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Linearization of Intrinsically Nonlinear Oscillators

Pirooz Mohazzabi[1*], Qinghua Luo[1], Ilirian Dhimitruka[2]

[1]Department of Mathematics and Physics, University of Wisconsin-Parkside, Kenosha, WI, USA
[2]Department of Chemistry, University of Wisconsin-Parkside, Kenosha, WI, USA
Email: *mohazzab@uwp.edu

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Abstract

We show that an intrinsically nonlinear oscillator can always be transformed into a linear or harmonic oscillator by addition of a constant force, which shifts the equilibrium position of the oscillator.

Keywords

Intrinsic, Nonlinear, Oscillator, Linearization

1. Introduction

Consider a particle of mass m moving in one dimension under a potential energy function $V(x)$, with a minimum at $x=0$, where the particle has stable equilibrium. This function can usually be expanded in a Taylor series about the equilibrium point [1]-[5],

$$V(x)=V(0)+\frac{V'(0)}{1!}x+\frac{V''(0)}{2!}x^2+\frac{V'''(0)}{3!}x^3+\cdots \tag{1}$$

Because the minimum of the potential energy can be chosen arbitrarily, we set $V(0)=0$. Also, at the minimum of the potential energy, the first derivative is zero, $V'(0)=0$. Therefore, the expansion starts with the quadratic term,

$$V(x)=\frac{V''(0)}{2!}x^2+\frac{V'''(0)}{3!}x^3+\cdots \tag{2}$$

Now, if the amplitude of oscillations is small, $|x|\ll 1$, we can neglect all the terms higher than the second-order term, and write

$$V(x)\approx\frac{1}{2}kx^2 \tag{3}$$

where $k=V''(0)$, is a positive constant. This, by definition, results in simple harmonic (or linear) oscillations of the particle.

Because of the above analysis, some references explicitly claim that for small amplitudes, *any* oscillator is approximately a simple harmonic (or linear) oscillator [6], and some others leave the reader with this impression. This, however, is not true and some oscillators, even in the zero-amplitude limit remain non-harmonic [7]. It has been shown that there are in fact many oscillators that are intrinsically non-linear. More specifically, there are many oscillators for which Taylor expansion of their potential energy function starts with higher than a quadratic term, or the expansion does not even exist, as we shall see in the following examples.

Consider a block of mass m, connected to a spring of spring constant k, which can slide on a horizontal frictionless rod, as shown in **Figure 1**. The potential energy of the system is

$$V(x)=\frac{1}{2}k\left(l-l_0\right)^2=\frac{1}{2}k\left(\sqrt{x^2+l_0^2}-l_0\right)^2=\frac{1}{2}kl_0^2\left[\sqrt{1+\left(\frac{x}{l_0}\right)^2}-1\right]^2 \tag{4}$$

Now, if the amplitude of the oscillations is small, $x/l \ll 1$, we have

$$\sqrt{1+\left(\frac{x}{l_0}\right)^2}\approx 1+\frac{1}{2}\left(\frac{x}{l_0}\right)^2 \tag{5}$$

Then, to a good approximation, we have

$$V(x)=\frac{k}{8l_0^2}x^4 \tag{6}$$

Therefore, the oscillator is intrinsically non-linear or anharmonic. We mention in passing that even though this oscillator is non-harmonic, its period can be calculated, which is [7]

$$T=\sqrt{\frac{2}{\pi}}\Gamma^2\left(\frac{1}{4}\right)\frac{l_0}{A}\sqrt{\frac{m}{k}} \tag{7}$$

where A is the amplitude of the oscillations.

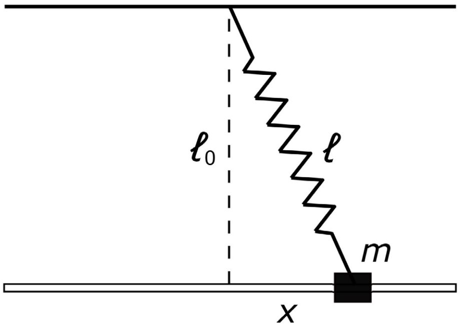

Figure 1. An intrinsically nonlinear oscillator.

We now consider a potential energy function that cannot be Taylor expanded about the equilibrium point. The potential energy function of the dynamically shifted oscillator is defined by

$$V(x)=\begin{cases}\frac{1}{2}k\left(x+x_0\right)^2, & x\ge 0\\ \frac{1}{2}k\left(x-x_0\right)^2, & x<0\end{cases} \tag{8}$$

where x_0 is a constant, which can be positive or negative, known as the dynamic shift. Dimensionless graphs of this potential energy function are shown in Figure 2. Clearly, for $x_0 = 0$ this reduces to the harmonic oscillator potential. As can be seen from Figure 2(a), for $x_0 > 0$ this potential energy function cannot be expanded about the equilibrium point in a Taylor series. Therefore, an oscillator under this potential energy function is intrinsically non-linear. However, for $x_0 < 0$, the potential energy has two stable equilibrium points, as can be seen from Figure 2(b), resulting in small amplitude harmonic oscillations about these points. Interestingly, the dynamically shifted oscillator potential with $x_0 < 0$ resembles that of an ammonia molecule [8].

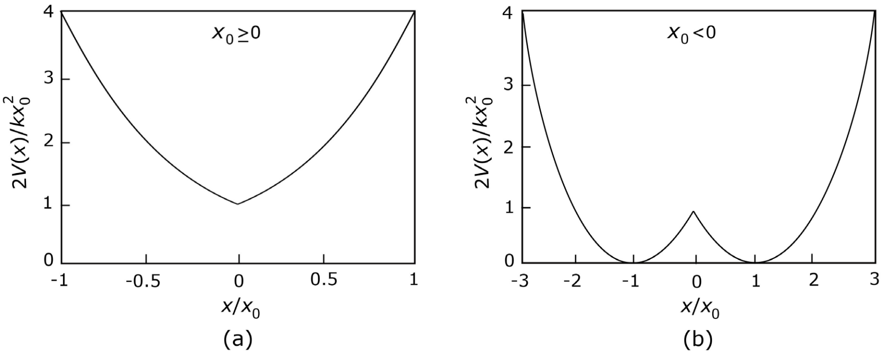

Figure 2. Dimensionless graphs of the dynamically shifted potential energy function.

2. Linearizing a Nonlinear Oscillator

We now show how an intrinsically nonlinear oscillator can, in general, be linearized by addition of a constant force. We begin with a simple example.

Consider a one-dimensional nonlinear oscillator with the equation of motion

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-kx^3 \tag{9}$$

The equilibrium position of this oscillator is at $x=0$. Now suppose we add a constant force f to it,

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-kx^3+f \tag{10}$$

This changes the equilibrium position to a new point, which is the solution of the equation

$$-kx^3+f=0 \tag{11}$$

which is $x=\left(f/k\right)^{1/3}$. Defining a new function $u=x-\left(f/k\right)^{1/3}$, transforms Equation (10) to

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2}=-k\left[u+(f/k)^{1/3}\right]^3+f \tag{12}$$

which reduces to

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2}=-k\left[u^3+3\left(\frac{f}{k}\right)^{1/3}u^2+3\left(\frac{f}{k}\right)^{2/3}u+\frac{f}{k}\right]+f \tag{13}$$

or

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2}=-ku^3-3k\left(\frac{f}{k}\right)^{1/3}u^2-3k\left(\frac{f}{k}\right)^{2/3}u \tag{14}$$

Now, if the amplitude of the oscillations is small, $|u|\ll 1$, the first and second terms on the right-hand side are negligible compared to the last term, and we have

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2}=-3k\left(\frac{f}{k}\right)^{2/3}u \tag{15}$$

which is the differential equation of a simple harmonic motion.

Now consider a particle oscillating under a general potential energy function $V(x)$ with a minimum at $x=0$ (equilibrium point). The force corresponding to this potential energy is

$$F(x)=-\frac{\mathrm{d}V}{\mathrm{d}x} \tag{16}$$

Then the equation of motion of the particle is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{\mathrm{d}V}{\mathrm{d}x} \tag{17}$$

If we now add a constant force *f*, the equation of motion becomes

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2}=-\frac{\mathrm{d}V}{\mathrm{d}x}+f \tag{18}$$

This shifts the equilibrium position to a new value x_0 which is the solution of the following equation,

$$-\frac{\mathrm{d}V}{\mathrm{d}x}+f=0 \tag{19}$$

The new potential energy function is now

$$U(x)=V(x)-fx \tag{20}$$

which has a minimum at $x=x_0$ if f is chosen small enough. Expanding the new potential energy function $U(x)$ in a Taylor series about x_0, we have

$$U(x)=U(x_0)+\frac{U'(x_0)}{1!}(x-x_0)+\frac{U''(x_0)}{2!}(x-x_0)^2+\frac{U'''(x_0)}{3!}(x-x_0)^3+\cdots \tag{21}$$

We chose our coordinate system so that $U(x_0)=0$. Also, $U'(x_0)=0$ because $U(x)$ is minimum at x_0. Therefore, we have

$$U(x)=\frac{U''(x_0)}{2!}(x-x_0)^2+\frac{U'''(x_0)}{3!}(x-x_0)^3+\cdots \tag{22}$$

For small oscillations, $|x-x_0| \ll 1$, we can ignore the third- and higher-order terms, and we have

$$U(x) \approx \frac{1}{2}U''(x_0)(x-x_0)^2 \tag{23}$$

Therefore, the equation of motion of the oscillator is

$$m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} = -U''(x_0)(x-x_0) \tag{24}$$

Finally, defining $u = x - x_0$, we obtain

$$m\frac{\mathrm{d}^2u}{\mathrm{d}t^2} = -U''(x_0)u \tag{25}$$

Therefore, the oscillator becomes linear (harmonic) even if with the initial potential energy $V(x)$ it was intrinsically nonlinear. Remember that $U''(x_0) > 0$ because x_0 is the minimum of $U(x)$.

As an example, let us consider the system shown in **Figure 1**, and turn it sideways so that the oscillator (block) is also acted on by the gravitational force, as shown in **Figure 3**. The potential energy function is now given by

$$U(x) = -mgx + \frac{1}{2}k(l-l_0)^2 = -mgx + \frac{1}{2}kl_0^2\left(\frac{l}{l_0}-1\right)^2 \tag{26}$$

From the figure, we have

$$l = \sqrt{x^2 + l_0^2} \tag{27}$$

Therefore,

$$U(x) = -mgx + \frac{1}{2}kl_0^2\left[\sqrt{\left(\frac{x}{l_0}\right)^2 + 1} - 1\right]^2 \tag{28}$$

To find the value of *x* at which $U(x)$ is a minimum,

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -mg + kx\left[1 - \frac{1}{\sqrt{\left(\frac{x}{l_0}\right)^2 + 1}}\right] = 0 \tag{29}$$

This equation has one root that can be investigated numerically. Let us denote this root by x_0, which corresponds to the minimum of the potential energy function (28). We now expand $U(x)$ in a Taylor series about x_0,

$$U(x) = U(x_0) + \frac{U'(x_0)}{1!}(x-x_0) + \frac{U''(x_0)}{2!}(x-x_0)^2 + \cdots \tag{30}$$

In this equation, the first term is zero by choosing the minimum of the potential energy to be zero, and the second term is zero because the first derivative of the function is zero at its minimum [also by Equation (29)]. Therefore, when the amplitude of the oscillations is small, we have

$$U(x) \approx \frac{1}{2} U''(x_0)(x - x_0)^2 \tag{31}$$

Here $U''(x_0)$ is the second derivative of Equation (28) evaluated at $x = x_0$. Also, $U''(x_0) > 0$ because $U(x_0)$ is a minimum. The equation of motion of the oscillator is then

$$m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -U''(x_0)(x - x_0) \tag{32}$$

and if we define $u = x - x_0$, we get

$$m \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} = -U''(x_0) u \tag{33}$$

and the oscillator becomes linear or harmonic.

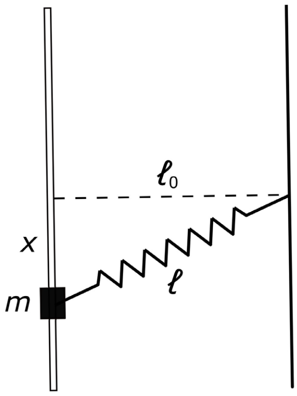

Figure 3. Same structure as in **Figure 1** but now the block is also acted on by the force of gravity *mg*.

Yet another example is the system shown in **Figure 4**, in which a particle of mass *m* is tethered by two identical springs, each of spring constant *k*. Based on the above discussion, one can easily show that in the absence of gravity oscillation of the system is intrinsically nonlinear, but with the effect of gravity it becomes linear or harmonic.

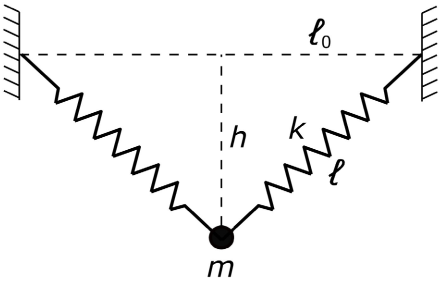

Figure 4. An oscillating system which is intrinsically nonlinear in the absence of gravity, but becomes linear due to gravity.

3. Discussion and Conclusion

There are many oscillators in nature that are intrinsically nonlinear. These oscillators remain non-harmonic or nonlinear even in the limit of zero amplitude because the leading term in Taylor expansion of their potential energy function does not start with a quadratic term in the displacement.

In this article, we have shown that these nonlinear systems can be linearized by addition of a constant force to the oscillator. The role of the constant force is to shift the position of the minimum of the potential energy function, resulting in its Taylor expansion to start with a quadratic term.

It is also possible for the leading term in Taylor expansion of an intrinsically nonlinear oscillator to start with higher powers of the displacement. However, the power of the leading term cannot be odd. This is because if the leading term has an odd power, the potential energy function cannot have a minimum at the equilibrium point. Therefore, the leading term must have an even power. However, regardless of this even power, the linearization by addition of a constant force described above is still applicable and can easily be verified.

In addition to physics and other sciences, oscillations play a crucial role in various fields of engineering. These include, but are not limited to, mechanical engineering, electronics and communication engineering, civil engineering, Aerospace Engineering, and Aerospace Engineering. Because in some cases these oscillations are intrinsically nonlinear, and because linear oscillating systems are the simplest ones and are easy to study, transforming a nonlinear oscillator to linear can be helpful.

Finally, according to the analysis provided in this work, as long as the potential energy function of an intrinsically nonlinear oscillator has a minimum at its equilibrium point, it can be linearized according to the above algorithm by adding a constant force to it.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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