

# Qualitative Properties of Equilibrium Point in a Discrete Predator-Prey Model

Weijie Lin, Minhong Yang, Chenhao Sun, Kaiwen Guan, Xiaoliang Zhou\*

School of Mathematics and Statistics, Lingnan Normal University, Zhanjiang, China

Email: zxlmath@163.com, \*710149570@qq.com

**How to cite this paper:** Lin, W.J., Yang, M.H., Sun, C.H., Guan, K.W. and Zhou, X.L. (2024) Qualitative Properties of Equilibrium Point in a Discrete Predator-Prey Model. *Journal of Applied Mathematics and Physics*, 12, 2920-2927.

<https://doi.org/10.4236/jamp.2024.128175>

**Received:** July 2, 2024

**Accepted:** August 23, 2024

**Published:** August 26, 2024

Copyright © 2024 by author(s) and Scientific Research Publishing Inc. This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

## Abstract

In this paper, the dynamic properties of a discrete predator-prey model are discussed. The properties of non-hyperbolic fixed points and hyperbolic fixed points of the model are analyzed. First, by using the classic Shengjin formula, we find the existence conditions for fixed points of the model. Then, by using the qualitative theory of ordinary differential equations and matrix theory we indicate which points are hyperbolic and which are non-hyperbolic and the associated conditions.

## Keywords

Discrete Predator-Prey Model, Non-Hyperbolic Fixed Point, Node, Saddle Point

## 1. Introduction

Predator-prey interaction is the most fundamental and important process in population dynamics. For populations with overlapping generations, the birth process occurs continuously, so predator-prey interactions are often modeled by ordinary differential equations. Very rich and complex dynamics and bifurcation have been observed in continuous time predator-prey systems. Many other species, such as monophyletic plants and semi-living animals, have discrete non-overlapping generations, and their births occur during the regular breeding season (see [1]). Their interaction is described or expressed by the difference equation as a discrete-time map. Discrete time predator-prey models can exhibit more complex dynamics than corresponding continuous time models (see [2]).

To describe predator-prey mite outbreak interactions on fruit trees in Washington, Wollkind *et al.* [3] investigated the following temperature-dependent model system for predator-prey mite outbreak interactions on fruit tree

$$\begin{cases} \frac{dx}{dt} = x(t)(a_1 - b_1x(t)) - p(x(t))y(t), \\ \frac{dy}{dt} = y(t)\left(a_2 - b_2\frac{y(t)}{x(t)}\right) \end{cases} \quad (1)$$

where,  $x(t)$  and  $y(t)$  are the density functions of prey and predator at time  $t$  respectively. The predator equation in model (1) was first proposed by Leslie [4], based on the assumption that the carrying capacity of predators in the environment is proportional to the number of prey. Obviously, the growth of predators is still Logistic, with an intrinsic growth rate  $a_2$ , the carrying capacity  $(a_2x)/b_2$  and the reduction measure of the number of predators  $(b_2y)/(a_2x)$ .

In model (1), the  $p(x)$  is called a response function which reflects the consumption rate of prey per predator. When  $p(x)$  is Holling type I (i.e.  $p(x) = ax$ ), Holling type II (i.e.  $p(x) = ax/(b+x)$ ) and Holling type III (i.e.  $p(x) = ax^2/(b+x^2)$ ), the consumption rate of prey per predator is monotone. For the dynamic properties of the model in these three types, we can refer to references [5]-[13]. However, some experiments and observations suggest that when nutrient food density reaches high levels, there may be an inhibitory effect on the growth rate of prey, that is, a non-monotonic response will occur. To model such an inhibitory effect, Andrews [14] suggested a function

$$p(x) = \frac{ax}{b + cx + x^2} \quad (2)$$

which is called a Holling type IV function. Later, Sokol and Howell [15] proposed a simplified Holling type IV function of the form

$$p(x) = \frac{ax}{b + x^2}. \quad (3)$$

For the dynamic properties of the model with nonlinear response functions (2) and (3), we can refer to references [14]-[19].

In this paper, inspired by articles [2] [14]-[19], we consider the discrete time type of model (1) with Holling type IV function of (3) as

$$\begin{cases} x_{n+1} = x_n + rx_n(1 - x_n) - \frac{ax_n y_n}{1 + bx_n^2}, \\ y_{n+1} = y_n + sy_n\left(1 - \frac{x_n}{hy_n}\right), \end{cases} \quad (4)$$

where  $a, b, r, s$  and  $h$  are positive numbers. We first solve for all the fixed points of model (4) by using the classic Shengjin formula, obtain the existence of fixed point and gave the conditions for the existence of one, two, and three positive fixed points. Then, we investigated the property of hyperbolicity of zero fixed point. Then discuss the dynamic properties of non-hyperbolic fixed points and hyperbolic fixed points of the model (4) by using the qualitative theory of ordinary differential equations and matrix theory.

## 2. The Fixed Points of Model (4)

The mapping  $F : R^2 \mapsto R^2$  corresponding to model (4)

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + rx(1-x) - \frac{axy}{1+bx^2} \\ y + sy\left(1 - \frac{x}{hy}\right) \end{pmatrix}.$$

To find fixed point of the mapping  $F$ , we set

$$x + rx(1-x) - \frac{axy}{1+bx^2} = x, \quad y + sy\left(1 - \frac{x}{hy}\right) = y. \tag{5}$$

It is easy to see that there is a trivial fixed point  $E_0(0,0)$ .

**Lemma 1.** (Shengjin formula) Consider a cubic equation

$$ax^3 + bx^2 + cx + d = 0 \tag{6}$$

and let

$$A = b^2 - 3ac, \quad B = bc - 9ad, \quad C = c^2 - 3bd, \quad \Delta = B^2 - 4AC.$$

Then there are the following conclusions:

1) When  $A = B = 0$ , the Equation (6) has a triple real root

$$x_1 = x_2 = x_3 = \frac{-b}{3a} = \frac{-c}{b} = \frac{-3d}{c}.$$

2) When  $\Delta = B^2 - 4AC > 0$ , Equation (6) has a real root and a pair of conjugate imaginary roots

$$x_1 = \frac{-b - \sqrt[3]{Y_1} - \sqrt[3]{Y_2}}{3a}, \quad x_{2,3} = \frac{-2b + \sqrt[3]{Y_1} + \sqrt[3]{Y_2} \pm \sqrt{3}(\sqrt[3]{Y_1} - \sqrt[3]{Y_2})i}{6a}$$

where

$$Y_{1,2} = Ab + 3a \left( \frac{-B \pm \sqrt{B^2 - 4AC}}{2} \right), \quad i^2 = -1.$$

3) When  $\Delta = B^2 - 4AC = 0$ , Equation (6) has three real roots, two of which are double roots

$$x_1 = -\frac{b}{a} + K, \quad x_2 = x_3 = -\frac{1}{2}K$$

where  $K = \frac{B}{A}, (A \neq 0)$ .

4) When  $\Delta = B^2 - 4AC < 0$ , Equation (6) has three real roots which are not equal to each other

$$x_1 = \frac{-b - 2\sqrt{A} \cos \frac{\theta}{3}}{3a}, \quad x_{2,3} = \frac{-b + \sqrt{A} \left( \cos \frac{\theta}{3} \pm \sqrt{3} \sin \frac{\theta}{3} \right)}{3a}$$

where

$$\theta = \arccos T, \quad T = \frac{2Ab + 3Ba}{2\sqrt{A^3}}, \quad (A > 0, -1 < T < 1).$$

**Theorem 1.** Let

$$A = (-hrb(hr + a) + 9h^2r^2b)^2 - 3hrb(hr + a),$$

$$B = -hrb(hr + a) + 9h^2r^2b,$$

$$C = (hr + a)^2 - 3h^2r^2b,$$

and

$$\Delta = B^2 - 4AC.$$

Then we have nontrivial fixed points of model (4) as follows:

1) When  $A = B = 0$ , model (4) has a fixed point  $E_1\left(\frac{1}{3}, \frac{1}{3h}\right)$ .

2) When  $\Delta = B^2 - 4AC > 0$ , model (4) has a real fixed point

$$E_2\left(\frac{hrb - \sqrt[3]{Y_1} - \sqrt[3]{Y_2}}{3hrb}, \frac{hrb - \sqrt[3]{Y_1} - \sqrt[3]{Y_2}}{3h^2rb}\right).$$

where

$$Y_{1,2} = -Ahrb + 3hrb\left(\frac{-B \pm \sqrt{B^2 - 4AC}}{2}\right).$$

3) When  $\Delta = B^2 - 4AC = 0$ , model (4) has two fixed points

$$E_3\left(1 + \frac{B}{A}, \frac{1}{h} + \frac{B}{hA}\right), E_4\left(-\frac{1}{2} \frac{B}{A}, -\frac{1}{2} \frac{B}{hA}\right).$$

4) When  $\Delta = B^2 - 4AC < 0$ , model (4) has three fixed points

$$E_5\left(\frac{hrb - 2\sqrt{A} \cos \frac{\theta}{3}}{3hrb}, \frac{hrb - 2\sqrt{A} \cos \frac{\theta}{3}}{3h^2rb}\right),$$

$$E_6\left(\frac{hrb + \sqrt{A}\left(\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3}\right)}{3hrb}, \frac{hrb + \sqrt{A}\left(\cos \frac{\theta}{3} + \sqrt{3} \sin \frac{\theta}{3}\right)}{3h^2rb}\right),$$

$$E_7\left(\frac{hrb + \sqrt{A}\left(\cos \frac{\theta}{3} - \sqrt{3} \sin \frac{\theta}{3}\right)}{3hrb}, \frac{hrb + \sqrt{A}\left(\cos \frac{\theta}{3} - \sqrt{3} \sin \frac{\theta}{3}\right)}{3h^2rb}\right).$$

where

$$\theta = \arccos T, T = \frac{hrb(2A + 3B)}{2\sqrt{A^3}} (A > 0, -1 < T < 1).$$

**Proof.** When  $xy \neq 0$ , we can obtain the following equation from (5)

$$hrbx^3 - hrbx^2 + (hr + a)x - hr = 0.$$

By using Lemma 1 we can easily get the conclusions of the theorem. The detail of

the proof will be omitted.

### 3. The Properties of Fixed Point $E_0(0,0)$

Define the following notations

$$\begin{aligned}\delta_1 &= \{(r,s) | r=0\}, \quad \{\delta_2 = (r,s) | r=-2\}, \\ \delta_3 &= \{(r,s) | s=0\}, \quad \{\delta_4 = (r,s) | s=-2\}, \\ \alpha_1 &= \{(r,s) | r < -2, s < -2\}, \quad \alpha_2 = \{(r,s) | r < -2, s > 0\}, \\ \alpha_3 &= \{(r,s) | r > 0, s < -2\}, \quad \alpha_4 = \{(r,s) | r > 0, s > 0\}, \\ \beta_1 &= (r,s) | -2 < r < 0, -2 < s < 0\}, \\ \gamma_1 &= \{(r,s) | -2 < r < 0, s < -2\}, \quad \gamma_2 = \{(r,s) | -2 < r < 0, s > 0\}, \\ \gamma_3 &= \{(r,s) | r < -2, -2 < s < 0\}, \quad \gamma_4 = \{(r,s) | r > 0, -2 < s < 0\}.\end{aligned}$$

**Theorem 2.** Fixed point  $E_0$  has the following properties

- 1) when  $(r,s) \in \delta_{1,2,3,4}$ ,  $E_0$  is a non-hyperbolic fixed point;
- 2) When  $(r,s) \in \alpha_{1,2,3,4}$ ,  $E_0$  is an unstable node;
- 3) When  $(r,s) \in \beta_1$ ,  $E_0$  is a stable node;
- 4) when  $(r,s) \in \gamma_{1,2,3,4}$ ,  $E_0$  is a saddle point.

**Proof.** The Jacobian matrix of the mapping  $F$  at fixed point  $E_0$  is

$$J(E_0) = \begin{pmatrix} 1+r & 0 \\ -\frac{s}{h} & 1+s \end{pmatrix}.$$

We easily know that it has two characteristic roots

$$\lambda_1 = 1+r, \quad \lambda_2 = 1+s.$$

Therefore

- 1) when  $(r,s) \in \delta_{1,2,3,4}$ , we get  $|\lambda_1|=1$  or  $|\lambda_2|=1$ . Thus fixed point  $E_0$  is a non-hyperbolic fixed point.
- 2) when  $(r,s) \in \alpha_{1,2,3,4}$ , we get  $|\lambda_1|>1$  and  $|\lambda_2|>1$ . Thus  $E_0$  is an unstable node.
- 3) when  $(r,s) \in \beta_1$ , we get  $|\lambda_1|<1$  and  $|\lambda_2|<1$ . Thus  $E_0$  is a stable node.
- 4) when  $(r,s) \in \gamma_{1,2,3,4}$ ,  $|\lambda_1|<1$  and  $|\lambda_2|>1$  or  $|\lambda_1|>1$  and  $|\lambda_2|<1$ . Thus  $E_0$  is a saddle point.

### 4. The Properties of Nontrivial Fixed Points $E_i$

The Jacobian matrix mapped at any fixed point  $E_i(x,y)$  is

$$J(E_i) = \begin{pmatrix} 1+r-2rx - \frac{ay}{1+bx^2} + \frac{2abx^2y}{(1+bx^2)^2} & -\frac{ax}{1+bx^2} \\ -\frac{s}{h} & 1+s \end{pmatrix}.$$

Set

$$T_i := \text{tr}(J(E_i)) = 2 + r + s - 2rx - \frac{ay}{1 + bx^2} + \frac{2abx^2y}{(1 + bx^2)^2},$$

$$D_i := \det(J(E_i)) = \left(1 + r - 2rx - \frac{ay}{1 + bx^2} + \frac{2abx^2y}{(1 + bx^2)^2}\right)(1 + s) - \frac{asx}{h + bhx^2}.$$

**Lemma 2.** Let  $F(\lambda) = \lambda^2 + T_i\lambda + D_i$ ,  $\Delta_i = T_i^2 - 4D_i$  and  $\lambda_{1,2}$  be two roots of equation  $F(\lambda) = 0$ , we have

- 1) if  $\Delta_i \geq 0$ ,  $-\frac{T}{2} < -1$ ,  $F(-1) > 0$ , then  $\lambda_1 < -1, \lambda_2 < -1$ .
- 2) if  $\Delta_i \geq 0$ ,  $-\frac{T}{2} > 1$ ,  $F(1) > 0$ , then  $\lambda_1 > 1, \lambda_2 > 1$ .
- 3) if  $F(-1) < 0$ ,  $F(1) < 0$ , then  $\lambda_1 < -1, \lambda_2 > 1$ .
- 4) if  $\Delta_i \geq 0$ ,  $-1 < -\frac{T}{2} < 1$ ,  $F(-1) > 0$ ,  $F(1) > 0$ , then  $-1 < \lambda_{1,2} < 1$ .
- 5) if  $F(-1) = 0$  or  $F(1) = 0$ , then  $|\lambda_1| = 1$  or  $|\lambda_2| = 1$ .
- 6) if  $\Delta_i < 0$ ,  $0 < D < 1$ , then conjugate complex roots  $|\lambda_1| < 1, |\lambda_2| < 1$ .
- 7) if  $\Delta_i < 0$ ,  $D = 1$ , then conjugate complex roots  $|\lambda_1| = 1, |\lambda_2| = 1$ .
- 8) if  $\Delta_i < 0$ ,  $D > 1$ , then conjugate complex roots  $|\lambda_1| > 1, |\lambda_2| > 1$ .

**Proof.** (1) The axis of symmetry of the quadratic function  $F(\lambda)$  is  $\lambda = -\frac{T}{2}$ .

Then if  $-\frac{T}{2} < -1$ , from assumption  $F(-1) > 0$  we know  $F(\lambda)$  increases monotonically and  $F(\lambda) > 0$  for  $\lambda \in (-1, +\infty)$ . Since  $\Delta_i \geq 0$ , we know equation  $F(\lambda) = 0$  has real solutions which can only fall in  $(-\infty, -1)$ , that is  $\lambda_1 < -1, \lambda_2 < -1$ . Therefore, the conclusion of (1) is proved.

By using similar methods, we can prove the other conclusions of (2)-(8) and the process of proof will be omitted.

**Theorem 3.** The fixed point  $E_i (i = 1, 2, \dots, 7)$  has the following properties

- 1) When  $\Delta_i \geq 0$ ,  $2 < T_i < 1 + D_i$ , the fixed point  $E_i$  is an unstable node.
- 2) When  $\Delta_i \geq 0$ ,  $T_i < \min(2, 1 + D_i)$  the fixed point  $E_i$  is an unstable node.
- 3) When  $T_i < -(1 + D_i)$ , the fixed point  $E_i$  is an unstable node.
- 4) When  $\Delta_i \geq 0$ ,  $\max(-2, -(1 + D_i)) < T_i$  and  $T_i < \min(2, 1 + D_i)$ , the fixed point  $E_i$  is a stable node.
- 5) When  $T_i = 1 + D_i$  or  $T_i = -(1 + D_i)$ , the fixed point  $E_i$  is a non-hyperbolic fixed point.
- 6) When  $\Delta_i < 0, 0 < D_i < 1$ , the fixed point  $E_i$  is a stable node.
- 7) When  $\Delta_i < 0, D_i = 1$ , the fixed point  $E_i$  is a non-hyperbolic fixed point.
- 8) When  $\Delta_i < 0, D_i > 1$ , the fixed point  $E_i$  is an unstable node.

**Proof.** Because the characteristic equation of Jacobian matrix corresponding to fixed point  $E_i$  is

$$F(\lambda) = \lambda^2 + T_i\lambda + D_i = 0,$$

then by Lemma 2 we easily obtain the conclusions of Theorem 3.

## 5. Conclusion and Discussion

There are many articles (e.g., [14]-[19]) on the continuous predator-prey model with Holling type IV function, but few on the discrete type. In this paper, we consider discrete models with Holling type IV function. We first analyzed the fixed points of the model. The main tool used is the classic Shengjin formula. We obtained the existence of fixed point and gave the conditions for the existence of one, two, and three positive fixed points. Then, we investigated the property of hyperbolicity of zero fixed point. The conclusion is that the zero fixed point is non-hyperbolic on the line  $r=0$ ,  $r=-2$ ,  $s=0$  or  $s=-2$  in the plane of parameters  $(r,s)$  and hyperbolic (including unstable node, stable node and saddle) on the other regions. Finally, we discussed the property of hyperbolicity of positive fixed points. The main tool used is eigenvalue method, that is, to judge the magnitude of the modulus of the eigenvalue corresponding to the fixed point.

The non-hyperbolic properties have better research significance and need more in-depth research, which is not involved in this paper. We hope that interested readers will pay attention to this aspect.

## Acknowledgements

This work has been supported by Guangdong Basic and Applied Basic Research Foundation (Grant No. 2022A1515010964, 2022A1515010193), the Key Project of Science and Technology Innovation of Guangdong College Students (Grant No. pdjh2023b0325).

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

## References

- [1] Hoyt, S.C. (1969) Integrated Chemical Control of Insects and Biological Control of Mites on Apple in Washington. *Journal of Economic Entomology*, **62**, 74-86. <https://doi.org/10.1093/jee/62.1.74>
- [2] May, R.M. (1973) *Stability and Complexity in Model Ecosystems*, Princeton University Press.
- [3] Wollkind, D., Collings, J. and Logan, J. (1988) Metastability in a Temperature-Dependent Model System for Predator-Prey Mite Outbreak Interactions on Fruit Trees. *Bulletin of Mathematical Biology*, **50**, 379-409. [https://doi.org/10.1016/s0092-8240\(88\)90005-5](https://doi.org/10.1016/s0092-8240(88)90005-5)
- [4] Leslie, P.H. (1948) Some Further Notes on the Use of Matrices in Population Mathematics. *Biometrika*, **35**, 213-245. <https://doi.org/10.1093/biomet/35.3-4.213>
- [5] Hsu, S. and Huang, T. (1995) Global Stability for a Class of Predator-Prey Systems. *SIAM Journal on Applied Mathematics*, **55**, 763-783. <https://doi.org/10.1137/s0036139993253201>
- [6] Sáez, E. and González-Olivares, E. (1999) Dynamics of a Predator-Prey Model.

- SIAM Journal on Applied Mathematics*, **59**, 1867-1878.  
<https://doi.org/10.1137/s0036139997318457>
- [7] Zhu, Y. and Wang, K. (2011) Existence and Global Attractivity of Positive Periodic Solutions for a Predator-Prey Model with Modified Leslie-Gower Holling-Type II Schemes. *Journal of Mathematical Analysis and Applications*, **384**, 400-408.  
<https://doi.org/10.1016/j.jmaa.2011.05.081>
- [8] Wang, X., Du, Z. and Liang, J. (2010) Existence and Global Attractivity of Positive Periodic Solution to a Lotka-Volterra Model. *Nonlinear Analysis: Real World Applications*, **11**, 4054-4061. <https://doi.org/10.1016/j.nonrwa.2010.03.011>
- [9] Lv, Y. and Du, Z. (2011) Existence and Global Attractivity of a Positive Periodic Solution to a Lotka-Volterra Model with Mutual Interference and Holling III Type Functional Response. *Nonlinear Analysis: Real World Applications*, **12**, 3654-3664.  
<https://doi.org/10.1016/j.nonrwa.2011.06.022>
- [10] Zhang, C., Huang, N. and O'Regan, D. (2013) Almost Periodic Solutions for a Volterra Model with Mutual Interference and Holling Type III Functional Response. *Applied Mathematics and Computation*, **225**, 503-511.  
<https://doi.org/10.1016/j.amc.2013.09.047>
- [11] Mo, J., Li, W., He, D., Wang, S. and Zhou, X. (2023) Dynamic Analysis of a Predator-Prey Model with Holling-II Functional Response. *Journal of Applied Mathematics and Physics*, **11**, 2871-2878. <https://doi.org/10.4236/jamp.2023.1110188>
- [12] Wang, D. and Ma, Y. (2024) Bifurcation and Turing Pattern Formation in a Diffusion Modified Leslie-Gower Predator-Prey Model with Crowley-Martin Functional Response. *Journal of Applied Mathematics and Physics*, **12**, 2190-2211.  
<https://doi.org/10.4236/jamp.2024.126133>
- [13] Wang, S., Yu, H., Dai, C. and Zhao, M. (2020) The Dynamical Behavior of a Certain Predator-Prey System with Holling Type II Functional Response. *Journal of Applied Mathematics and Physics*, **8**, 527-547. <https://doi.org/10.4236/jamp.2020.83042>
- [14] Andrews, J.F. (1968) A Mathematical Model for the Continuous Culture of Microorganisms Utilizing Inhibitory Substrates. *Biotechnology and Bioengineering*, **10**, 707-723. <https://doi.org/10.1002/bit.260100602>
- [15] Sokol, W. and Howell, J.A. (1981) Kinetics of Phenol Oxidation by Washed Cells. *Biotechnology and Bioengineering*, **23**, 2039-2049.  
<https://doi.org/10.1002/bit.260230909>
- [16] Cui, J. and Song, X. (2004) Permanence of Predator-Prey System with Stage Structure. *Discrete & Continuous Dynamical Systems B*, **4**, 547-554.  
<https://doi.org/10.3934/dcdsb.2004.4.547>
- [17] Baek, H. (2010) A Food Chain System with Holling Type IV Functional Response and Impulsive Perturbations. *Computers & Mathematics with Applications*, **60**, 1152-1163. <https://doi.org/10.1016/j.camwa.2010.05.039>
- [18] Yang, W., Li, X. and Bai, Z. (2008) Permanence of Periodic Holling Type-IV Predator-Prey System with Stage Structure for Prey. *Mathematical and Computer Modelling*, **48**, 677-684. <https://doi.org/10.1016/j.mcm.2007.11.003>
- [19] Liu, C. and Zhou, X. (2019) Existence and Global Attractivity of Positive Periodic Solutions for a Delayed Predator-Prey Model with Mutual Interference and Functional Response. *Filomat*, **33**, 43-64. <https://doi.org/10.2298/fil1901043l>