&2\\1&2&3\\2&3&4\end{pmatrix}\Rightarrow$$

$$\begin{array}{lll}\text{Anti-triangles} & & \text{Anti-diagonal}\quad .\\ \text{Past}:\begin{pmatrix}0&1&*\\1&*&*\\*&*&*\end{pmatrix} & \text{Future}:\begin{pmatrix}*&*&*\\*&*&3\\*&3&4\end{pmatrix} & \text{Present}:\begin{pmatrix}*&*&2\\*&2&*\\2&*&*\end{pmatrix}\end{array}$$

To empirically test the strong Goldbach conjecture, one does not need to know the whole matrix $\mathbf{M}_D$. One only needs to define the Boolean pro-prime vector $\langle\boldsymbol{\pi}_D|$ at the considering dimension.

The following algorithm can be constructed to prove the Goldbach conjecture empirically.

```
Start with some dimension $D$ and End with $D_{\max}$
do $G = D, D_{\max}$ :
    Known $\langle \boldsymbol{\pi}_G |$; Gold=.false.
    do $I = 0, G-1$:
        $J = G - (I+1)$
        if $J \ge I$ then :
            if $\pi(I) \wedge \pi(J)$ then : Gold=.true. ; exit
    if .not.Gold then :
        write("Goldbach Failed at Dimension : ")$G$; exit
if Gold : write("Goldbach Holds in the Interval : ")$D, D_{\max}$
```

This algorithm is simpler than the previously proposed one on the squeezed matrix **M** based on testing the present anti-diagonals in a chosen dimension range. To test the strong Goldbach conjecture, it only needs computing the Boolean pro-prime vector $\langle \boldsymbol{\pi}_D |$ in the interval $\{D, D_{\max}\}$ and finding an adequate representation of large natural numbers, as large as one can computationally reach.

The algorithm output is absent unless the currently tested dimension appears to be a natural number not expressible with the sum of two pro-prime numbers. If this happens, a failure warning will be issued, and the dimension will be written. If not, the algorithm steps forward to test the next dimension. If the Goldbach conjecture is true, the program sequence will run without any signal until the last dimension is tested.

Running this algorithm will certainly be a way to empirically test the strong Goldbach conjecture, which is not exempt from problems when large natural numbers are reached.

7.6. Conclusions

This study exhaustively analyzes the strong Goldbach conjecture from a simple point of view. Such a choice proves that it is not immediate to solve the conjecture, except in case one admits an empirical computer proof performed as far as possible in the natural number sequence. Even so, some problems when reaching large natural numbers will persist due to the construction and handling of such numbers and the filling of the Boolean prime vector, as large prime numbers are difficult to assess.

Thus, there is the possibility of empirically testing the strong Goldbach conjecture. However, in doing so, employing an indefinite natural number sequence will cause technical difficulties in constructing large natural numbers and the knowledge of prime generator numbers positions in large sets.

Throughout the present discussion, the attempt to prove the strong Goldbach conjecture using the various tools one has designed here for this purpose has led to a situation where it seems impossible to construct a conjecture proof by inductive reasoning. The culprit of such an impenetrable logical wall seems to be the unpredictable appearance of the prime generators and, thus, of prime numbers in the natural number sequence.

There is no doubt evidence of the difficulties in obtaining a useful past anti-triangle of the natural vector $\langle \mathbf{N}_0 |$ tensor sum to induce the present anti-diagonal elements. Even more difficult seems to be obtaining a rule to define the shape of the future anti-triangle from the past anti-triangle and present anti-diagonal.

Perhaps there are other ways to tackle the problem that the present author has not imagined or known, but still, he hopes this work might be a shy step toward some new insights into it. Meanwhile, the strong Goldbach conjecture will remain as it has been for two centuries, simple and unsolved in a dark corner of our minds, waiting quietly for the light of some new developments.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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