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The Application of Thermomechanical Dynamics (TMD) to Thermoelectric Energy Generation by Employing a Low Temperature Stirling Engine

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Abstract

A thermoelectric generation Stirling engine (TEG-Stirling engine) is discussed by employing a low temperature Stirling engine and the dissipative equation of motion derived from the method of thermomechanical dynamics (TMD). The results and mechanism of axial flux electromagnetic induction (AF-EMI) are applied to a low temperature Stirling engine, resulting in a TEG-Stirling engine. The method of TMD produced thermodynamically consistent and time-dependent physical quantities for the first time, such as internal energy $\mathcal{E}(t)$, thermodynamic work $W_{th}(t)$, the total entropy (heat dissipation) $Q_d(t)$ and measure or temperature of a nonequilibrium state $\tilde{T}(t)$. The TMD analysis produced a lightweight mechanical system of TEG-Stirling engine which derives electric power from waste heat of temperature ($40^{\circ}\mathrm{C} < T < 100^{\circ}\mathrm{C}$) by a thermoelectric conversion method. An optimal low rotational speed about $30 < \theta'(t)/(2\pi) < 60$ (rpm) is found, applicable to devices for sustainable, clean energy technologies. The stability of a thermal state and angular rotations of TEG-Stirling engine are specifically shown by employing properties of nonequilibrium temperature $\tilde{T}(t)$, which is also applied to study optimal fuel-injection and combustion timings of heat engines.

Keywords

Thermoelectric Generation Stirling Engine (TEG-Stirling Engine), Thermomechanical Dynamics (TMD), Time-Dependent Nonequilibrium Temperature, Stability of Heat Engines in a Thermal State, Optimal Fuel-Injection and Combustion Timings

1. A Low Temperature Stirling Engine as a Thermomechanical Motion Converter

The huge power generations and consumptions of human societies and industries in modern world demand energy harvesting technologies with the least pollution to environment [1]. Energy harvesting devices are supposed to extract a very small amount of electric power from clean energy sources in a low temperature, such as the use of vibration-based power generations [2] [3], Seebeck and Peltier effects using crystal and chemical materials [4], semiconductors, thermocouple devices, solar electric power and car waste heat recovery systems [5] [6].

These devices extract the electric energy by mechanoelectric and thermoelectric conversions, using electromagnetic induction and microscopic temperature gradients as driving force. On the contrary, the macroscopic power stations, such as wind [7], geothermal, thermal and nuclear power stations, employ high heat and temperature, high-speed revolution per minute (rpm), heavy-duty turbines which are classified as radial flux generators (RFGs). Therefore, these conventional energy-generation technologies are categorized as the extraction of electric power by using (A) microscopic semiconductor devices fabricated from lightweight, integrated circuits (ICs); (B) the huge macroscopic systems of high-speed and power, heavyweight, thermoelectric RFGs (*i.e.*, steam turbines).

On the other hand, a possible thermoelectric conversion method was proposed by thermomechanical dynamics (TMD) for nonequilibrium irreversible states (NISs). A drinking bird and a low temperature Stirling engine are shown in **Figure 1** and **Figure 2**, as thermomechanical energy-converter devices. They respectively change thermal energy into mechanical motion (swinging-drinking motion and rotations of a flywheel) and extract thermodynamic work in NISs. They are different from mechanisms of semiconductors, Seebeck and Peltier mechanism and moreover, high-pressure steam and a high speed turbine are not required.

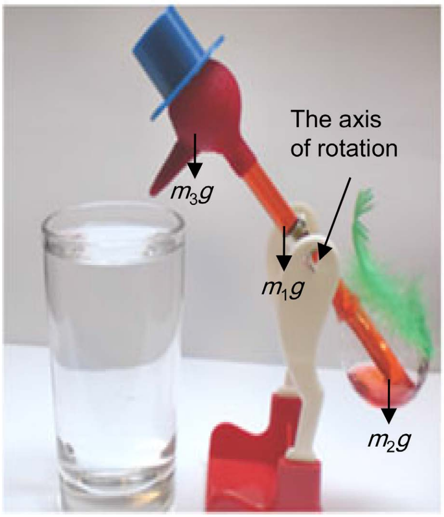

Figure 1. Note that drinking bird's motion is performed in nonequilibrium irreversible states (NISs).

Figure 2. A low temperature Stirling engine (LTSE) works with heat flows (entropy flows) in NISs.

A possible thermoelectric conversion mechanism of a drinking bird and a theoretical low temperature Stirling engine were proposed [8]-[10] and extended to a thermoelectric generator by employing the method of axial flux electromagnetic induction (AF-EMI) [11] [12], denoted as a thermoelectric generation Stirling engine (TEG-Stirling engine). The thermoelectric conversion mechanism of TEG-Stirling engine is fundamentally different from (A) microscopic semiconductors and ICs, and (B) macroscopic RFGs, as explained above. Hence, we emphasize that TEG-Stirling engine is the third kind of thermoelectric generator, different from conventional generators (A) and (B).

In traditional physical analyses, thermomechanical energy conversions of the drinking bird and a low temperature Stiling engine are approximately explained by the equation of state in thermodynamic equilibrium. However, the drinking bird and Stirling engine are heat engines, and they produce work by way of thermal phase transitions from equilibrium to nonequilibrium irreversible states (NISs) and vice versa. Therefore, it is essential to understand that fundamental modifications to Newtonian mechanics and thermodynamics are required in NISs. This is the physical reason why the method of thermomechanical dynamics (TMD) is proposed. The dissipative equation of motion, conservation laws of heat flows, thermodynamic consistency with the measure or temperature in NISs, $\tilde{T}(t) = T_0\tau(t)$, are introduced [10] [12]. The dissipative equation of motion of Stirling engine, conservation laws of heat flows and nonequilibrium temperature are briefly explained in Section 2.

The computer simulations of a disk-magnet electromagnetic induction (DM-EMI) is first performed on the assumption that a stable, optimal flywheel rotational speed exists [9]-[11] in a low temperature Stirling engine. Based on the proof of optimal flywheel rotations by the method of TMD [12], a thermoelectric generation Stirling engine (TEG-Stirling engine) and improvements of electric-power generations are investigated in detail. The TEG-Stirling engine with DM-EMI yields tiny pulse electric currents via a low stable flywheel rotation, resulting in a thermoelectric conversion which depends on physical parameters and materials, such as masses, moment of inertia, magnet and coils, thermal conductivity and so forth. The time-dependent thermodynamic quantities, such as internal energy $\mathcal{E}(t)$, work $W(t)$, the total entropy (heat dissipation) $Q_d(t)$, nonequilibrium temperature $\tilde{T}(t)$ are self-consistently calculated in Section 3.

A drinking bird and a low temperature Stirling engine make use of a low temperature heat flows; so it is reasonable to apply these heat engines to renewable technologies for boiled water and hot springs ($T = 50 \sim 100^{\circ}\text{C}$), wind and solar systems. Hence, TEG-Stirling engine can be well applied to energy harvesting technologies (EHTs). It is important to understand that high temperature vaporized steam, sophisticated materials and semiconductors are not needed in TEG-Stirling engine. The thermal transition of motion between equilibrium and NISs is applied to the concept of stability of thermal states using time-dependent temperature $\tilde{T}(t)$, and optimal fuel-injection and combustion timings of flywheel rotations are studied in Section 4.

The theoretically proposed TEG-Stirling engine is lightweight and requires a low temperature heat-flow, a low rotational optimal-velocity and axial flux electromagnetic induction (AF-EMI). Therefore, the heat-production in coil caused by high rotation velocities of a turbine (Joule heating) could not be a problem because a low rotational speed is efficient for thermoelectric conversions of TEG-Stirling engine. The thermoelectric power generation of TEG-Stirling engine can be improved so that it is applicable to assist the power generation of wind, geothermal and thermal energy plants.

As a brief review of TMD, the dissipative equation of motion, self-consistency of time-dependent thermodynamic quantities and the concept of temperature are reviewed in Section 2. Time-dependent properties of nonequilibrium thermodynamic quantities, internal energy $\mathcal{E}(t)$, work $W(t)$, the total entropy (heat dissipation) $Q_d(t)$, nonequilibrium temperature $T(t)$ and stability of thermal state are explained in Section 3. The stability of thermal work and applications to study optimal fuel-injection and combustion timings are shown in Section 4. Conclusions and perspectives are discussed in Section 5.

2. Thermomechanical Dynamics (TMD) and the Heat-Picture ($Q(t)$-Picture) Analysis

A schematic structure of a low temperature Stirling engine (LTSE) is shown in Figure 3, and LTSE is a thermomechanical rotation-motor, or a rotating motion converter from reciprocating motion of pistons into mechanical work (rotations of a flywheel) using heat flows [13]. The mechanical motion of LTSE is explained by four thermodynamic processes (conjugate variable processes): isothermal expansion, isovolumetric (isochoric) heat-removal, isothermal compression and isovolumetric heat-addition.

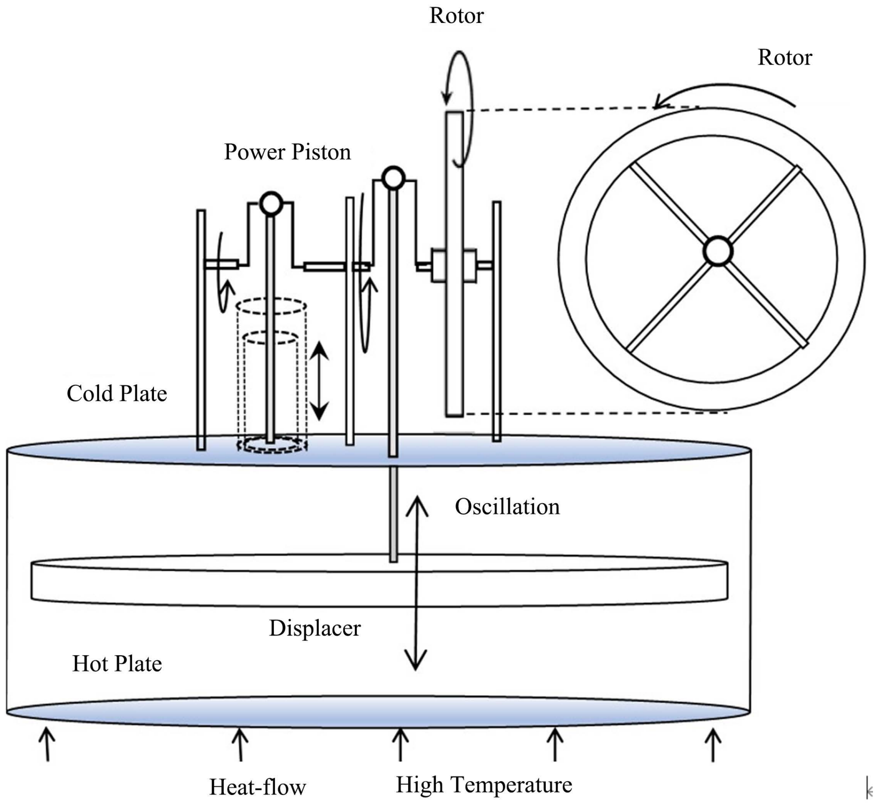

Figure 3. A schematic structure of a low temperature Stirling engine (LTSE) [13].

It is essential to note that the motion of LTSE is not performed in thermodynamic equilibrium. The rotations of the flywheel of LTSE are driven by heat flows coming from the hot plate to the cold plate (see **Figure 3**); in other words, the motion of LTSE is performed by heat flows in nonequilibrium irreversible states (NISs). Therefore, the equation of motion for flywheel rotations driven by the heat-flow in NISs must be constructed, meaning that the time-symmetry of equation of motion is not maintained. The equation of motion for LTSE driven by flywheel rotations is termed as *the dissipative equation of motion*. The dissipative equation of motion is not derivable from Lagrangian or Hamiltonian method, because the equation couples to heat, or dissipative processes.

We proposed a model of thermomechanical dynamics (TMD) to study nonequilibrium irreversible states, and the method of TMD has consistently solved two independent problems: the dissipative equations of motion for a drinking bird and a low temperature Stirling engine. Thermomechanical motion and time-dependent changes of thermodynamic quantities, such as internal energy, thermodynamic work, heat dissipations (entropy flows) and temperature in NISs, are self-consistently obtained and studied as thermodynamic functions in heat picture (Q(t)-picture [10]).

There are three propositions in the TMD method [10]:

1) The dissipative equation of motion for thermodynamic work must be constructed.

The dissipative equation of motion for work must be found by considering thermomechanical motion and phenomenological effects of frictional variations, time-dependent changes of physical quantities, thermal conductivity and efficiency. It would be helpful to make use of Hamiltonian or Lagrangian approach, however, one should note that the time-symmetry is broken in NISs and hence, there is no guaranty that purely mechanical Hamiltonian, Lagrangian and Euler-Lagrange equations can exist.

2) The total heat-energy flow conservation law at time *t*.

The time-dependent progress of thermodynamic work $\mathrm{d}W_{th}(t)$, internal energy $\mathrm{d}\mathcal{E}(t)$ and total entropy $T(t)\mathrm{d}\mathcal{S}(t)$ are assumed by the law of thermodynamics in the following heat-energy flow form as,

$$\mathrm{d}\mathcal{E}(t)/\mathrm{d}t = T(t)\mathrm{d}\mathcal{S}(t)/\mathrm{d}t + \mathrm{d}W_{th}(t)/\mathrm{d}t. \quad (2.1)$$

However, the conventional expression is inconvenient in nonequilibrium thermomechanical processes, because heat dissipations occur in the process of work and internal energy (friction, viscosity among working fluid, wear and warming-up of internal systems, *etc.*). Therefore, it is essential in TMD that the law of thermodynamics should be understood as the differential form of time-dependent total energy-heat flow:

$$\mathrm{d}Q_\varepsilon(t)/\mathrm{d}t = \mathrm{d}Q_d(t)/\mathrm{d}t + \mathrm{d}Q_w(t)/\mathrm{d}t, \quad (2.2)$$

where internal energy $\mathcal{E}(t) \Rightarrow Q_\varepsilon(t)$, entropy flow $T(t)\mathrm{d}\mathcal{S}(t)/\mathrm{d}t \Rightarrow \mathrm{d}Q_d(t)/\mathrm{d}t$ and thermodynamic work $W_{th}(t) \Rightarrow Q_w(t)$ are understood. This is the $Q(t)$-picture to discuss nonequilibrium irreversible states [10]. Thermodynamic equilibrium is defined by $\mathrm{d}Q_w(t)/\mathrm{d}t = 0$: neither thermodynamic work nor heat dissipation into external system exists in thermodynamic equilibrium.

The $Q(t)$-picture is more than a simple variable replacement. For example, the heat flow of time-dependent thermodynamic work, $\mathrm{d}Q_w(t)/\mathrm{d}t$, is essentially different from mechanical work in thermodynamic equilibrium, because $\mathrm{d}Q_w(t)/\mathrm{d}t$ should be regarded as thermal work composed of both *thermally conserved energy* and *thermally dissipating energy*. The concept is fundamentally important to realize thermomechanical, nonequilibrium processes. Therefore, in $Q(t)$-picture, we assume that thermal energies are decomposed into two parts as:

$$Q(t) = \text{thermally conserved energy} + \text{thermally dissipating energy}. \quad (2.3)$$

The time-dependent internal energy, $Q_\varepsilon(t)$ is written as:

$$Q_\varepsilon(t) = Q_{\varepsilon i}(t) + Q_{\varepsilon d}(t). \quad (2.4)$$

The heat $Q_{\varepsilon i}(t)$ is thermal energy used to change internal energy (*thermal internal energy*), and $Q_{\varepsilon d}(t)$ is the associating heat dissipation. Similarly, thermodynamic work $Q_w(t)$ is written as:

$$Q_w(t) = Q_{wk}(t) + Q_{wd}(t). \quad (2.5)$$

The heat $Q_{wk}(t)$ is thermal energy used for the kinetic energy of the flywheel rotations, the displacer and power piston oscillations, and $Q_{wd}(t)$ is the associating heat dissipation. The total heat into the system of heat engine is denoted by $Q_{in}(t)$, and the heat-energy conservation law is now rewritten as:

$$\begin{aligned} Q_{in}(t) &= Q_{\varepsilon}(t) + Q_{w}(t) \\ &= Q_{\varepsilon i}(t) + Q_{\varepsilon d}(t) + Q_{wk}(t) + Q_{wd}(t) \\ &= Q_{\varepsilon i}(t) + Q_{wk}(t) + Q_{d}(t). \end{aligned} \tag{2.6}$$

The total heat dissipation is defined by

$$Q_d(t) = Q_{\varepsilon d}(t) + Q_{wd}(t), \tag{2.7}$$

with $Q_d(0) = 0$. In $Q(t)$-picture, the direct use of non-equilibrium thermodynamic variables, such as pressure, volume, friction, stress, chemical potentials are suppressed, but those variables can be determined when each corresponding $Q(t)$ is obtained.

3) The temperature in nonequilibrium irreversible states, $\tilde{T}(t)$.

The measure of a nonequilibrium irreversible state is defined by the ratio of entropy-flow against internal energy-flow:

$$\tau(t) = \frac{\mathrm{d}Q_d(t)/\mathrm{d}t}{\mathrm{d}Q_{\varepsilon}(t)/\mathrm{d}t}. \tag{2.8}$$

The value of $\tau(t)$ is a dimensionless, positive-definite function, $\tau(t) > 0$, and the *temperature* in NISs is defined by,

$$\tilde{T}(t) \equiv T_0 \tau(t), \tag{2.9}$$

where $\tilde{T}(0) = T_0$ and $\tau(0) = 1$ are assumed at $t = 0$. The true thermodynamic equilibrium is defined when the measure $\tau(t) = 1$ holds identically for all $t \geq 0$, and it leads to the conclusion that no work exists, $\mathrm{d}W_{th}(t)/\mathrm{d}t = 0$ for $t \geq 0$ from the Equation (2.2) in the proposition (2). The measure (2.8) suggests that the very minute heat-energy fluctuation relation could exist only in the true thermodynamic equilibrium as $\mathrm{d}Q_d(t)/\mathrm{d}t = \mathrm{d}Q_{\varepsilon}(t)/\mathrm{d}t$. The other stable thermal equilibriums are all understood as metastable states: $\tau(t) \sim 1$.

The conditions of near equilibrium, local equilibrium defined by linearity of fluxes [14]-[16] and time-dependent physical quantities of NISs are studied by the metastable condition,

$$\tau(t) = \left(\mathrm{d}Q_d(t)/\mathrm{d}t\right)/\left(\mathrm{d}Q_{\varepsilon}(t)/\mathrm{d}t\right) \sim 1, \tag{2.10}$$

in the TMD model. The values of $\tau(t) > 1$ and $1 > \tau(t) > 0$ progress respectively to equilibrium temperature, $\tilde{T}(t) \searrow T_0$ (from above) and $\tilde{T}(t) \nearrow T_0$ (from below) [10]. The measure, $\tau(t)$, or nonequilibrium temperature, $\tilde{T}(t)$, demonstrates empirically well-known results, such as warming-up of heat engines, optimal fuel-injection timings and instability of thermal states; they are exhibited in Sections 3 and 4.

2.1. The Dissipative Equation of Motion for TEG-Stirling Engine

An important result of TEG-Stirling engine is that a low optimal velocity of rotation can be employed for sensitive thermoelectric conversions different from those of high velocity turbines; the optimal angular velocities can be chosen about $30 < \theta'(t)/(2\pi) < 60$ (rpm) in the numerical simulations. This is important for technical applications of thermoelectric conversions [9] [11]. The equation of motion and heat dissipation for thermomechanical conversion are briefly explained.

The appropriate *dissipative equation of motion* for TEG-Stirling engine with thermomechanical driving force is discussed in detail [12], which is expressed as:

$$I_0\theta''(t) + c\theta'(t) - \lambda_w Q_w(t)\left|\sin\theta(t)\right| = 0, \tag{2.11}$$

where the rotation angle $\theta(t)$ is shown in **Figure 4** and $c\theta'(t)$ is an overall friction term; λ_w is a dimensionless coupling constant for heat and mechanical work. The piecewise continuous function $\left|\sin\theta(t)\right|$ is assumed to originate from viscous pumping and friction of gas inside the displacer room, which couples to heat of work $Q_w(t)$. The initial conditions of (2.11) is tentatively chosen as $\theta(0) = 0$ and $\theta'(0) = 0.1\pi$ (rad/s).

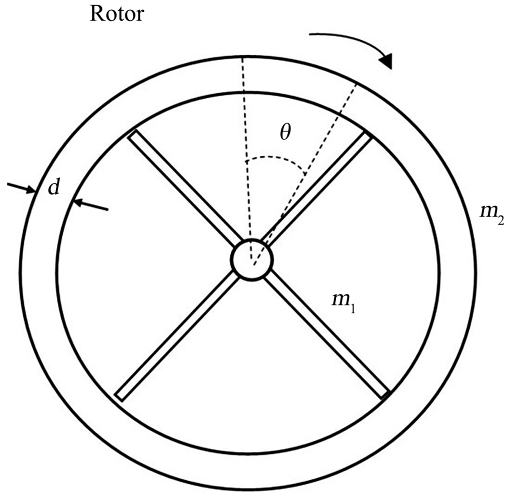

Figure 4. The rotational angle, *θ*, of the rotor in **Figure 3**, starting from the vertical axis.

The coupling constant, λ_w, and work $Q_w(t)$ are important to determine $\theta(t)$ and $\theta'(t)$, and so, the initial starting values of λ_w and $Q_w(t)$ must be chosen carefully for self-consistent numerical calculations. The heat coming-in from a hot plate, $Q_{in}(t)$, is employed as:

$$Q_{in}(t) = Q_{1H}\left(1.0 - e^{-\xi_1 t}\right), \tag{2.12}$$

and $Q_{1H} = 100.0$ (cal). The coupling heat is arbitrarily supplied initially as $Q_w(t) = \eta_0 Q_{in}(t)$. Starting from arbitrary parameters, η_0, ξ_1 and λ_w, the stable maximum velocities, $\theta'(t)/(2\pi) = 30 \sim 60$ (rpm), should be searched by maintaining the full self-consistent calculations in the TMD requirements (2) and (3). Since thermodynamic heat $Q_w(t)$ couples to the dissipative equation of motion with $Q_w(t) = Q_{wk}(t) + Q_{wd}(t)$ and $Q_w(t) = \eta_0 Q_{in}(t)$, appropriate values of parameters have to be adjusted by repeating self-consistent numerical calculations; for example, $\eta_0 = 5.15 \times 10^{-4}$, $\xi_1 = 0.285$ (1/s) and $\lambda_w = 3.20 \times 10^{-3}$, in order to find an optimal angular velocities around $30 < \theta'(t)/(2\pi) < 60$ (rpm) in the current calculations.

An example of heat coming-in, $Q_{in}(t)$, and a uniform heat flow $\mathrm{d}Q_{in}(t)/\mathrm{d}t$ are respectively shown in **Figure 5** and **Figure 6**. Thermodynamic responses, such as thermal changes of internal energy, entropy flow, thermodynamic work are calculated in $Q(t)$-picture in the unit of calorie. The *dissipative* equation of motion is fundamental though it looks simple. However, one should realize that the concept of force and acceleration are fundamentally modified by accepting the coupling of heat and friction as shown in (2.11). The modification to Newtonian mechanics and thermodynamics in equilibrium results in a fundamental change in understanding physical phenomena [10], which directed us to propose the theory of thermomechanical dynamics (TMD) for NISs.

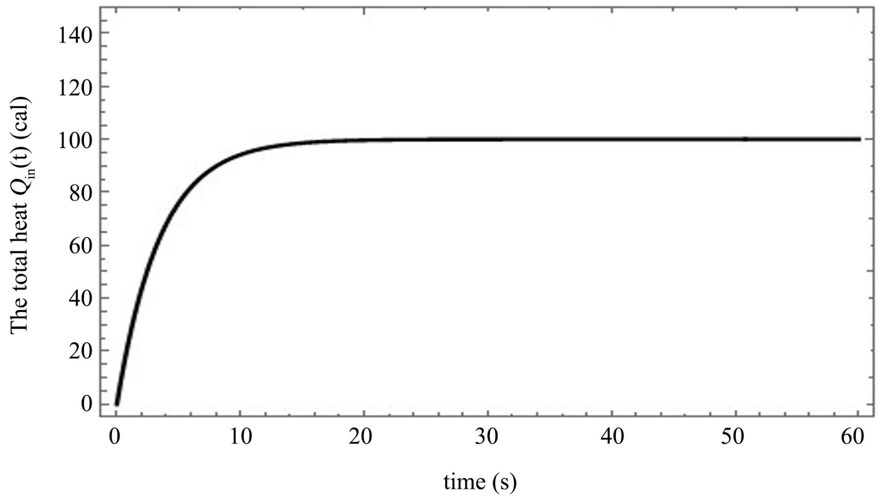

Figure 5. Thermal heat generated at the side of hot plate, $Q_{in}(t)$, is used for computer simulations.

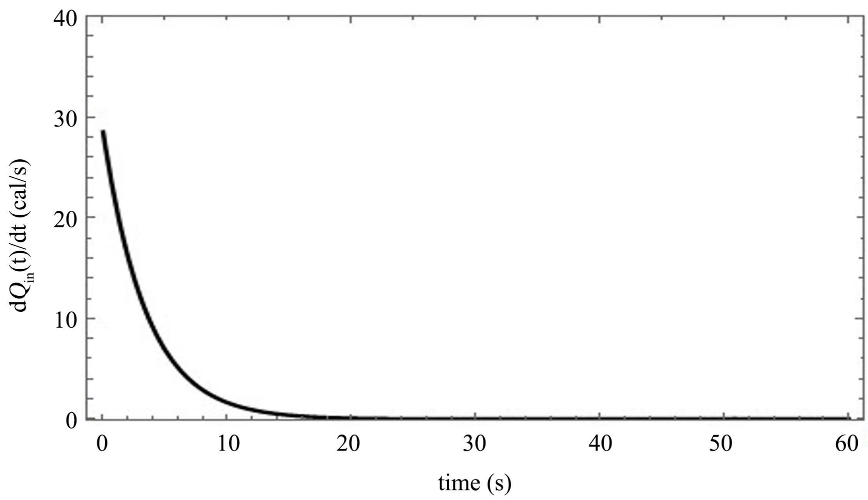

Figure 6. Thermal heat flow, $\mathrm{d}Q_{in}(t)/\mathrm{d}t$. Note that $\mathrm{d}Q_{in}(t)/\mathrm{d}t \sim 0$ for $t > 20$ (s), corresponding to a stable plateau line in **Figure 5**.

2.2. Large-Scale and Small-Scale Views of the Flywheel Rotations

The motion of heat engines is performed in a thermomechanical state with thermal and frictional fluctuations. Hence, the idea of velocity and acceleration in classical mechanics based on the class of differentiable function C^2 and thermodynamic potential functions at equilibrium should be modified, and in addition, the motion is not statistical nor probabilistic. The flywheel motion seems deterministic in a macroscopic scale (**Figure 7**) as Newtonian mechanics requires, but the motion seems to be statistically determined in the microscopic scale (**Figure 8**) because of viscosity, frictional and thermal fluctuations.

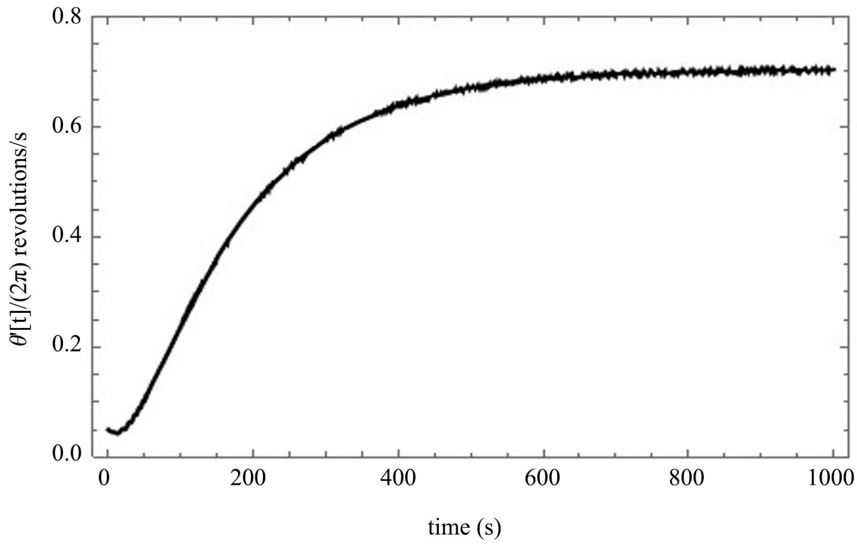

Figure 7. The large-scale view of an angular velocity, $\theta'(t)/2\pi$ (revolutions/s), in the time range $0 < t < 1000$ (s). Note that the tiny fluctuations appear to be smooth [12].

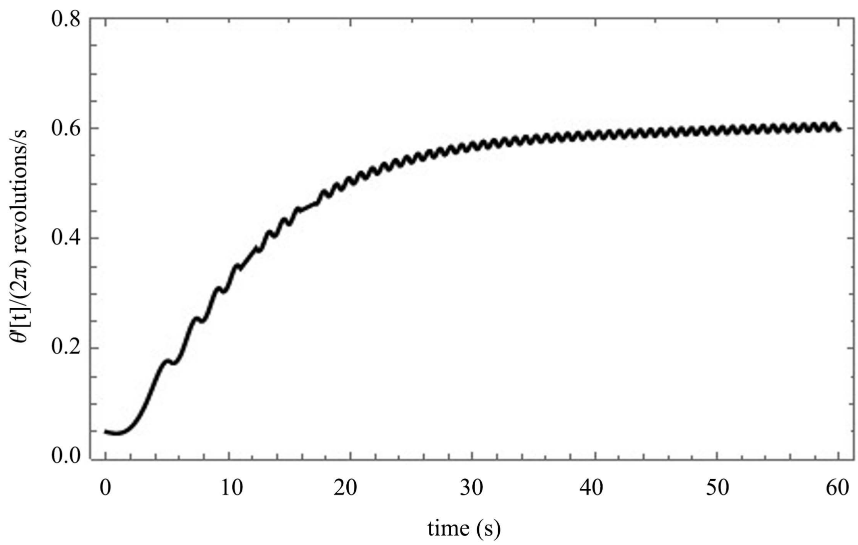

Figure 8. The angular velocity, $\theta'(t)/2\pi$ (revolutions/s), in the time range $0 < t < 60$ (s). Note that tiny frictional fluctuations are observed in a magnified view.

The piecewise continuous driving force in (2.11) immediately indicates that the acceleration is not defined as differentiable and continuous quantity (the class C^2) as supposed in Newtonian mechanics. The acceleration cannot be determined as the second-order derivative derived from the trajectory of motion, and the jump discontinuities of accelerations are not avoidable, which emerges from friction and viscosity of working fluid, sheer stress and machine structure, temperature and thermal fluctuations. Although force changes directions and velocities of motion, accelerations are not associated with $mass \times acceleration$ [12].

An angular velocity of the flywheel rotation, $\theta'(t)$, shown in **Figure 7**, reaches a maximum stable velocity and persists a long time for a cup of hot water (50˚C - 100˚C). The velocity $\theta'(t)$ seems to be a smooth curve, but it has tiny, spiny changes along the line, which is the new distinctive result derived from the dissipative equation of motion (2.11). The spiny macroscopic changes should be considered caused by frictional and thermal fluctuations among working fluid, a displacer, mechanical components of the heat engine. The phenomena of **Figure 7** and **Figure 8** remind us of Brownian motion, which is the random motion of tiny particles and pollens, recognized as random motions with magnified views under a microscope, which would be considered smooth, continuous motion without a microscope.

The dissipative equation of motion would reflect a realistic flywheel motion, though one usually assumes the flywheel rotations smooth and continuous. The method of TMD manifests physical and empirical motion of thermomechanical phenomena and generates a fundamental revision to the method of Newtonian mechanics and thermodynamics. In the TMD model, the concept of force is physical, and force only changes directions of motion or velocities of particles. The calculations result in the assembly of hedgehog-like spiny lines if one tries to calculate accelerations numerically, as shown in **Figure 9**, and this is first discussed in detail in the paper [12].

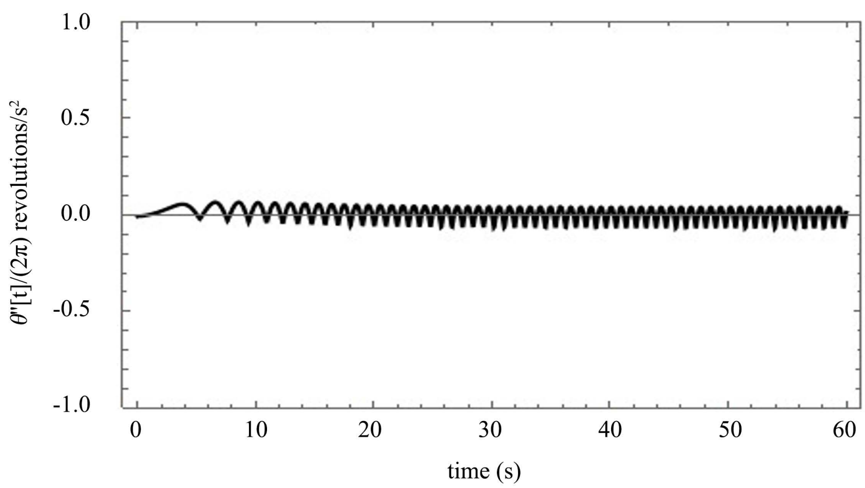

Figure 9. The acceleration of the flywheel revolution $\theta''(t)$ results in a piecewise and continuous, hedgehog-like line [12].

3. The Time-Dependent Thermodynamic Quantities, $\mathcal{E}(t)$, $Q_d(t)$, $W_{th}(t)$, $\tilde{T}(t)$ and Stability of Work

The time-dependent thermodynamic quantities of TEG-Stirling engine, $\mathcal{E}(t)$, $\mathcal{S}(t)$, $W_{th}(t)$, and $\tilde{T}(t)$ or $\tau(t)$, are obtained as explained in Section 2, and the duration of time t is controlled by changing parameters and coupling constants, η_0, ξ_1 and λ_w, which is important to study a long-time and a short-time properties of heat engines [12]. Especially, the properties of $W_{th}(t)$ and $\tilde{T}(t)$ are important to understand stability of heat engines. The upper gas and the lower gas of the displacer in the cylinder room are supposed to be mixed uniformly and simultaneously during the cycle of oscillations. In other words, time-retardation effect during heat transfer is not considered in the theoretical calculations. The time-retardation effects and spatial heat flows and distributions can be incorporated with Fourier's heat conduction method, but empirical results of heat engines are clearly reproduced and confirmed in the numerical analyses.

The time-dependent nonequilibrium temperature $\tilde{T}(t) = T_0\tau(t)$ is defined by the condition (3) in the TMD method so that the whole calculations maintain conservation laws of thermodynamics, and $\tilde{T}(t)$ has characteristic properties to check stability or instability of a thermal state of heat engines. The flywheel rotation is stable and persists for a long time as shown in **Figure 10**, and the corresponding dissipation of heat is shown in **Figure 11**. The dissipation of heat increases for the first period of time; this is interpreted as warming-up of the system of a heat engine, but the dissipation of heat becomes a minimum value when the flywheel rotation reaches a stable maximum rotation. The result is fundamental for energy *efficiency* and *stability* of heat engines, which is specifically applied to the analysis of optimal fuel-injection and combustion timings in Section 4.

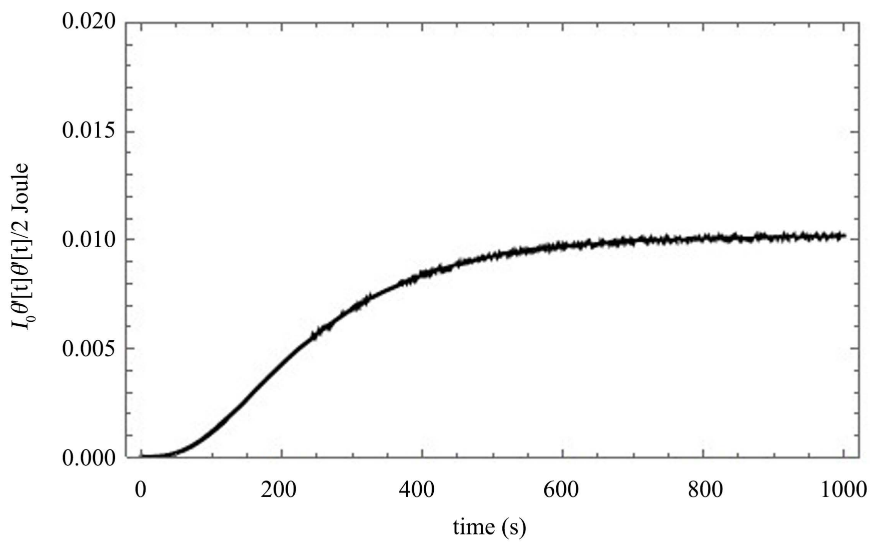

Figure 10. Thermomechanical work, $I_0'\theta'(t)^2/2$ (joule) reaches a stable maximum value ($0 < t < 1000$ (s)) [12].

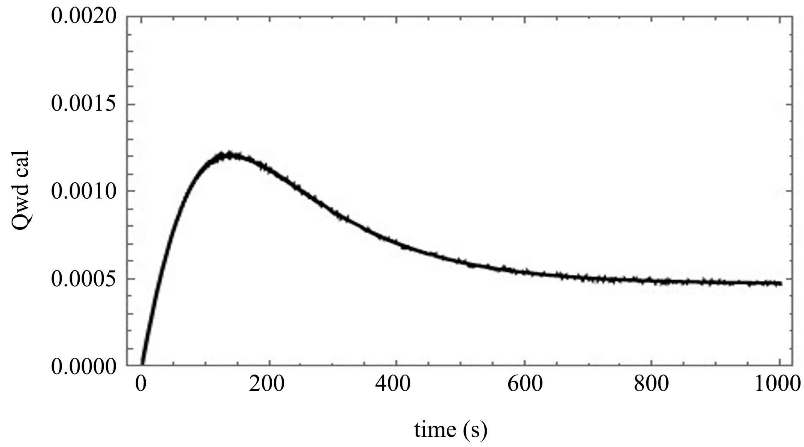

Figure 11. The dissipation of heat, $Q_{wd}(t)$, from thermodynamic work reaches a minimum value [12].

The kinetic energies of displacer and piston oscillations are effectively included in the flywheel kinetic energy, and the thermal kinetic work is expressed as $Q_{wk}(t)$ and the associating heat dissipation as $Q_{wd}(t)$. The total heat-energy of thermal work is given by $Q_w(t) = Q_{wk}(t) + Q_{wd}(t)$ as defined in (2.5). The thermal kinetic work $Q_{wk}(t)$ and associating heat dissipation $Q_{wd}(t)$ are shown in **Figure 12** and **Figure 13**, respectively. The thermal work ratio is $Q_{wk}(t)/(Q_{wk}(t)+Q_{wd}(t)) \sim 0.123$, while the ratio to the total heat dissipation is $Q_{wk}(t)/Q_d(t) \sim 0.875 \times 10^{-3}$ for a stable maximum region. An experimental device of low-temperature Stirling engine continues some back-and-force small swingings before it stops completely, but the last small flywheel swingings are completely ignored in computer simulations, because they do not make a rotation. This is the reason why the rotation ends suddenly at time $t \sim 60$ (s). The time *t* can be adjusted by changing external parameters, such as incoming heat, coupling constants, masses and moment of inertia.

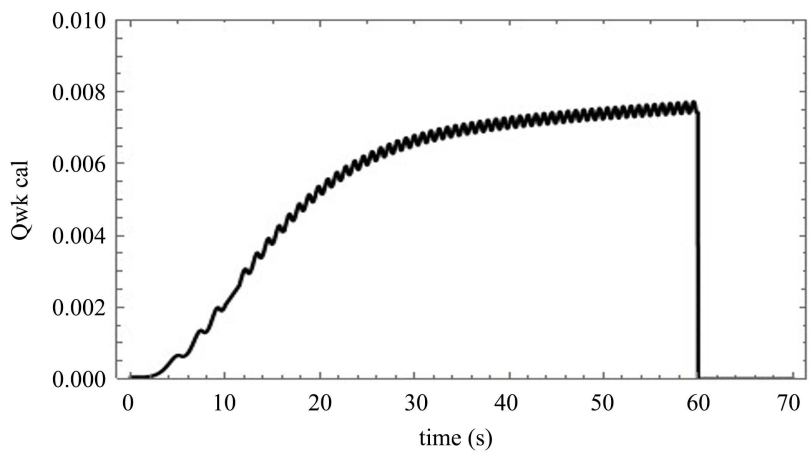

Figure 12. Thermomechanical work of the flywheel, $Q_{wk}(t) = I_0'\theta'(t)^2/2$ (joule). The flywheel rotation ends about $t = 60$ (s).

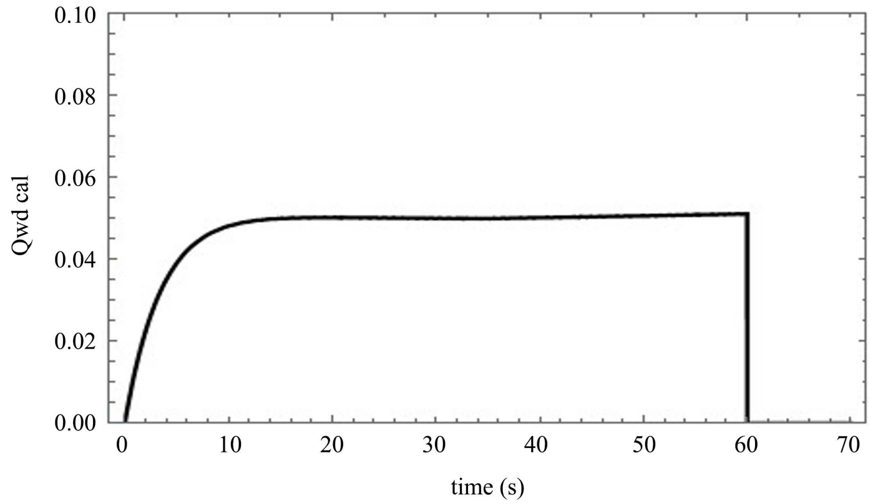

Figure 13. The heat dissipation from flywheel motion, $Q_{wd}(t)$. Thermomechanical work is $Q_w(t) = Q_{wk}(t) + Q_{wd}(t)$.

The internal energy $Q_\varepsilon(t)$ and the total heat dissipation $Q_d(t)$ are shown in **Figure 14** and **Figure 15**, and they both come to a stable maximum as the flywheel rotation reaches a stable maximum speed shown in **Figure 10**. It indicates that thermodynamic quantities, $Q_\varepsilon(t)$, $Q_d(t)$ and $Q_w(t)$ are in a stable thermal state, and the mechanical rotations seem to progress until the end $t \sim 60$. However, the nonequilibrium temperature, $\tilde{T}(t) = T_0\tau(t)$, is very sensitive to changes of thermal states, and unstable thermal states can be observed directly as shown in **Figure 16**. The figure shows that the thermal state slowly becomes unstable from $t \sim 30$ and divergent and meaningless above $t > 40$, because $\tau(t)$ must be positive definite [10].

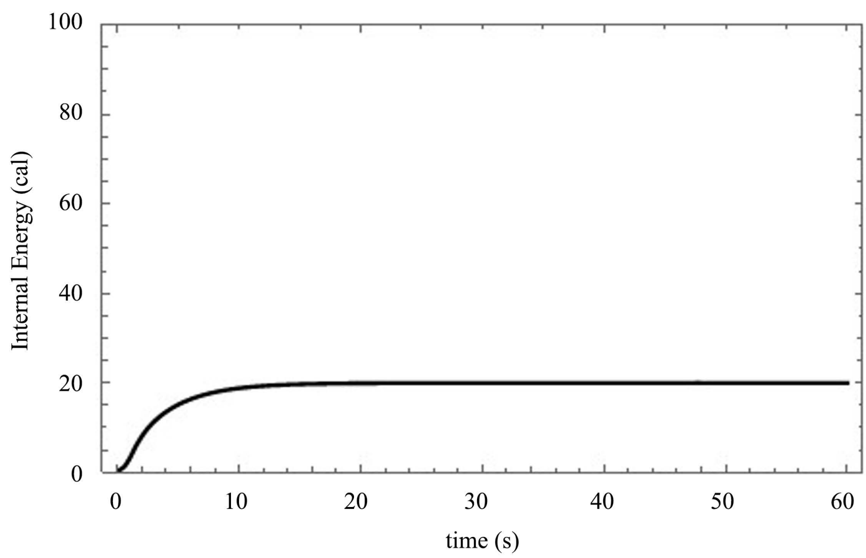

Figure 14. The internal energy, $Q_\varepsilon(t) = Q_{\varepsilon i}(t) + Q_{\varepsilon d}(t)$, for a rapidly-decreasing, short-time uniform heat flow.

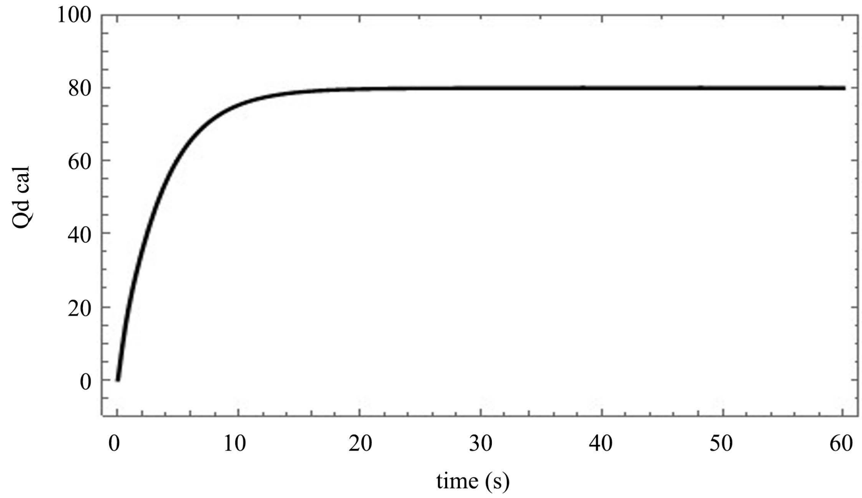

Figure 15. The total heat dissipation, $Q_d(t)$, for a rapidly-decreasing, short-time uniform heat flow.

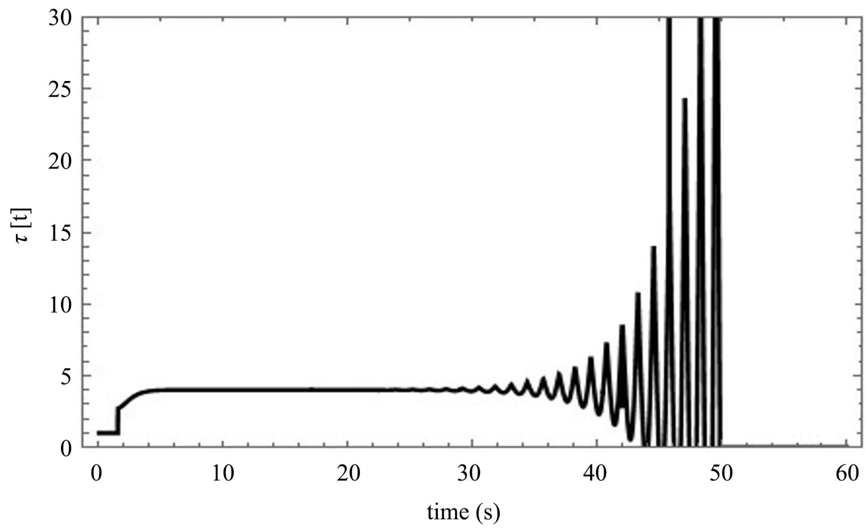

Figure 16. The emergence of instability before a halt of rotations, shown by $\tau(t)$. Note that the measure $\tau(t)$ is dimensionless quantity and must be positive definite: $\tau(t) > 0$ [10].

Thermodynamic properties of heat engines for ignition and detonation mechanisms are investigated and time-dependent physical quantities are compatible with empirical results of heat engines [12]. It is useful to study internal states of piston and combustion engines and applicable even if they use one or more reciprocating pistons for energy conversions. The characteristic properties of $\tilde{T}(t)$ at the beginning for ignition or detonation, intermediate stable thermal state, and final termination phenomena of a long-time and a short-time operations of heat engines are shown in Figure 17 and Figure 18, respectively. The thermal temperature $\tilde{T}(t)$ increases and changes rapidly at the first period of time as seen from Figure 17, interpreted as the warming-up of heat engines. The temperature soon reaches a thermal equilibrium shown by a stable horizontal line, which also corresponds to a stable maximum rotation. The system of heat engine is apt to keep a stable temperature when a flywheel rotation is in a stable maximum. The heat engine becomes unstable and stops abruptly when heat flow is weak to make a rotation, which is exhibited by $\tau(t)$ in **Figure 18** around $10<t<18$. The time-dependent thermal temperature is considered to be a fundamental function to study stability of a thermal state inside heat engines [12].

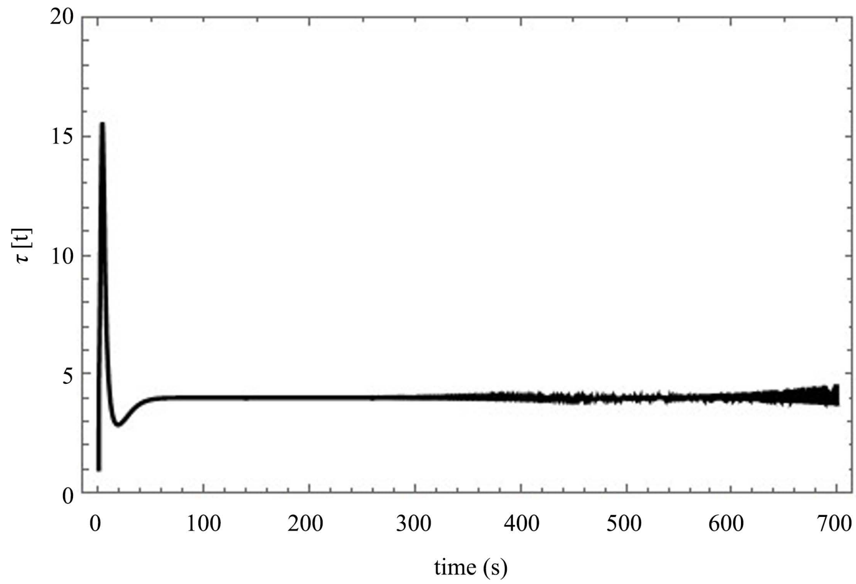

Figure 17. The variation of $\tau(t)$ or temperature $\tilde{T}(t)=T\tau(t)$ in a long-time rotation. Note the variation and fluctuations at the beginning and intermediate time.

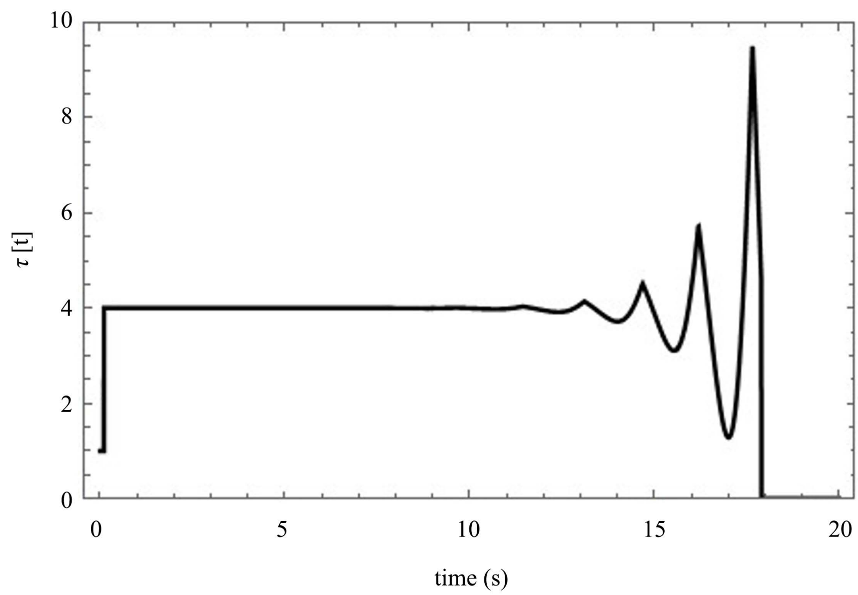

Figure 18. The variation of $\tau(t)$ or temperature in a short-time rotation; rotations end at $t\sim 18$. This could be a model for ignition-like short-time heat flow [12].

Now, the dissipative equation of motion (2.11) for a low temperature Stirling engine is coupled with an axial flux electromagnetic induction (AF-EMI), resulting in the thermoelectric generation Stirling engine (TEG-Stirling engine) [9] [11]. In terms of the flux-categorization, TEG-Stirling engine is classified as an AF-EMI device, whereas the turbines of wind, water, geothermal, thermal and nuclear plants are classified as a radial flux electromagnetic induction (RF-EMI) device. The TEG-Stirling engine is different from the mechanism of turbine in everyday life. The TEG-Stirling engine operates in a very low speed, 30 - 40 (rpm), which is applicable to thermoelectric energy conversions from a low temperature heat source. The TEG-Stirling engine produces tiny pulsating electric currents, originating from the AF-EMI conversion of flywheel rotations.

The angular velocity curves, $\theta'(t)$, produced in TEG-Stirling have tiny fluctuations, shown in **Figures 7-11**, and these frictional disturbances and fluctuations become evident when a microscopic scale is enlarged, displaying piecewise continuous changes. The differential analysis breaks down, because $\theta(t)$ has the finite number of continuous but not differentiable points, which makes numerical calculations of electric power too difficult and hard to produce physically conspicuous results. Therefore, we employ a continuous function similar to $\theta'(t)$ as a sense of an averaged function in macroscopic scale, which enables precise numerical evaluations. It facilitates reproducing electric currents and powers similar to those in papers [9] [11]. The produced electric currents and powers by TEG-Stirling engine of 8-magnet-8-coil thermoelectric conversions are shown in **Figure 19** and **Figure 20**, which should be checked by experiment.

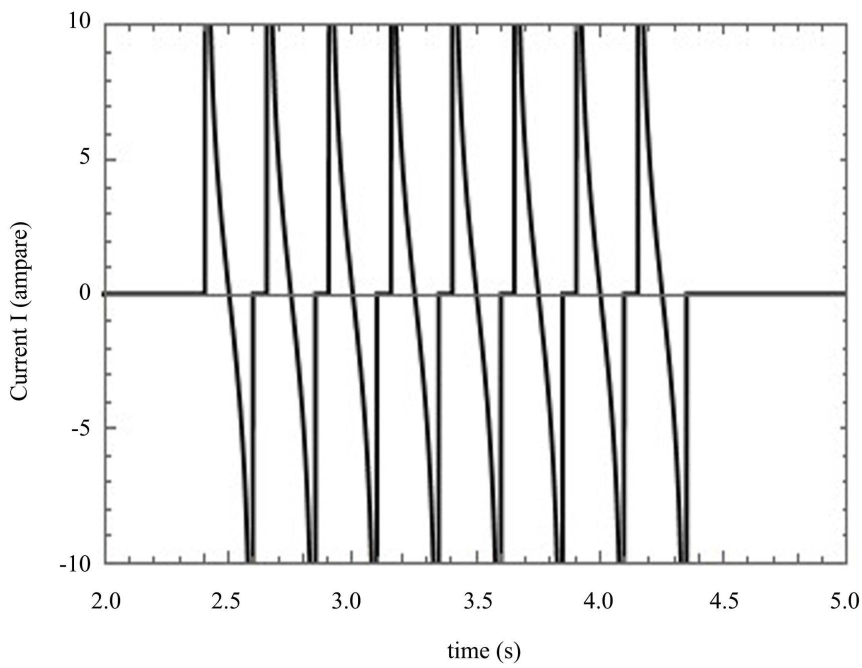

Figure 19. The pulse currents of the axial flux electromagnetic induction (AF-EMI). (8-magnet 8-coil TEG-Stirling engine).

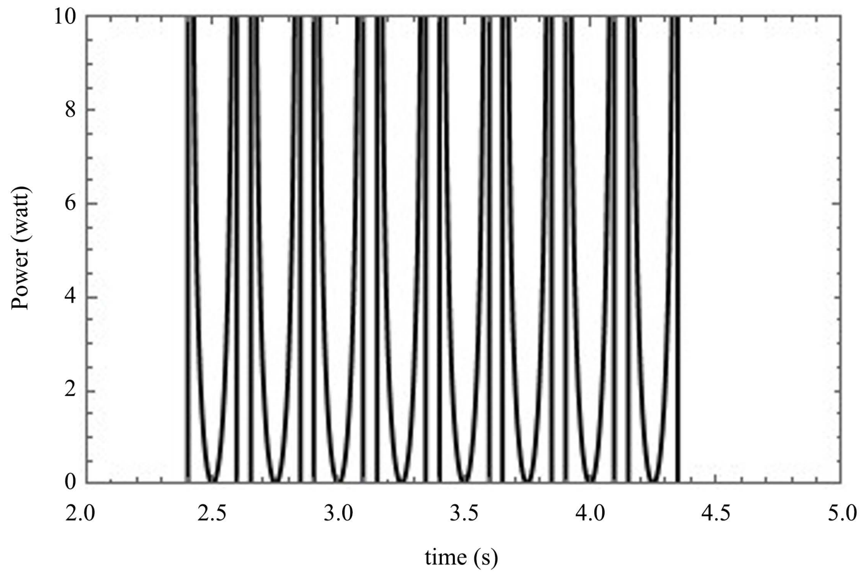

Figure 20. The power of the axial flux electromagnetic induction (AF-EMI). (8-magnet 8-coil TEG-Stirling engine).

4. The Optimal Fuel-Injection and Combustion Timings for Stable Rotations of Heat Engines

The TMD numerical simulations exhibit important properties for stability of a thermal state and a long-time and a short-time uniform rotations. The results are briefly summarized as follows:

1) The initial warming-up for inducing flywheel rotations:

Since a flywheel is at rest in the beginning, a heat engine needs some excess heat to warm up. This is shown by the result that the dissipation of heat from work, $Q_{wd}(t)$, increases at the beginning for certain time and then, gradually develops into a stable minimum. It suggests a well known fact that heat engines in general demand a short time for warming-up before reaching a stable rotation (see **Figure 11**). It signifies a well-known fact that a slight assistance of an initial flywheel motion is helpful for heat engines getting to a stable revolution (indicated by the initial condition $\theta'(0) \neq 0$).

2) Stability and inertia of rotations:

The kinetic work $Q_{wk}(t) = (I_0'/2)\theta'(t)^2$ and the associated heat-flow $Q_{wd}(t)$ give the total thermomechanical work: $Q_w(t) = Q_{wk}(t) + Q_{wd}(t)$. When $Q_{wd}(t)$ reaches a minimum value, the kinetic work $Q_{wk}(t)$ reaches a stable maximum for flywheel rotations, indicating that the rotation persists for a long time with the minimum value of dissipation (**Figure 10** and **Figure 11**). This is an important result for thermoelectric energy conversions derived from TMD analyses for heat engines and essential for constructing practical energy-conversion devices.

3) The time-dependent temperature $\tilde{T}(t)$ to study stability and thermal transitions:

Though the dynamic property of a thermal state is investigated by thermodynamic quantities, $Q_\varepsilon(t)$, $Q_d(t)$ and $Q_w(t)$, the time-dependent temperature

$\tilde{T}(t)$ is more sensitive to stability and changes of thermal states of heat engines. The temperature $\tilde{T}(t)$ reacts conspicuously on fuel-injection at the beginning and an engine-halt at the end, but $\tilde{T}(t)$ proceeds definitely to a stable thermal state; they are explicitly shown in **Figure 17** and **Figure 18**. Based on the numerical simulations, the applications to *fuel-injection* and *combustion* timings are discussed as follows.

The discrete ignition-heat, $Q_{ig}(t)$, is used by assuming:

$$Q_{ig}(t) = Q_1(t,t_1) + Q_2(t,t_2) + Q_3(t,t_3) + \cdots, \quad (t_1 < t_2 < t_3 \cdots). \tag{4.1}$$

where $Q_i(t,t_i)$ ($i = 1,2,\cdots$) could be taken as a rectangular step-function [12]. We used specifically the following step-function,

$$Q_{ig}(t) = Q_{1H}\left(1.0 - e^{-\xi_1 t}\right) + Q_1(t - t_1)\left(\theta(t - t_1) - \theta(t - t_2)\right), \tag{4.2}$$

where the first term is (2.12), and the second term is the ignition term to recover a stable flywheel revolution; Q_1 (cal) is a tiny ignition-heat. The ignition starts at t_1 and ends at t_2.

The recovery and restoration of flywheel revolutions, possible fuel-injection and combustion timings can be investigated. The time-dependent quantities $Q_\varepsilon(t)$, $Q_d(t)$ and $\tau(t)$ displayed in **Figures 14-16** are used to investigate a fuel-injection effect. The fuel-injection time is chosen at $t_1 = 37$ and the duration time is $t_2 - t_1 \sim 5$ seconds. The numerical results are given by **Figure 21** for $\tau(t)$, **Figure 22** for internal energy $Q_\varepsilon(t)$, **Figure 23** for total heat flow $Q_d(t)$. The heat Q_1 can be very small, $Q_1/Q_{1H} \sim 0.001$, meaning a small fuel-injection suffices to keep the stable flywheel motion. The internal energy $Q_\varepsilon(t)$ and total heat flow $Q_d(t)$ slightly increase for $37 < t$ compared to **Figure 14** and **Figure 15**, but the fluctuations of $\tau(t)$ are clearly diminished compared to **Figure 21**, and $\tau(t)$ is back to a stable thermal state and then, rotations stall for $45 < t$. The result suggests that temperature $\tilde{T}(t)$ expresses how, slowly or violently, an internal thermal state responses against external heat flows. This is a new concept for physical interpretation of temperature.

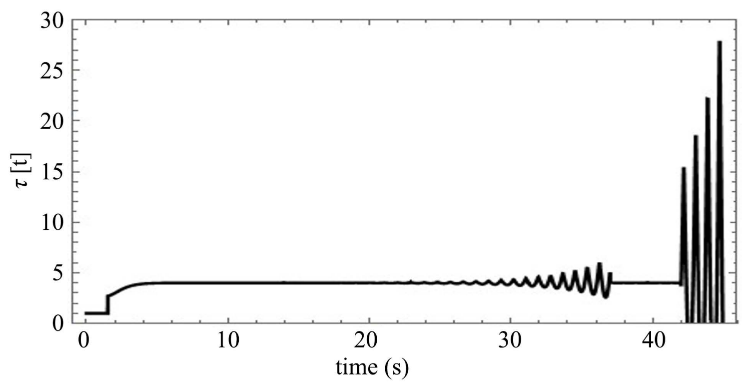

Figure 21. The nonequilibrium temperature, $\tilde{T}(t) = T_0\tau(t)$ (T_0 is an initial temperature at $t = 0$). Note the short time restoration of $\tau(t)$ by a tiny fuel-injection at $t = 36$ (s), and a restoration of thermal state or rotations of engine follows. But, $\tau(t)$ becomes unstable and comes to a halt for $t > 41$ (s).

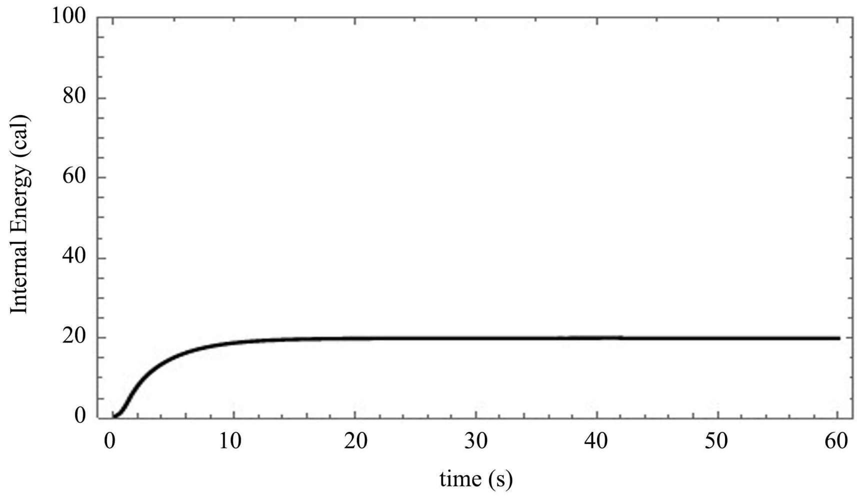

Figure 22. The internal energy, $Q_\varepsilon(t)$ with a fuel-injection at $t = 37$ (s). A tiny straight line is continued as it is in **Figure 14**.

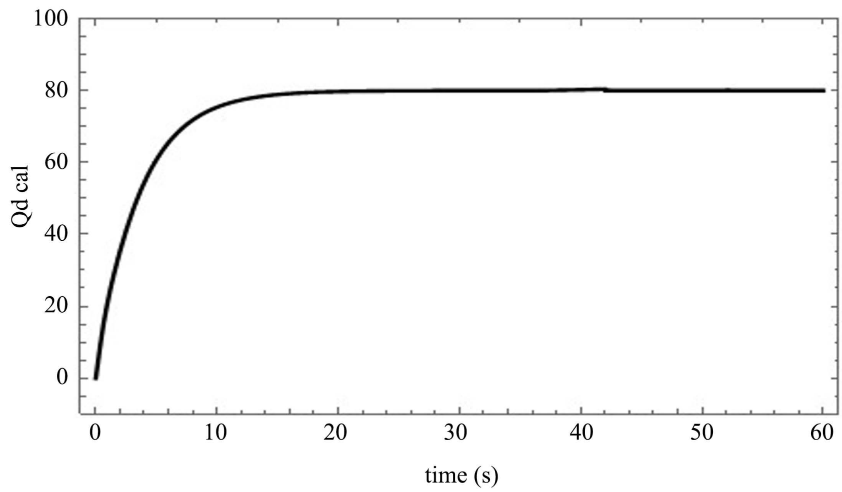

Figure 23. The total heat dissipation, $Q_d(t)$ with a fuel-injection. The heat dissipation continues with a tiny increase about the end (compare with **Figure 15**).

The fuel-injection timings for the stable flywheel rotations are useful to continuously propel, move or power heat engines in general. The constant motion and work for repeated operations of machines are necessary for practical engineering applications. Hence, the mechanism of the fuel-injection and combustion timings is essential to understand efficiency of heat engines. A constant stable motion can be recovered and continued by a very small fuel-injection when the system of heat engines is at a maximum stable thermal state. In addition, it is important that TEG-Stirling engine should be lightweight and require a low, stable maximum angular velocity ($\dot{\theta}/(2\pi) = 30 \sim 50$ rpm). This is the reason why TEG-Stirling engine can be well applied to thermoelectric conversions of waste heat and hot spring water. The property is useful for technologies of sustainable development goals (SDGs).

5. Conclusions and Perspectives

Newtonian mechanics is rigorous when the dissipations of energy (entropy flows) caused by frictional variations and thermal fluctuations do not affect the mechanical motion, trajectory, velocity and acceleration of particles so much. Statistical mechanics is described by states that fluctuate about average values and are characterized by a probability distribution. However, the fundamental requirements of the TMD method are different from Newtonian mechanics, statistics and probability approaches. The dissipative equation of motion for heat engines in the TMD method requires the external driving force constructed by mechanical motions coupled with frictional, viscous and thermal fluctuations. Though frictions are generally considered impractical and useless, heat engines cannot function without friction and viscosity. It is important to know that the coupling of thermodynamic work $Q_w(t)$ and mechanical motion constitute thermomechanical and frictional forces.

The method of TMD has self-consistently solved reciprocating motions of a drinking bird and a low temperature Stirling engine, which contributed to construct thermoelectric generation devices. The time-dependent thermodynamic quantities ($\mathcal{E}(t)$, $W_{th}(t)$, $\mathcal{S}(t)$, $\tilde{T}(t)$) in NISs, stability of a thermal state, and the fuel-injection and combustion timings are studied in the current paper. The time-dependent thermodynamic quantities enable one to investigate responses of heat engines to external driving heat. The thermal temperature, $\tilde{T}(t)$, has been consistent with empirical results of heat engines, and it is very useful to study stability and changes of thermal states, ignition and detonation, friction and diffusion mechanism. The results show a role of temperature as a response function to external disturbances.

The method of TMD has explicitly demonstrated the physical mechanism of coupling between thermal and mechanical states, and the thermomechanical processes of a drinking bird and a low temperature Stirling engine can be self-consistently discussed. The time-dependent characteristics of thermal quantities are consistently calculated by requiring the dissipative equation of motion, thermodynamic laws and measure $\tau(t)$ for NISs. The physical and mathematical concepts derived from the thermodynamic system of a drinking bird and a low temperature Stirling engine are fundamental. The TMD method has established both a physical approach for phase transitions from a thermodynamic equilibrium to nonequilibrium irreversible states and a mathematical approach by introducing nonlinear differential equations with time-dependent coefficients (NDE-TC) [17].

The TMD method is compatible with empirical results of heat engines, thermal and mechanical phenomena. It should be applied to Fourier's theory of heat conduction, reciprocated engines, ignition, combustion and detonation [18]-[20]. The very high-temperature pressurized steam is not necessarily required for thermoelectric energy conversions for the current AF-EMI, TEG-Stirling engine. In addition, it demands a lightweight and an optimum low angular velocity. Therefore, it is possible that thermoelectric energy conversion devices from hot water and waste heat about $T = 50^\circ C$ - $100^\circ C$ can be constructed by TEG-Stirling engine. The results are essential for applications to sustainable environmental technologies [6] [21] [22]. The theory of TMD must be tested by applying it to fundamental physics, such as the third law of thermodynamics and Maxwell's demon [23]-[26], which will be studied in the near future.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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