

High-Order Solitons and Hybrid Behavior of (3 + 1)-Dimensional Potential Yu-Toda-Sasa-Fukuyama Equation with Variable Coefficients

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Abstract

In this paper, some exact solutions of the (3 + 1)-dimensional variable-coefficient Yu-Toda-Sasa-Fukuyama equation are investigated. By using Hirota's direct method and symbolic computation, we obtained N-soliton solution. By using the long wave limit method, the N-order rational solution can be obtained from N-order soliton solution. Then, through the paired complexification of parameters, the lump solution is obtained from N-order rational solution. Meanwhile, we obtained a hybrid solution between 1-lump solution and N-soliton ($N = 1, 2$) by using the long wave limit method and parameter complex. Furthermore, four different sets of three-dimensional graphs of solitons, lump solutions and hybrid solutions are drawn by selecting four different sets of coefficient functions which include one set of constant coefficient function and three sets of variable coefficient functions.

Keywords

Variable-Coefficient YTSF Equation, Hirota Bilinear Method, N-Soliton, Hybrid Solution

1. Introduction

It is well known that the exact solutions of nonlinear partial differential equations play an important role in the study of many complex physical phenomena and other nonlinear engineering problems. Constructing the exact solution of nonlinear wave equations is of great significance in science and engineering applications. With the rapid development of science and technology, many methods for solving nonlinear partial differential equations have been developed,

such as the long wave limit method [1] [2], the multiple exp-function algorithm [3], inverse scattering [4], Lie group method [5], Hirota bilinear method [6], Darboux transformation [7] [8].

Recently, Yu *et al.* [9] extended Bogoyavlenskii-Schif equation to be the (3 + 1)-dimensional potential Yu-Toda-Sasa-Fukuyama equation

$$u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z + 3u_{yy} - 4u_{xt} = 0. \quad (1)$$

This equation combines contributions from multiple scholars and represents an effort to gain a deeper understanding of nonlinear wave phenomena, involving the intersection of different subject areas. The study of (3 + 1)-dimensional potential Yu-Toda-Sasa-Fukuyama equation helps to explore nonlinear phenomena that are widely seen in physics, engineering and mathematics, such as the formation and propagation of waves. As a classical nonlinear partial differential equation, the constant-coefficient (3 + 1)-dimensional YTSF equation has been studied by many scholars [9]-[14]. Manafian *et al.* investigated the periodic wave solutions for the (3 + 1)-dimensional potential-Yu-Toda-Sasa-Fukuyama equation, from its bilinear form, obtained using the Hirota operator [15]. By using the exp-function method, Wang obtained new generalized solitary solutions and periodic solutions with free parameters [16]. Younis investigated some new exact solutions and mixed lump wave solutions by using the extended three soliton test approach [17]. By selecting specific parameters, Ma obtained some local waves from multi-soliton solutions [18]. Tan investigated the dynamics of kinky wave by using extended homoclinic test technique [19] [20]. Scholars have studied many methods to analyze the constant coefficient equations, and the exact solutions of the constant coefficient equations have been studied thoroughly. However, due to the fact that variable-coefficient equations are more complicated and difficult to solve than the constant-coefficient equations, research on variable-coefficient equations is relatively scarce.

In order to seek new soliton structures and study the diversity of solitons, based on [9], we study the (3 + 1)-dimensional potential Yu-Toda-Sasa-Fukuyama equation with variable coefficients as follows:

$$u_{xxxz} + 4u_x u_{xz} + 2u_{xx} u_z + \alpha(t)u_{yy} + \beta(t)u_{xt} = 0, \quad (2)$$

where $\alpha(t), \beta(t)$ are free function of t , and u is the function of x, y, z, t .

This paper is organized as follows. In Section 2, we get the bilinear form of the (3 + 1)-dimensional vc-YTSF equation. In Section 3, first, N-soliton solutions can be obtained by using the Hirota bilinear method. Second, by using the long wave limit method, we get the N-order rational solutions, lump solutions and hybrid solutions between 1-lump and N-solitons from N-solitons. Finally, some conclusions can be given in Section 4.

2. The N-Soliton Solutions and Lump Solutions

We take $\sigma = x + \lambda z$ [19], then Equation (2) can be reduced to the following form

$$\lambda u_{\sigma\sigma\sigma\sigma} + 6\lambda u_{\sigma}u_{\sigma\sigma} + \alpha(t)u_{yy} + \beta(t)u_{\sigma t} = 0 \tag{3}$$

According to [21], we take the following transformation:

$$u = 2(\ln f)_{\sigma} \tag{4}$$

where f is an unknown test function. Substituting (4) into (3), we get the following bilinear form of (3 + 1)-dimensional vc-YTSE equation:

$$(\lambda D_{\sigma}^4 + \alpha(t)D_y^2 + \beta(t)D_{\sigma}D_t)f \cdot f = 0, \tag{5}$$

where the operator D_{-} is the Hirota's bilinear differential operator defined by [6].

2.1. The N -Soliton Solutions

Based on the Hirota's bilinear Equation (5), the N -order soliton solutions of (3 + 1)-dimensional vc-YTSE Equation (2) can be obtained,

$$f = f_N = \sum_{\mu=0,1} \exp\left(\sum_{i=1}^N \mu_i \eta_i + \sum_{i<j}^{(N)} A_{ij} \mu_i \mu_j\right), \tag{6}$$

where the $\sum_{\mu=0,1}$ represents summation over all possible combinations of $\mu_1 = 0, 1, \mu_2 = 0, 1, \dots, \mu_N = 0, 1$, and the $\sum_{i<j}^{(N)}$ is over all possible combinations of the N elements with the specific condition $i < j$.

By combining (6) and (5), we get the parameter relationships as

$$\begin{cases} \eta_i = k_i(m_i\sigma + p_i y + w_i(t) + \gamma_i, \\ w_i(t) = -\int \frac{\lambda k_i^2 m_i^4 + \alpha(t)p_i^2}{\beta(t)m_i} dt, \\ \exp(A_{ij}) = \frac{\alpha(t)(m_i p_j - m_j p_i)^2 - 3\lambda(k_i m_i^2 m_j - k_j m_j^2 m_i)^2}{\alpha(t)(m_i p_j - m_j p_i)^2 - 3\lambda(k_i m_i^2 m_j + k_j m_j^2 m_i)^2} \end{cases} \tag{7}$$

where $m_i \neq 0$ and k_i, p_i, γ_i are some free parameters. Substituting (6) with (7) into (4), we can obtain the N -soliton solutions of (3 + 1)-dimensional vc-YTSE Equation (2).

By consulting references [22]-[24] and combining the equation, we choose the following four functions for $\alpha(t)$ and $\beta(t)$.

Case 1:

$$\alpha(t) = 3, \beta(t) = -4 \tag{8}$$

Case 2:

$$\alpha(t) = t, \beta(t) = 1 \tag{9}$$

Case 3:

$$\alpha(t) = \frac{1}{2} \cos\left(\frac{t}{2}\right), \beta(t) = 1 \tag{10}$$

Case 4:

$$\alpha(t) = \frac{t}{2} \cos\left(\frac{t}{2}\right), \beta(t) = 1 \tag{11}$$

Among the above four different sets of functions, there is one set of constant coefficients and the remaining three of variable coefficients. Through these four sets of functions, we will get one set of three-dimensional diagrams of constant coefficient functions and three sets of variable coefficient functions. Through the comparison of the diagrams, we can clearly see the difference in the exact solutions between the variable coefficient equation and the constant coefficient equation.

We give four different three-dimensional diagrams of 1-soliton solution in **Figure 1**, with different parameter values as (a):

$$(\lambda, k_1, m_1, p_1, \gamma_1) = \left(-1, \frac{1}{5}, \frac{2}{5}, \frac{3}{2}, 1\right), \text{ (b): } (\lambda, k_1, m_1, p_1, \gamma_1) = \left(-1, \frac{1}{4}, \frac{4}{5}, \frac{1}{4}, 1\right), \text{ (c), (d):}$$

$$(\lambda, k_1, m_1, p_1, \gamma_1) = \left(-1, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 1\right). \text{ In Figure 2, we obtain four different three-dimensional diagrams of 2-soliton solutions when the parameters are: (a):}$$

$$\lambda = -1, k_1 = -\frac{1}{4}, k_2 = \frac{3}{2}, m_1 = 1, m_2 = \frac{2}{5}, p_1 = 3, p_2 = \frac{1}{5}, \gamma_1 = \gamma_2 = 1; \text{ (b):}$$

$$\lambda = -1, k_1 = \frac{1}{4}, k_2 = \frac{1}{2}, m_1 = \frac{3}{5}, m_2 = \frac{2}{5}, p_1 = 3, p_2 = \frac{1}{5}, \gamma_1 = \gamma_2 = 1; \text{ (c),}$$

$$\text{(d): } \lambda = -1, k_1 = \frac{1}{2}, k_2 = \frac{3}{2}, m_1 = 3, m_2 = \frac{2}{5}, p_1 = \frac{5}{2}, p_2 = \frac{1}{3}, \gamma_1 = \gamma_2 = 1.$$

Similarly, we get the diagrams of 3-soliton solutions in **Figure 3** by taking the

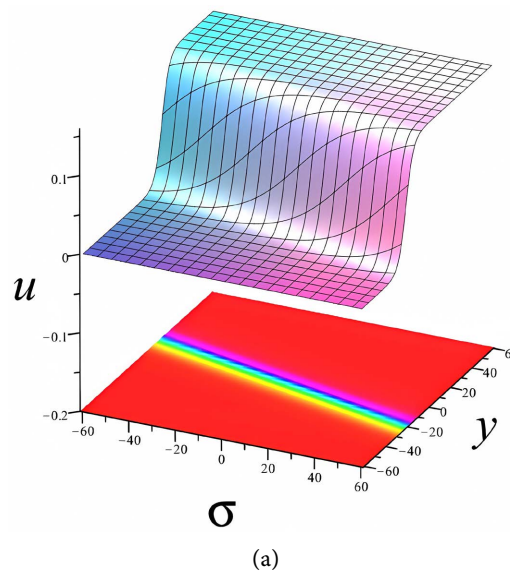
$$\text{following parameters: (a): } \lambda = -1, k_1 = \frac{1}{2}, k_2 = k_3 = \frac{3}{2}, m_1 = \frac{1}{2}, m_2 = \frac{3}{4},$$

$$m_3 = \frac{2}{3}, p_1 = \frac{3}{2}, p_2 = \frac{1}{5}, p_3 = \frac{1}{2}, \gamma_i = 1(i=1,2,3), \text{ (b): } \lambda = -1, k_1 = 1,$$

$$k_2 = 2, k_3 = \frac{5}{2}, m_1 = \frac{4}{5}, m_2 = \frac{1}{2}, m_3 = \frac{2}{3}, p_1 = \frac{1}{4}, p_2 = 3, p_3 = \frac{1}{2},$$

$$\gamma_i = 1(i=1,2,3), \text{ (c), (d): } \lambda = -1, k_1 = \frac{1}{2}, k_2 = 2, k_3 = \frac{3}{2}, m_1 = \frac{7}{2}, m_2 = \frac{1}{4},$$

$$m_3 = \frac{2}{3}, p_1 = \frac{5}{2}, p_2 = \frac{1}{5}, p_3 = \frac{1}{2}, \gamma_i = 1(i=1,2,3).$$



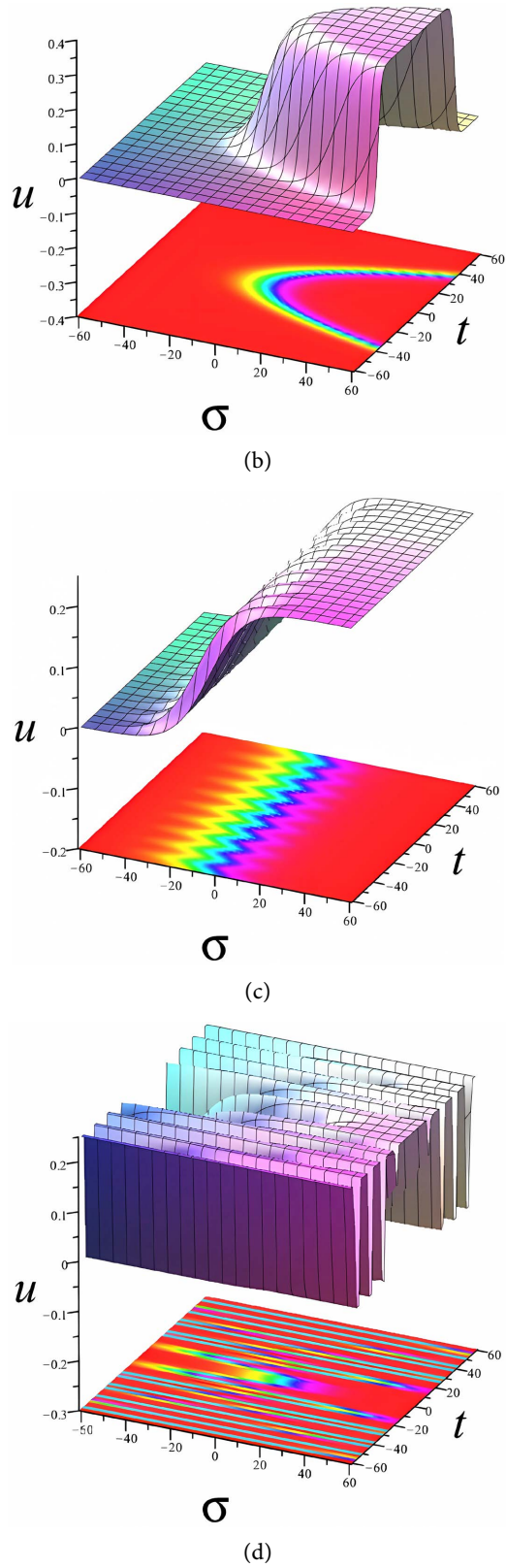
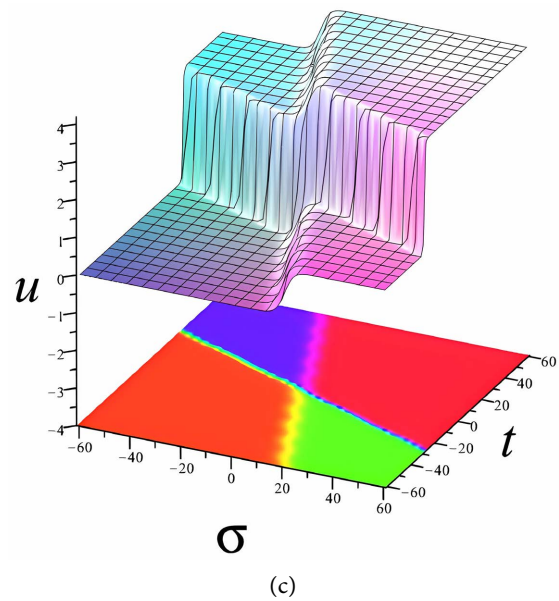
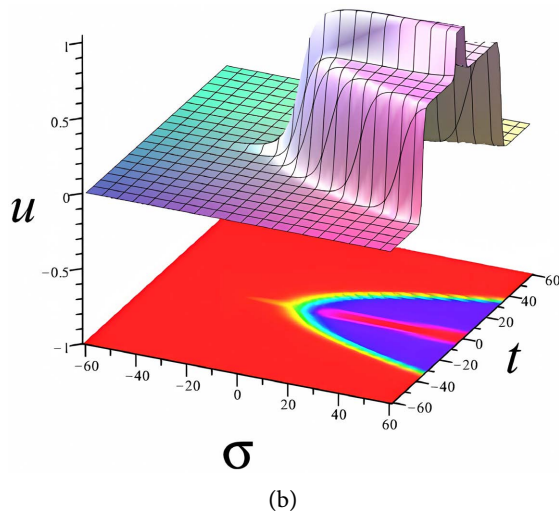
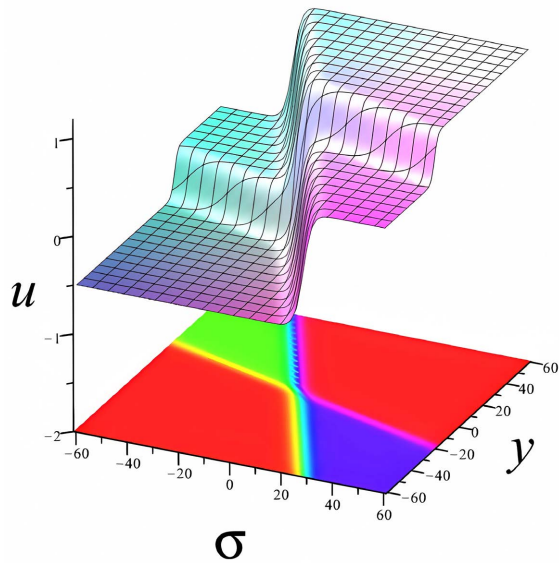


Figure 1. Plots of 1-soliton solution at $y=0$: (a) $\alpha(t)=3$, $\beta(t)=-4$; (b) $\alpha(t)=t$, $\beta(t)=1$; (c) $\alpha(t)=\frac{1}{2}\cos\left(\frac{t}{2}\right)$, $\beta(t)=1$; (d) $\alpha(t)=\frac{t}{2}\cos\left(\frac{t}{2}\right)$, $\beta(t)=1$.



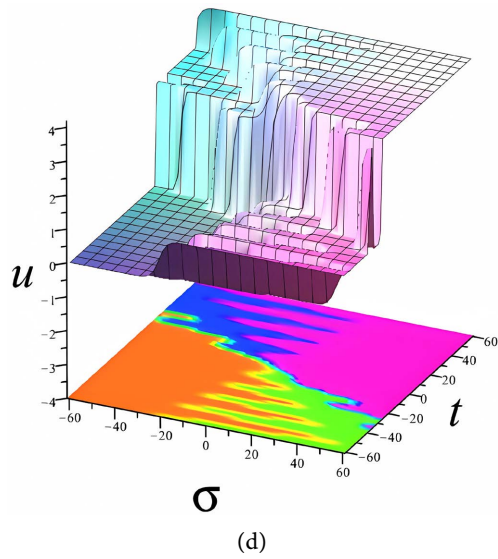
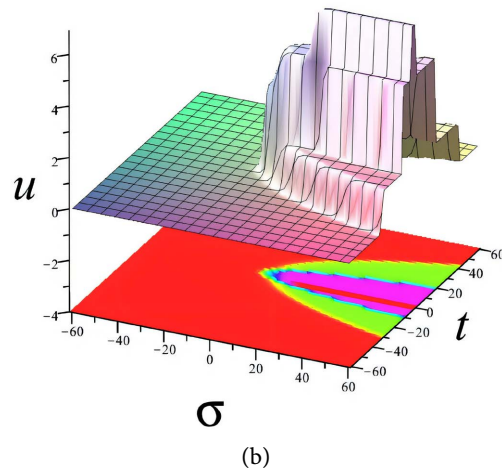
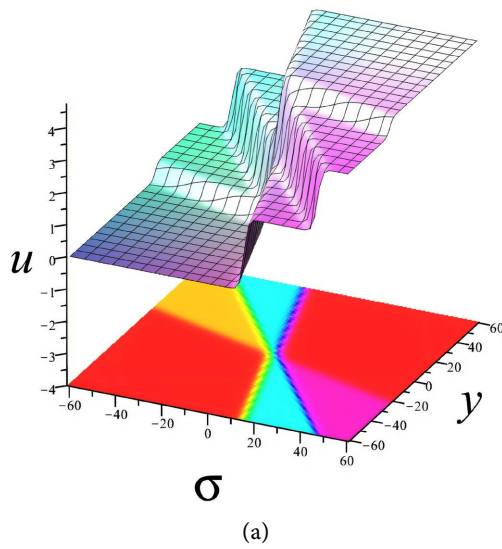


Figure 2. Plots of 2-soliton solution at $y=0$: (a) $\alpha(t)=3, \beta(t)=-4$; (b) $\alpha(t)=t, \beta(t)=1$; (c) $\alpha(t)=\frac{1}{2}\cos\left(\frac{t}{2}\right), \beta(t)=1$; (d) $\alpha(t)=\frac{t}{2}\cos\left(\frac{t}{2}\right), \beta(t)=1$.



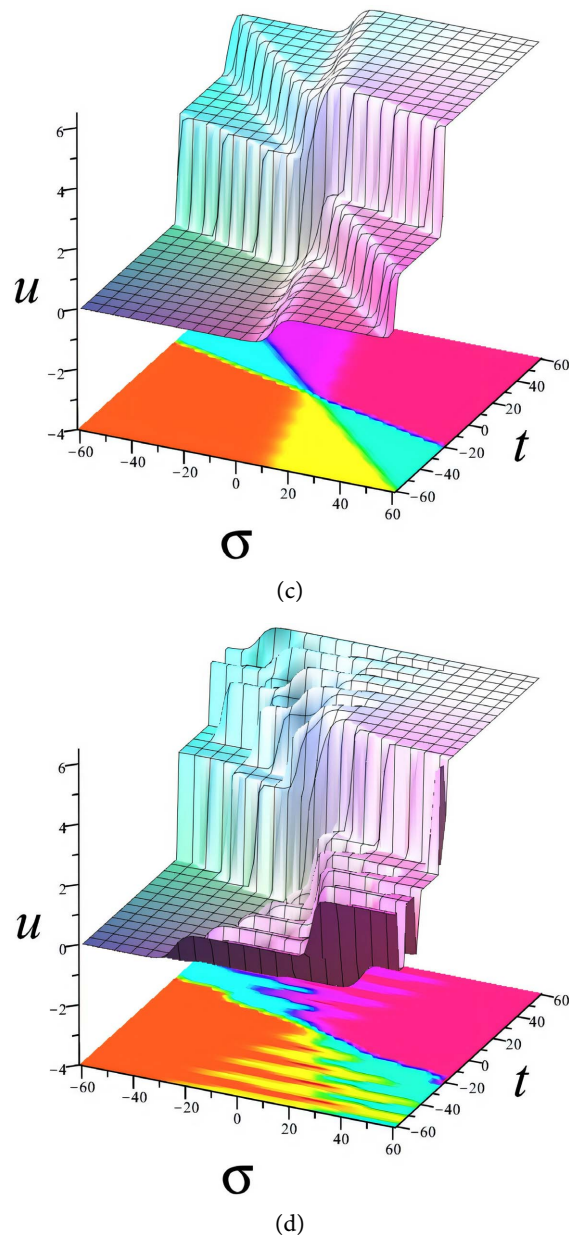


Figure 3. Plots of 3-soliton solution at $y=0$: (a) $\alpha(t)=3, \beta(t)=-4$; (b) $\alpha(t)=t, \beta(t)=1$; (c) $\alpha(t)=\frac{1}{2}\cos\left(\frac{t}{2}\right), \beta(t)=1$; (d) $\alpha(t)=\frac{t}{2}\cos\left(\frac{t}{2}\right), \beta(t)=1$.

2.2. The Lump Solution

In this section, we first get the N-order rational solution from the N-order soliton solutions by using the long wave limit method, and then complex the parameters of the N-order rational solutions to obtain the lump solutions. In order to get the N-order rational solution of (3 + 1)-dimensional vc-YTSF Equation (2), we take $\gamma_i = I\pi$ in (6) and (7), then f_N can be written as follows

$$f_N = \sum_{\mu=0,1} \prod_{i=1}^N (-1)^{\mu_i} \exp(\mu_i \xi_i) \prod_{i < j}^{(N)} \exp(\mu_i \mu_j A_{ij}), \tag{12}$$

with $\xi_i = k_i \left(m_i \sigma + p_i y - \int \frac{\lambda k_i^2 m_i^4 + \alpha(t) p_i^2}{\beta(t) m_i} dt \right)$. Letting k_i tend to 0 and taking $k_i/k_j = O(1)$ in (12) and (6), f_N can be rewritten as

$$f_N = \sum_{\mu=0,1} \prod_{i=1}^N (-1)^{\mu_i} (1 + \mu_i k_i \theta_i) \prod_{i < j}^{(N)} \exp(1 + \mu_i \mu_j k_i k_j B_{ij}) + O(k^{N+1}), \tag{13}$$

with

$$\begin{cases} \theta_i = m_i \sigma + p_i y - \int \frac{\alpha(t) p_i^2}{\beta(t) m_i} dt, \\ B_{ij} = \frac{12 \lambda m_i^3 m_j^3}{\alpha(t) (m_i p_j - m_j p_i)^2}. \end{cases} \tag{14}$$

Combining the above Equation (13) with the transformation u given by Equation (4), we can get the following form of rational solutions,

$$\begin{aligned} f_N = & \prod_{i=1}^N \theta_i + \frac{1}{2} \sum_{i,j}^{(N)} B_{ij} \prod_{r \neq i,j}^N \theta_r + \frac{1}{2! 2^2} \sum_{i,j,p,s}^{(N)} B_{ij} B_{ps} \prod_{r \neq i,j,p,s}^N \theta_r + \dots \\ & + \frac{1}{M! 2^M} \sum_{i,j,\dots,m,n}^{(N)} \overbrace{B_{ij} B_{kl} \dots B_{mn}}^M \prod_{s \neq i,j,k,l,\dots,m,n}^N \theta_s + \dots, \end{aligned} \tag{15}$$

where $\sum_{i,j,\dots,m,n}^{(N)}$ denotes the summation over all possible combinations of i, j, \dots, m, n . We can get the corresponding N -order rational solution of (3 + 1)-dimensional vc-YTSF Equation (2) by taking $N = 1, 2, \dots$ in (15) and substituting it into (4).

For example, take $N = 2$ in (15), then, we get the following form

$$f_2 = \theta_1 \theta_2 + B_{12}. \tag{16}$$

The following form of 2-order rational solution can be obtained by bringing (14) and (16) into (4)

$$u(x, y, t) = \frac{2m_1 \theta_2 + 2m_2 \theta_1}{\theta_1 \theta_2 + B_{12}}. \tag{17}$$

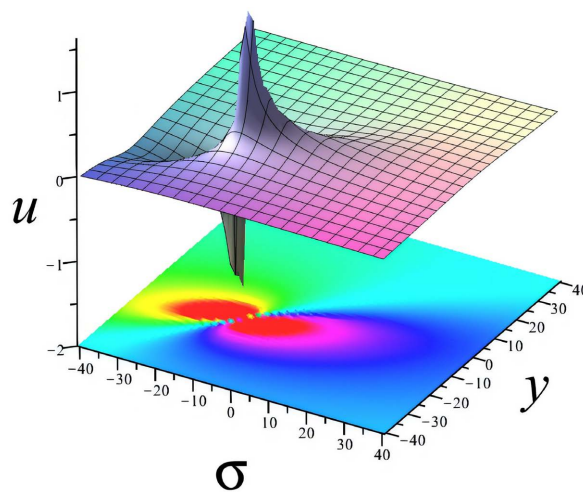
In order to obtain 1-order lump solution, first, we set $m_2 = m_1^* = a_1 - b_1 I$, $p_2 = p_1^* = c_1 - d_1 I$ in (16), so f_2 can be rewritten as

$$\begin{aligned} \check{f}_2 = & \left(a_1 \sigma + c_1 y + \frac{a_1 (d_1^2 - c_1^2) - 2b_1 c_1 d_1}{a_1^2 + b_1^2} \int \frac{\alpha(t)}{\beta(t)} dt \right)^2 \\ & + \left(b_1 \sigma + d_1 y + \frac{b_1 (c_1^2 - d_1^2) - 2a_1 c_1 d_1}{a_1^2 + b_1^2} \int \frac{\alpha(t)}{\beta(t)} dt \right)^2 \\ & + \frac{-3\lambda (a_1^2 + b_1^2)^3}{\alpha(t) (b_1 c_1 - a_1 d_1)^2} \\ & \stackrel{\text{def}}{=} \varphi^2 + \phi^2 + \Omega. \end{aligned}$$

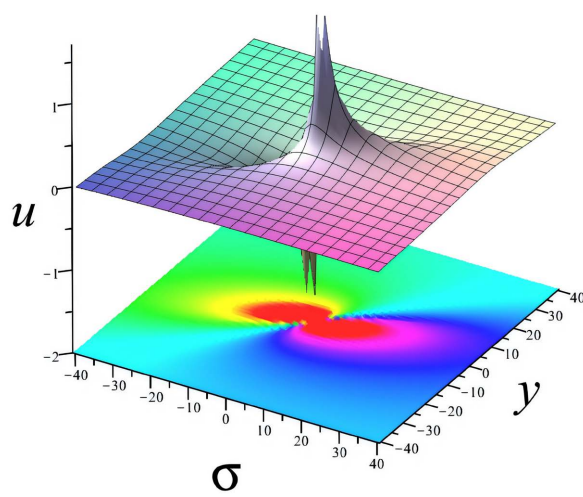
Then, substitute the above formula into transformation (4), we can the following solution.

$$u(x, y, t) = \frac{4(a_1\varphi + b_1\phi)}{\varphi^2 + \phi^2 + \Omega}. \quad (18)$$

Similar to the calculation procedure for $N=2$, when $N=4, 6, \dots, M$, we can get M-lump solutions. In **Figures 4-7**, we can obtain different diagrams of 1-lump solution with different $\alpha(t)$ and $\beta(t)$ by selecting appropriate parameters. We take $\lambda = -1$, $a_1 = d_1 = -\frac{1}{2}$, $b_1 = \frac{5}{4}$, $c_1 = 2$ in (18), then we get the 1-lump solutions for constant coefficient (3 + 1)-YTSF equation in **Figure 4**. Although lump solutions for constant coefficient equations have been widely studied [25]-[27], **Figure 4** is given as a special case. Of course, by taking the following three different parameters, we can obtain some diagrams of 1-lump solutions with different variable coefficients in **Figures 5-7**. In **Figure 5**, the parameters are $\lambda = -1$, $a_1 = -\frac{1}{2}$, $b_1 = \frac{5}{4}$, $c_1 = \frac{5}{2}$, $d_1 = -1$. The parameters in **Figure 6** are $\lambda = -1$, $a_1 = -\frac{3}{4}$, $b_1 = \frac{3}{5}$, $c_1 = 2$, $d_1 = 1$. It can be seen from **Figure 6** that



(a)



(b)

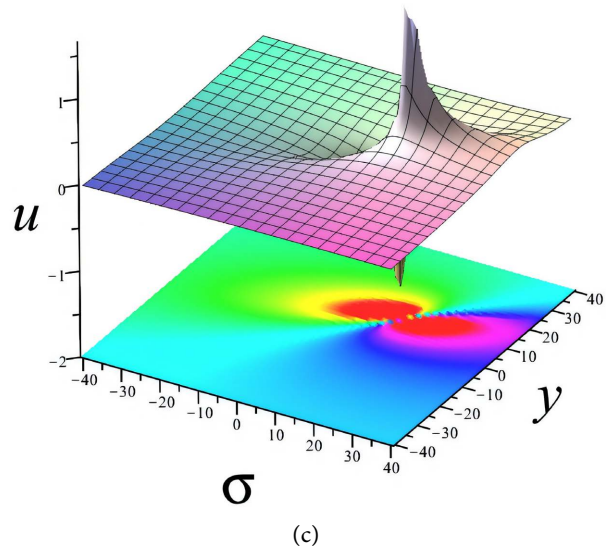
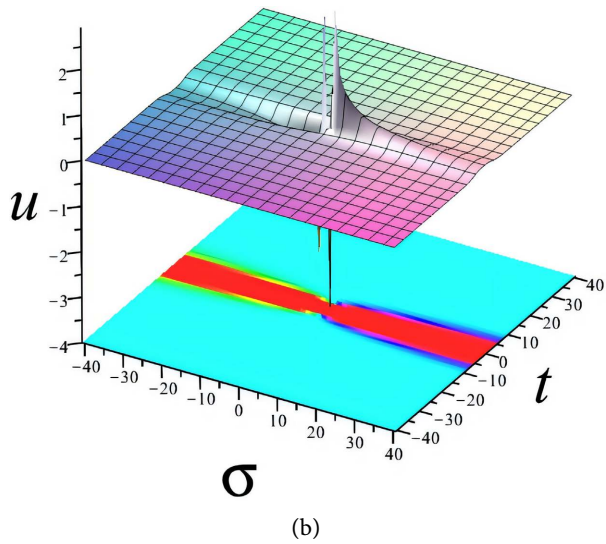
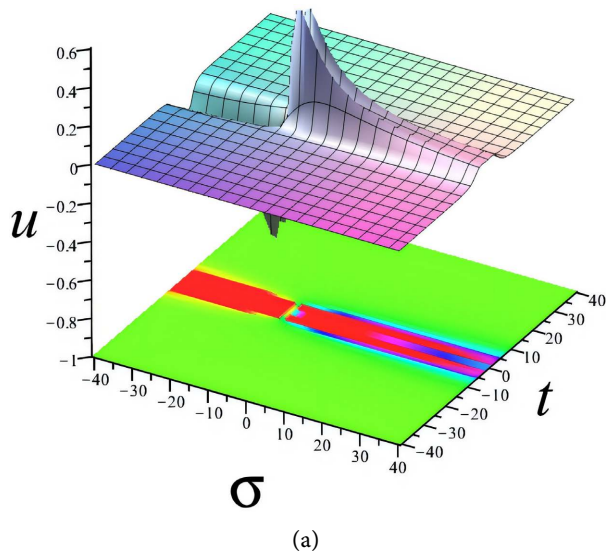


Figure 4. Plots of the 1-lump solution (18) with (8) at: (a) $t = -8$; (b) $t = 0$; (c) $t = 8$.



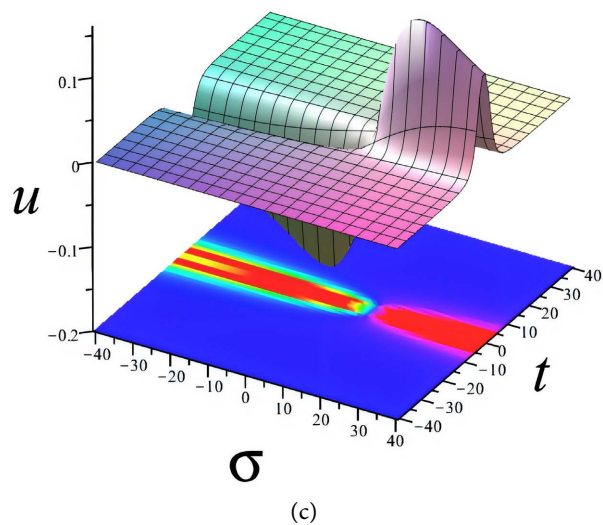
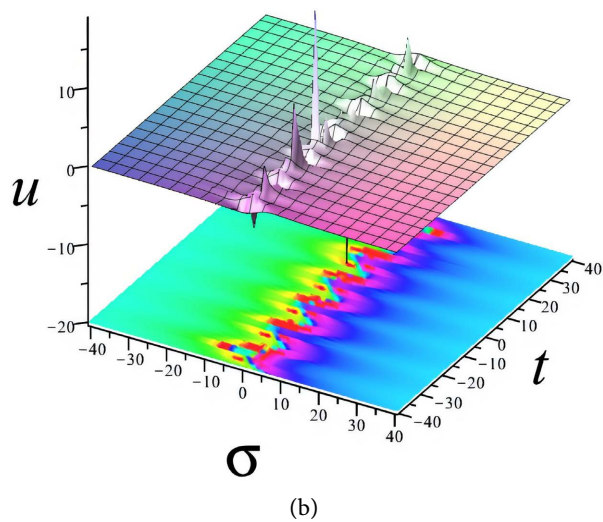
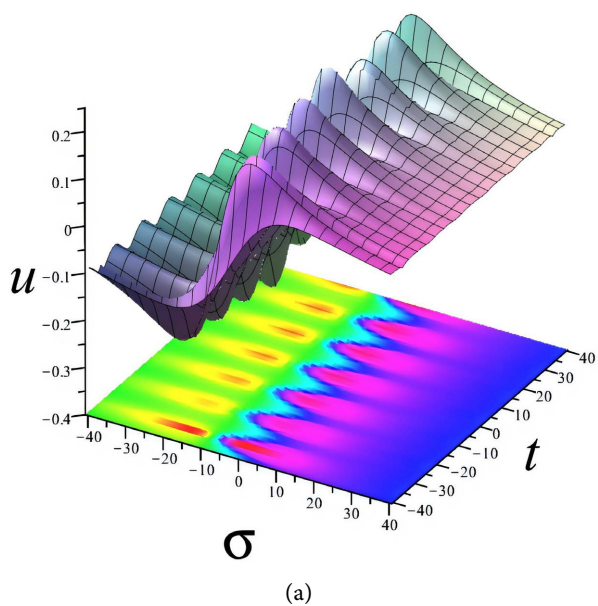


Figure 5. Plots of the 1-lump solution (18) with (9) at: (a) $y = -8$; (b) $y = 0$; (c) $y = 8$.



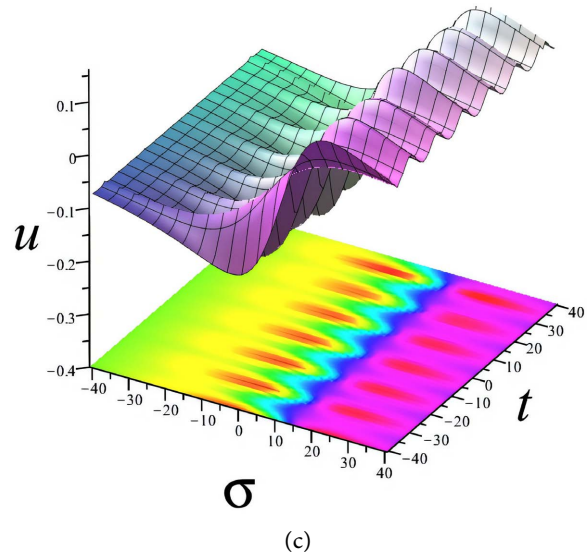
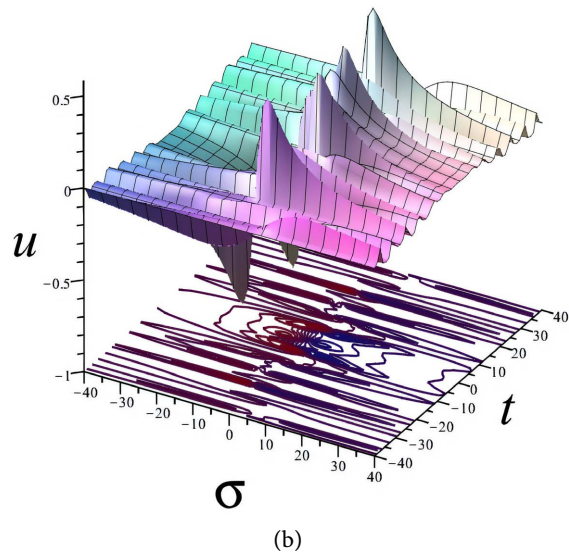
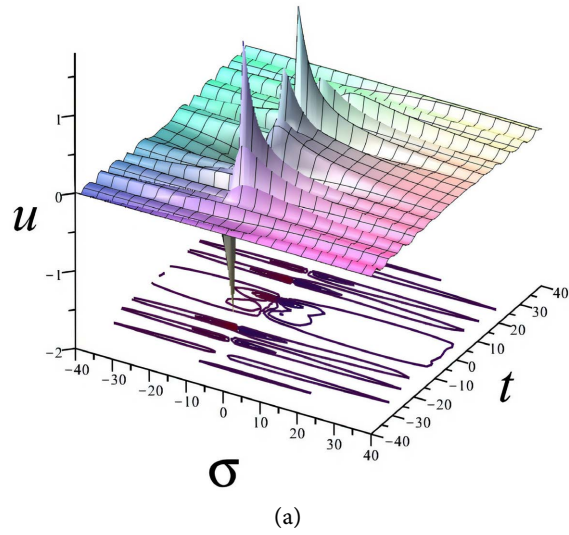


Figure 6. Plots of the 1-lump solution (18) with (10) at: (a) $y = -8$; (b) $y = 0$; (c) $y = 8$.



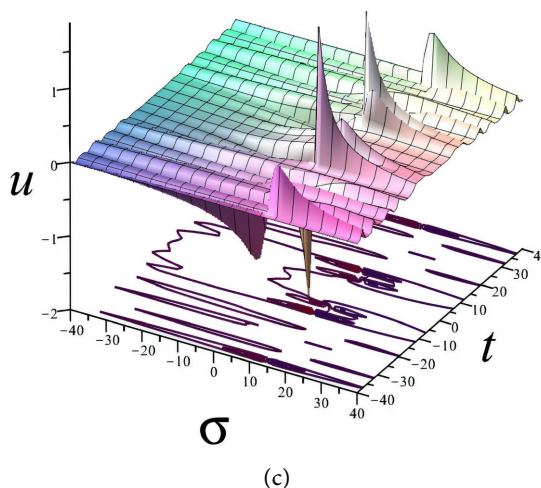


Figure 7. Plots of the 1-lump solution (18) with (11) at: (a) $y = -8$; (b) $y = 0$; (c) $y = 8$.

when $y = 0$, the diagrams of 1-lump solution change dramatically, which is worth focusing on in our future research work. When the parameters are $\lambda = -1$, $a_1 = -\frac{3}{4}$, $b_1 = \frac{3}{5}$, $c_1 = \frac{3}{2}$, $d_1 = \frac{1}{2}$, we can get diagrams of 1-lump solution in **Figure 7**.

3. The H -Order Hybrid Solution

In this section, we study the hybrid solution between lump solution and solitons in variable coefficient (3 + 1)-dimensional YTSF Equation (2) by using the long-wave limit method and parametric complex. We will mainly study the hybrid solution between the 1-lump solution and 1-soliton, the hybrid solution between the 1-lump solution and 2-soliton, and give a large number of three-dimensional diagrams to show this behavior.

3.1. A hybrid Solution between 1-Lump Solution and 1-Soliton

To obtain the hybrid solution between 1-lump solution and 1-soliton, first, we take $N = 4$, $\gamma_1 = \gamma_2 = I\pi$ in (5) and let k_1, k_2 tend to 0 to get \tilde{f}_3 as follows.

$$\tilde{f}_3 = \theta_1 \theta_2 + B_{12} + (C_{13} C_{23} + C_{13} \theta_2 + C_{23} \theta_1 + \theta_1 \theta_2 + B_{12}) e^{\eta_3}, \quad (19)$$

where

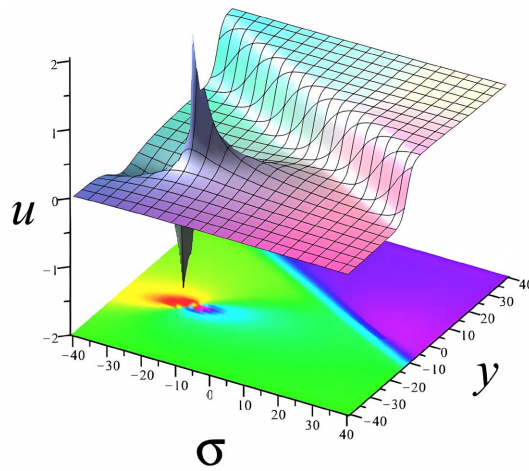
$$\begin{cases} \theta_i = \theta_i = m_i \sigma + p_i y - \int \frac{\alpha(t) p_i^2}{\beta(t) m_i} dt, \\ \eta_3 = k_3 \left(m_3 \sigma + p_3 y - \int \frac{\lambda k_3^2 m_3^4 + \alpha(t) p_3^2}{\beta(t) m_3} dt \right) + \gamma_3, \\ B_{12} = \frac{12 \lambda m_1^3 m_2^3}{\alpha(t) (m_1 p_2 - m_2 p_1)^2}, \\ c_{ij} = \frac{12 \lambda k_j m_i^3 m_j^3}{-3 \lambda k_j^2 m_i^2 m_j^4 + \alpha(t) (m_i p_j - m_j p_i)^2} (i = 1, 2; j = 3). \end{cases} \quad (20)$$

Then take $m_2 = m_1^* = a_1 - b_1 I$, $q_2 = q_1^* = c_1 - d_1 I$ in (19) and (20), and substitute it into (4), the final formula is the hybrid solution between 1-lump solution and 1-soliton (see **Figures 8-11**). By taking different and appropriate parameters, we give four different sets of diagrams for hybrid solution between 1-lump solution and 1-soliton, including one set of constant coefficients and three sets of variable coefficients. The values of the parameters in **Figure 8** are $\lambda = -1$, $a_1 = d_1 = -\frac{1}{2}$, $b_1 = \frac{5}{4}$, $c_1 = \frac{5}{2}$, $k_1 = \frac{1}{2}$, $m_1 = \frac{4}{5}$, $p_1 = \frac{3}{2}$, $\gamma_3 = 0$.

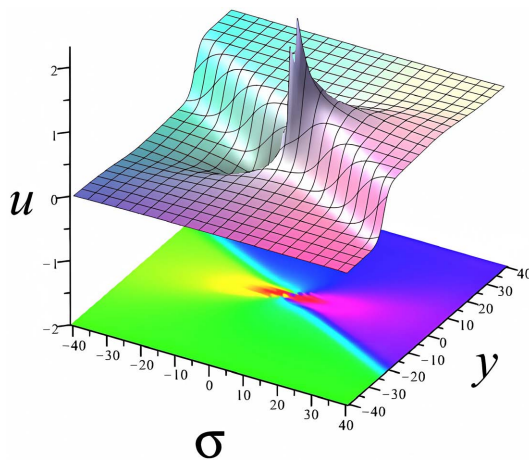
When we take $\lambda = -1$, $a_1 = -\frac{1}{2}$, $b_1 = \frac{5}{4}$, $c_1 = \frac{5}{2}$, $d_1 = -1$, $k_1 = \frac{1}{2}$, $m_1 = \frac{4}{5}$, $p_1 = -\frac{1}{4}$, $\gamma_3 = 0$, we can get the hybrid solution between 1-lump solution and 1-soliton in **Figure 9**. We give the diagrams of the hybrid solution between 1-lump solution and 1-soliton in **Figure 10** and **Figure 11**, with different parameters as **Figure 10**:

$(\lambda, a_1, b_1, c_1, d_1, k_1, m_1, p_1, \gamma_1) = \left(-1, -\frac{3}{4}, \frac{3}{5}, 2, 1, \frac{3}{2}, \frac{1}{2}, \frac{1}{2}, 0\right)$, **Figure 11**:

$(\lambda, a_1, b_1, c_1, d_1, k_1, m_1, p_1, \gamma_1) = \left(-1, -\frac{3}{4}, \frac{3}{5}, \frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \frac{1}{4}, \frac{1}{2}, 0\right)$



(a)



(b)

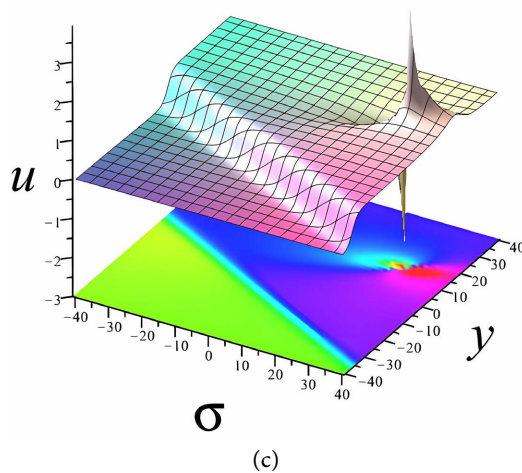
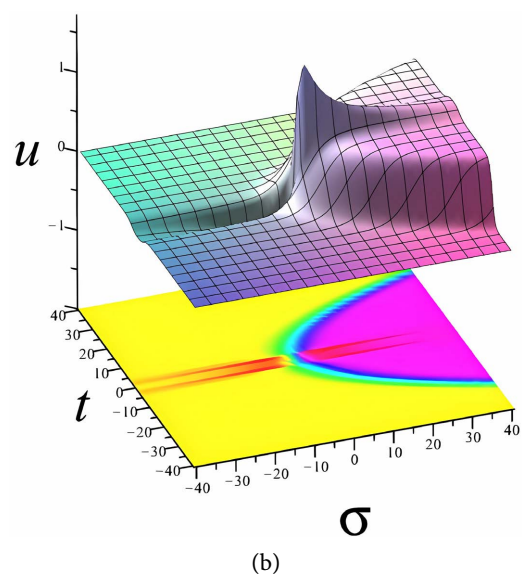
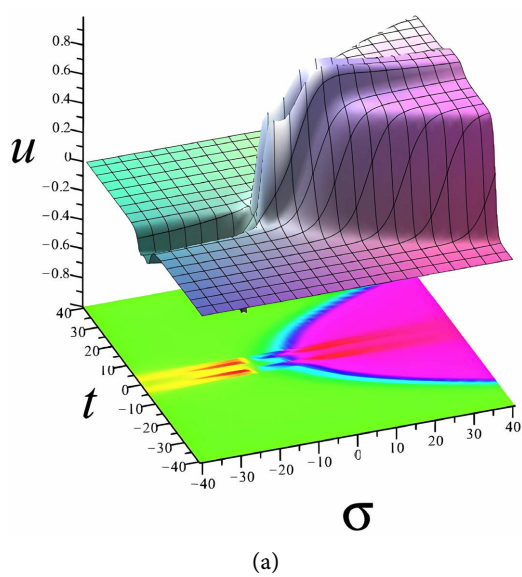


Figure 8. Plots of the hybrid solution between 1-lump solution and 1-soliton with (8) at different times: (a) $t = -8$, (b) $t = 0$, (c) $t = 10$.



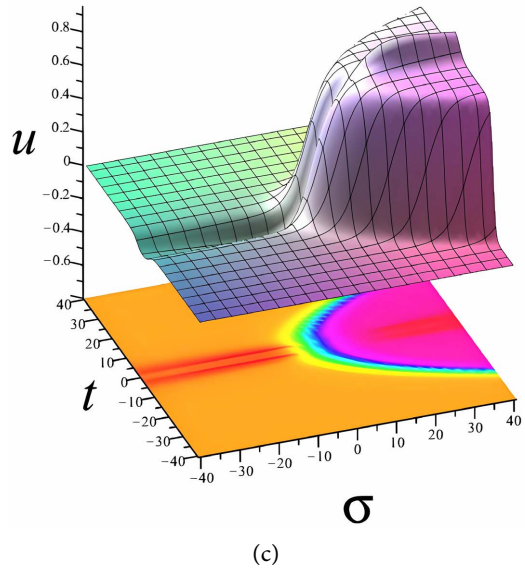
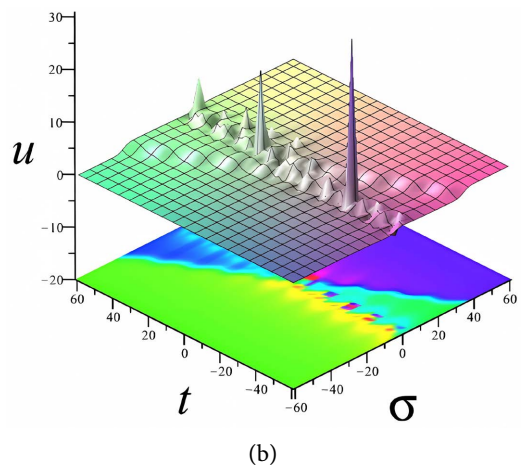
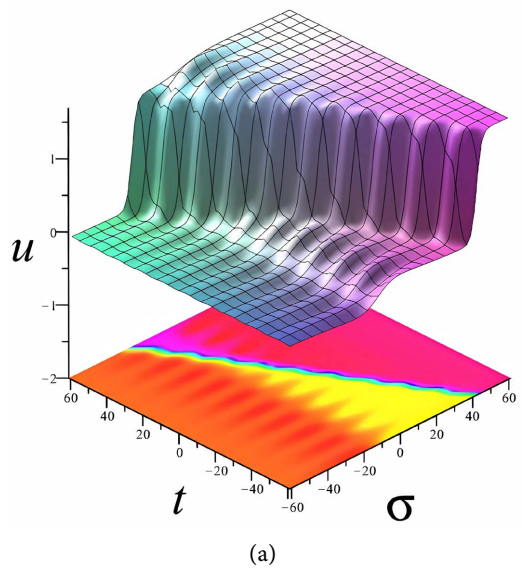


Figure 9. Plots of the hybrid solution between 1-lump solution and 1-soliton with (9) at different values of y : (a) $y = -8$, (b) $y = 0$, (c) $y = 10$.



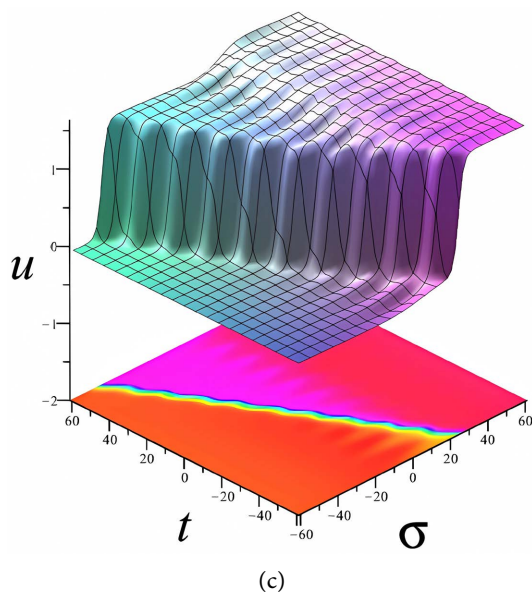
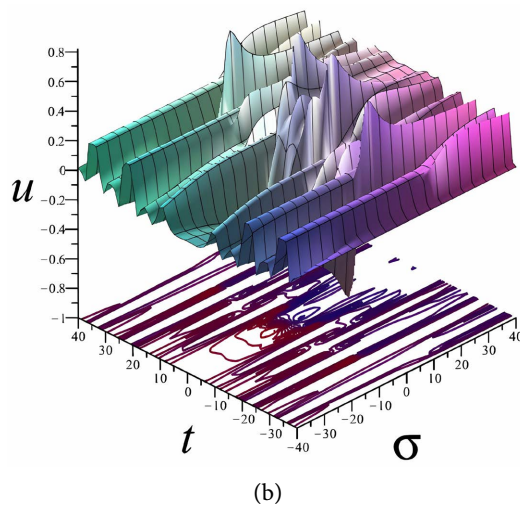
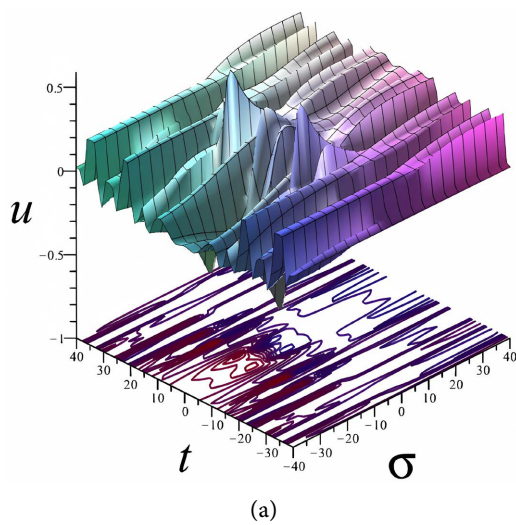


Figure 10. Plots of the hybrid solution between 1-lump solution and 1-soliton with (10) at different values of γ : (a) $\gamma = -8$, (b) $\gamma = 0$, (c) $\gamma = 10$.



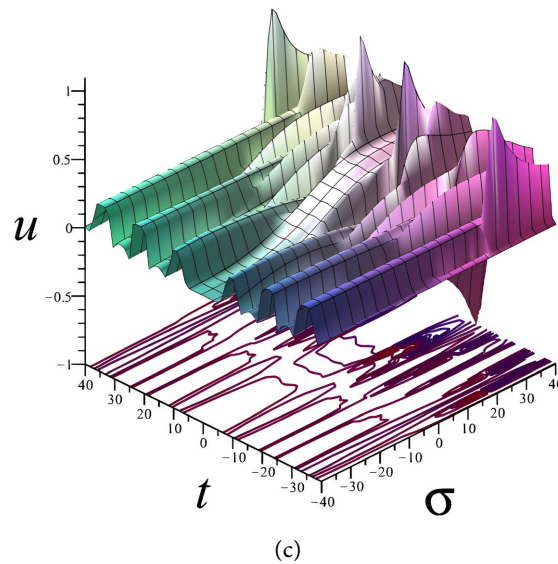


Figure 11. Plots of the hybrid solution between 1-solution and 1-soliton with (11) at different values of γ : (a) $\gamma = -8$, (b) $\gamma = 0$, (c) $\gamma = 10$.

3.2. A Hybrid Solution between the 1-Lump and 2-Soliton

Similar to the method of finding the hybrid solution between the 1-lump solution and 1-soliton, first, we take $N = 4$, $\gamma_1 = \gamma_2 = I\pi$ in (5) and let $k_1 \rightarrow 0, k_2 \rightarrow 0$ to get

$$\begin{aligned} \tilde{f}_4 = & \theta_1\theta_2 + B_{12} + (C_{13}C_{23} + C_{13}\theta_2 + C_{23}\theta_1 + \theta_1\theta_2 + B_{12})e^{\eta_3} \\ & + (C_{14}C_{24} + C_{14}\theta_2 + C_{24}\theta_1 + \theta_1\theta_2 + B_{12})e^{\eta_4} \\ & + (\theta_1\theta_2 + (C_{23} + C_{24})\theta_1 + (C_{13} + C_{14})\theta_2 + B_{12} \\ & + C_{13}C_{23} + C_{14}C_{23} + C_{13}C_{24} + C_{14}C_{24})e^{\eta_3+\eta_4} \exp(A_{34}), \end{aligned} \tag{21}$$

where $\eta_3, B_{12}, \exp(A_{34})$ have been mentioned previously, and

$$\begin{cases} \eta_4 = k_4 \left(m_4\sigma + p_4\gamma - \int \frac{\lambda k_4^2 m_4^4 + \alpha(t) p_4^2}{\beta(t) m_4} dt \right) + \gamma_4, \\ m_2 = m_1^* = a_1 - b_1 I, q_2 = q_1^* = c_1 - d_1 I, \\ c_{ij} = \frac{12\lambda k_j m_i^3 m_j^3}{-3\lambda k_j^2 m_i^2 m_j^4 + \alpha(t)(m_i p_j - m_j p_i)^2} (i=1, 2; j=3, 4). \end{cases} \tag{22}$$

Then, substituting (21) and (22) into (4), we get the hybrid solution between 1-lump solution and 2-soliton. We give the diagrams of the hybrid solution between 1-lump solution and 2-soliton in **Figures 12-15**, with different parameters as **Figure 12**:

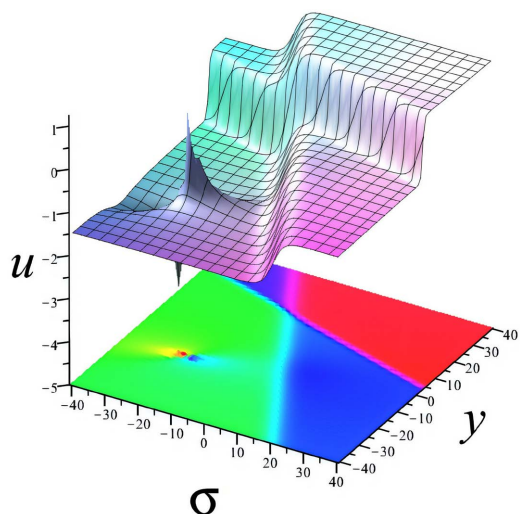
$$(\lambda, a_1, b_1, c_1, d_1, k_3, k_4, m_3, m_4, p_3, p_4, \gamma_3, \gamma_4) = \left(-1, -\frac{1}{2}, \frac{5}{4}, \frac{5}{2}, -\frac{1}{2}, -\frac{3}{4}, \frac{3}{2}, 1, \frac{2}{5}, 3, \frac{1}{5}, 0, 0 \right),$$

Figure 13:

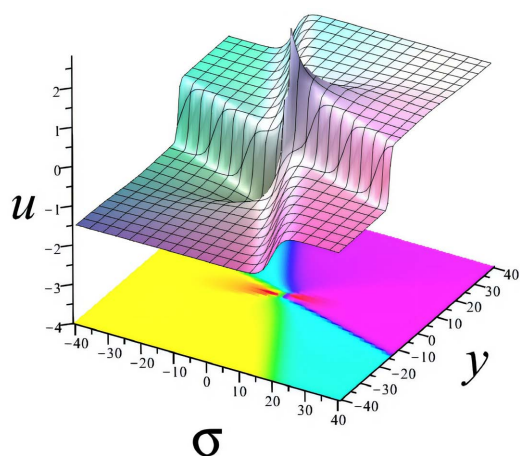
$$(\lambda, a_1, b_1, c_1, d_1, k_3, k_4, m_3, m_4, p_3, p_4, \gamma_3, \gamma_4) = \left(-1, -\frac{1}{2}, \frac{5}{4}, \frac{5}{2}, -1, \frac{1}{4}, \frac{1}{2}, \frac{3}{5}, \frac{3}{2}, \frac{1}{4}, 0, 0 \right),$$

Figure 14, Figure 15:

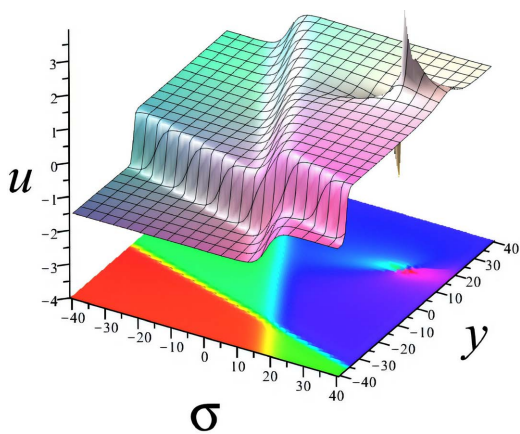
$$(\lambda, a_1, b_1, c_1, d_1, k_3, k_4, m_3, m_4, p_3, p_4, \gamma_3, \gamma_4) = \left(-1, -\frac{3}{4}, \frac{3}{5}, \frac{3}{2}, \frac{3}{4}, \frac{1}{2}, \frac{3}{2}, 3, \frac{2}{5}, \frac{5}{2}, \frac{1}{3}, 0, 0\right).$$



(a)

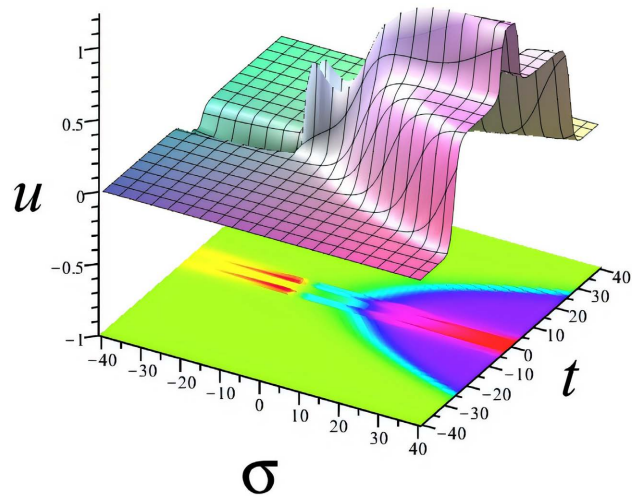


(b)

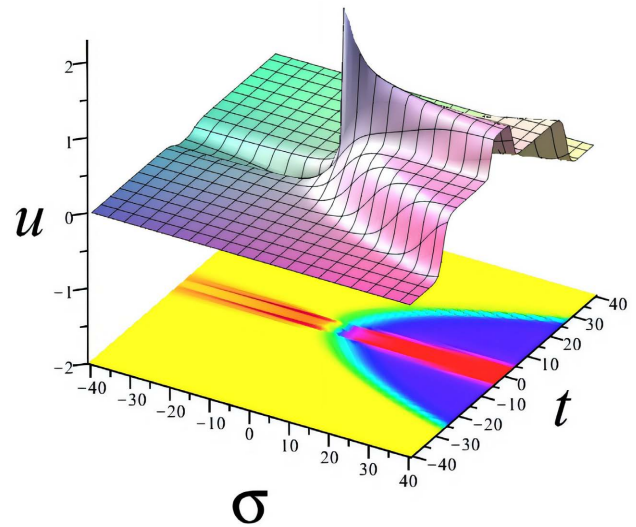


(c)

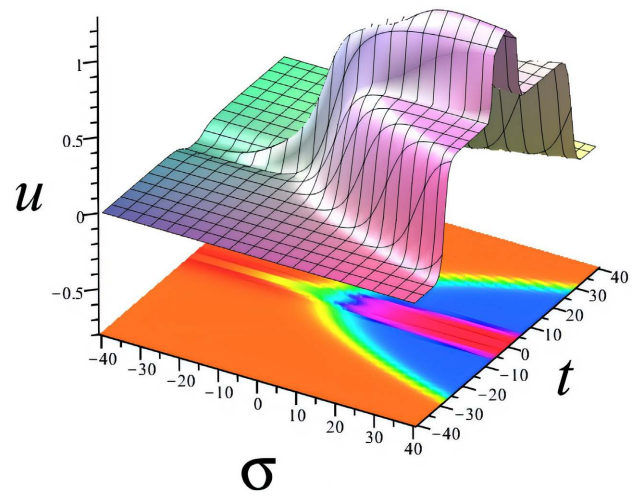
Figure 12. Plots of the hybrid solution between 1-lump solution and 2-soliton with (8) at different times: (a) $t = -8$, (b) $t = 0$, (c) $t = 10$.



(a)

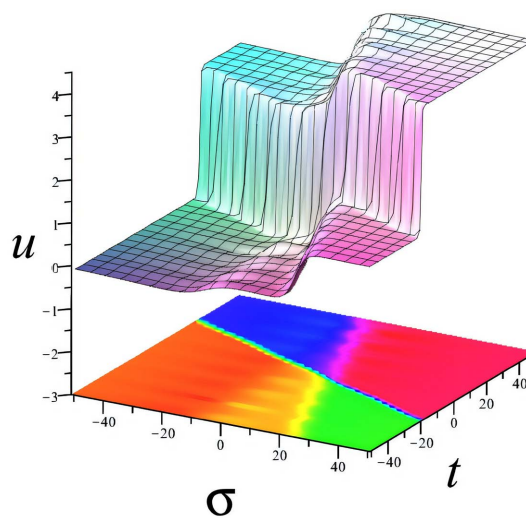


(b)

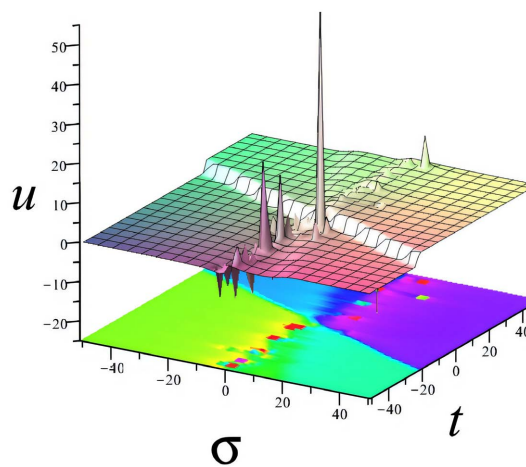


(c)

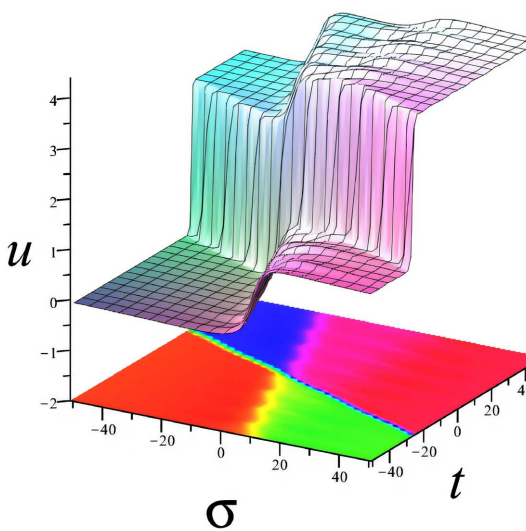
Figure 13. Plots of the hybrid solution between 1-lump solution and 2-soliton with (9) at different values of y : (a) $y = -8$, (b) $y = 0$, (c) $y = 10$.



(a)

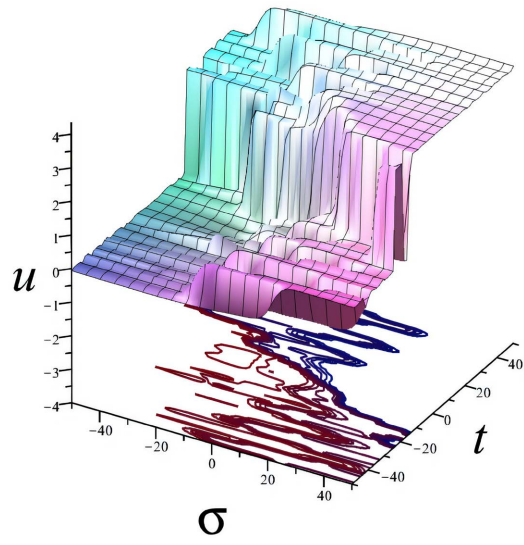


(b)

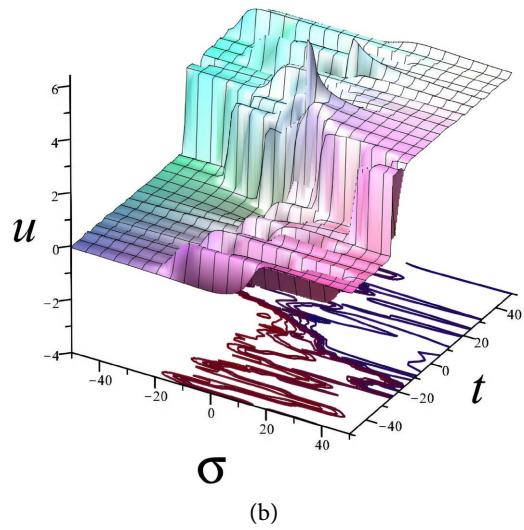


(c)

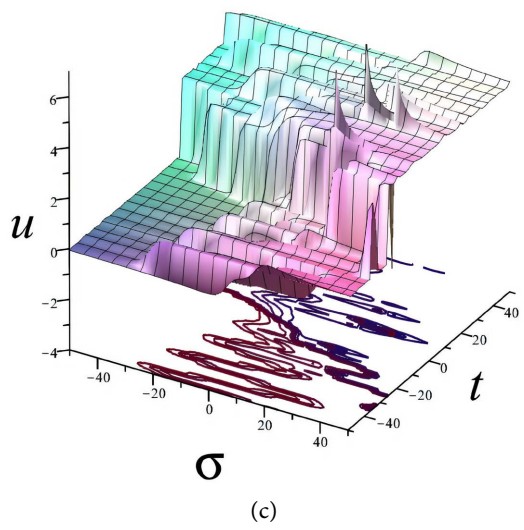
Figure 14. Plots of the hybrid solution between 1-lump solution and 2-soliton with (10) at different values of y : (a) $y = -8$, (b) $y = 0$, (c) $y = 10$.



(a)



(b)



(c)

Figure 15. Plots of the hybrid solution between 1-lump solution and 2-soliton with (11) at different values of y : (a) $y = -8$, (b) $y = 0$, (c) $y = 10$.

4. Conclusion

In this paper, the N -soliton solution, lump solution and hybrid solution of the $(3 + 1)$ -dimensional variable-coefficient Yu-Toda-Sasa-Fukuyama (vc-YTSE) equation are studied. Firstly, Hirota's direct method is used to obtain the N -soliton solution, then the N -order rational solution is obtained from N -soliton solution by the long-wave limit method, and lump solutions are obtained by parametric complex. Finally, we also discuss the hybrid solution of lump solutions and solitons, and show the process of obtaining the hybrid solution between 1-lump and 1-soliton, 1-lump and 2-soliton. In addition, by providing four different sets of coefficient functions, four different sets of three-dimensional graphs of solitons, lump and hybrid solutions are drawn.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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