Journal of Applied Mathematics and Physics, 2024, 12, 2723-2737
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Dynamics and Exact Solutions of (1 + 1)-Dimensional Generalized Boussinesq Equation with Time-Space Dispersion Term

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How to cite this paper: Feng, D.H., Li, J.B. and Jiao, J.J. (2024) Dynamics and Exact Solutions of (1 + 1)-Dimensional Generalized Boussinesq Equation with Time-Space Dispersion Term. *Journal of Applied Mathematics and Physics*, **12**, 2723-2737.
https://doi.org/10.4236/jamp.2024.128163

Received: June 29, 2024
Accepted: August 3, 2024
Published: August 6, 2024

Abstract

We study exact solutions to (1 + 1)-dimensional generalized Boussinesq equation with time-space dispersion term by making use of improved sub-equation method, and analyse the dynamical behavior and exact solutions of the sub-equation after constructing the nonlinear transformation and constraint conditions. Accordingly, we obtain twenty families of exact solutions such as analytical and singular solitons and singular periodic waves. In addition, we discuss the impact of system parameters on wave propagation.

Keywords

Generalized Boussinesq Equation, Improved Sub-Equation Method, Bifurcation, Soliton Solution, Periodic Solution

1. Introduction

Many natural phenomena can be described through an algebraic or differential equation. Studying the dynamical properties and exact solutions of these equations can help deepen understanding and explain these natural phenomena. For example, it helps to deepen the understanding of Ohmic losses caused by friction and temperature rise, thereby improving some production processes in electrical engineering [1]. It can help one find the necessary standards for process materials [2]. It helps one understand the water waves in seas and oceans and therefore predict their risks after studying the speed, shapes and height of these waves [3]-[6].

The Boussinesq equation as a fundamental generalized wave equation that arises in fluid mechanics and plasma physics, and describes the propagation of small amplitude long waves in shallow water, is viewed as

$$u_{tt} - u_{xx} - \beta\left(u^2\right)_{xx} - \gamma u_{xxxx} = 0, \tag{1}$$

where β is the nonlinear parameter representing the depth of the fluid, γ is the characteristic speed of the long wave and $u = u(x,t)$ is the elevation of the free surface of the fluid. x and t are the scaled spatial and temporal variables, respectively.

The standard Boussinesq Equation (1) was extended to many Boussinesq-type models since it possesses the properties of retaining quadratic nonlinearity and weak dispersion [7]-[9]. As is well known, Gravity waves have been generated in the fluids when the force of gravity or buoyancy tries to restore the equilibrium, the study of which is helpful to the oceanic and atmospheric sciences. In 2019, Wazwaz and Kaur [10] considered the propagation of gravity waves over water surface, more specifically, the head-on collision of oblique wave profiles, and introduced a (1 + 1)-dimensional generalized Boussinesq equation with time-space dispersion term which is of the new form

$$u_{tt} - u_{xx} - \beta\left(u^2\right)_{xx} - \gamma u_{xxxx} + \alpha u_{xt} = 0, \tag{2}$$

where α is a nonzero time-space dispersion parameter related to the head-on interaction of oblique wave profiles. The complete integrability was investigated via Painlevé test and some multiple soliton solutions were obtained by using simplified Hirota's method [10]. The nonlocal symmetries, Bäcklund transformation and interaction solutions were derived in [11]. The bilinear form was obtained and the higher-order rogue waves, breather and hybrid solutions in the determinant form were constructed by means of the Hirota bilinear method and KP hierarchy reduction [12]. Recently, bidirectional bell-shaped solutions, bidirectional lump-solitons and periodic solutions were gained via the Hirota bilinear method, Kudryashov expansion method and the Cole-Hopf transformation [13].

It has always been a major challenge to find more exact solutions to nonlinear equations that describe various phenomena and applications arising in the science of chemistry, physics, engineering, biology and so on. In recent decades, with the development of computer technology, many methods have been developed for solving nonlinear problems. Such methods are Lie group method [14]-[16], Jacobi elliptic function method [17] [18], tanh-function approach and extended tanh-function approach [19], Hirota bilinear method [20] [21], generalized bilinear method [22], modified exponential-expansion [23] [24], (G′/G)-expansion method [25], Kudryashov-expansion [26] [27], extended auxiliary equation approach [28], improved sub-equation method [29]-[31], simplified homogeneous balance method [32] and many other methods [33]-[37].

This work aims to construct more exact solutions of (2) by means of improved sub-equation method, and analyze the impact of system parameters on wave propagation by investigating the dynamics of selected exact solutions.

The rest of the manuscript is organized as follows. The nonlinear transformation and constraint conditions are constructed in Section 2. The bifurcations and phase portraits of sub-equation are analysed in Section 3. Many explicit and exact solutions of (2) are obtained via solving the sub-equation in Section 4. The simulations and discussions are presented in Section 5. Some conclusions are given in the final section.

2. Construction of Nonlinear Transformation

The purpose of this article is to solve more types of exact solutions of (2) by means of improved sub-equation method. We give the mathematical framework of the improved method and construct a nonlinear transformation between the generalized Boussinesq Equation (2) and the sub-equation in this section.

Step 1: Conversion from the given NPDE to an ODE.

We introduce the wave transformation

$$u(x,t)=u(\xi),\ \xi=x-ct, \tag{3}$$

where c is a wave speed. The resulting ODE is

$$\left(c^2-\alpha c-1\right)u''-\beta\left(u^2\right)''-\gamma u^{(4)}=0, \tag{4}$$

Step 2: Assumption of nonlinear transformation.

Assume that the solutions of Equation (4) can be expressed in the form

$$u(\xi)=\sum_{i=0}^{n}a_i\phi^i(\xi), \tag{5}$$

where $a_i\,(i=0,1,\cdots,n)$ are constants to be determined later, and the variable $\phi=\phi(\xi)$ satisfies the sub-equation

$$\phi'(\xi)=\varepsilon\sqrt{b_0+b_2\phi^2+b_4\phi^4}, \tag{6}$$

where $\varepsilon=\pm1$ and b_0,b_2,b_4 are constants.

Step 3: Determination of positive integer n.

Substituting (5) and (6) into (4) and then by the homogeneous balance method, balancing the highest order nonlinear term $\left(u^2\right)''$ with the highest derivative term $u^{(4)}$, we have $2n+2=n+4$ and $n=2$. From (5) we assume that the solutions of (4) are of the form

$$u(\xi)=a_0+a_1\phi(\xi)+a_2\phi^2(\xi), \tag{7}$$

where a_0,a_1 and $a_2\neq0$ are constants and $\phi(\xi)$ satisfies (6).

Step 4: Establishment of constraint parametric conditions.

By substituting (6) and (7) into (4) again, we derive a sixth algebraic equation of the variable ϕ. Setting the coefficients of all powers of ϕ to be zero, we arrive at the following algebraic equations

$$\begin{cases}\beta a_2+6\gamma b_4=0,\\ a_1\left(\beta a_2+\gamma b_4\right)=0,\\ 8\beta a_2^2b_2-3\left(c^2-\alpha c-1-2\beta a_0-20\gamma b_2\right)a_2b_4+3\beta a_1^2b_4=0,\\ a_1\left[b_4\left(c^2-\alpha c-1-2\beta a_0-10\gamma b_2\right)-9\beta a_2b_2\right]=0,\\ 3\beta a_2^2b_0-\left[b_2\left(c^2-\alpha c-1-2\beta a_0-4\gamma b_2\right)-18\gamma b_0b_4\right]a_2+\beta a_1^2b_2=0,\\ a_1\left[\left(c^2-\alpha c-1-2\beta a_0-\gamma b_2\right)b_2-12b_0\left(\beta a_2+\gamma b_4\right)\right]=0,\\ \left(c^2-\alpha c-1-2\beta a_0-4\gamma b_2\right)a_2-\beta a_1^2=0.\end{cases} \tag{8}$$

Solving the above system yields

$$a_2 = -\frac{6\gamma b_4}{\beta},\ a_1 = 0,\ a_0 = \frac{c^2 - \alpha c - 1 - 4\gamma b_2}{2\beta}, \tag{9}$$

and other parameters b_0, b_2 and $b_4 \neq 0$ are all arbitrary constants.

Thus the nonlinear transformation (7) between (4) and (6) has been successfully established under the constraint parameter conditions (9). The corresponding exact solutions of (4) can be constructed by solving the sub-Equation (6).

Step 5: Bifurcations analysis of the sub-equation.

The next crucial step is to solve the explicit and exact solutions of the sub-Equation (6).

Clearly, Equation (6) is equivalent to the following system

$$(\phi')^2 = b_0 + b_2\phi^2 + b_4\phi^4. \tag{10}$$

Setting $\phi' = y$, Equation (10) is reduced to a planar system

$$\frac{d\phi}{d\xi} = y,\ \frac{dy}{d\xi} = b_2\phi + 2b_4\phi^3, \tag{11}$$

which is a Hamiltonian system and has the Hamiltonian

$$H(\phi, y) = y^2 - (b_2\phi^2 + b_4\phi^4) = b_0. \tag{12}$$

The sub-Equation (6) is equivalent to the Hamiltonian system (11). Next we use bifurcation method of planar dynamical systems to analyse the bifurcations and phase portraits of the Hamiltonian system. As a result, we get not only all possible explicit and exact solutions but also their dynamical behavior.

Step 6: Construction of exact solutions of NPDE.

The corresponding exact solutions of the given NPDE can be constructed with the general form (7) and the constraint conditions (9) by solving the Hamiltonian system (11) according to the phase portraits obtained in Step 5.

3. Bifurcations and Phase Portraits of (11)

In the section we analyse the bifurcations and phase portraits of Hamiltonian system (11) by using the bifurcation theory of planar dynamical systems.

Denote that $f(\phi) = b_2\phi + 2b_4\phi^3$, then all equilibrium points of (11) lie on the ϕ-axis in the (ϕ, y)-phase plane and their abscissas are the real roots of $f(\phi)$. We have the proposition on the distribution of equilibria of (11).

Proposition 1. Suppose that $b_4 \neq 0$, then

1) Equation (11) has three equilibrium points at $O(0, 0)$ and $S^{\pm}\left(\pm\sqrt{-\frac{b_2}{2b_4}}, 0\right)$ for $b_2 b_4 < 0$;

2) Equation (11) has a unique equilibrium point at $O(0, 0)$ for $b_2 b_4 \geq 0$.

Let $M(\phi_e, 0)$ be the coefficient matrix of the linearized system of (11) at the equilibrium point $(\phi_e, 0)$ and let $J(\phi_e, 0) = \det M(\phi_e, 0)$, then

$$J(\phi_e,0) = -f'(\phi_e) = -\left(b_2 + 6b_4\phi_e^2\right). \tag{13}$$

It follows that

$$J(0,0) = -b_2,\ \left(\pm\sqrt{-\frac{b_2}{2b_4}},0\right) = 2b_2. \tag{14}$$

By the bifurcation theory of planar dynamical systems [38]-[40], an equilibrium point of a planar Hamiltonian system, $M(\phi_e,0)$ is a center (or saddle) if $J(\phi_e,0)>0$ (or $J(\phi_e,0)<0$). It is a high-order equilibrium point when $J(\phi_e,0)=0$.

On the other hand, in order to analyse further the bifurcations and the phase portraits of (11) in the (b_2,b_4)-parametric plane, we must study their bifurcation curves. For the equilibrium points $O(0,0)$ and $S^{\pm}\left(\pm\sqrt{\frac{-b_2}{2b_4}},0\right)$, their Hamiltonian defined by (11) are of the form

$$h_0 = H(0,0) = 0,\ h^{\pm} = H\left(\pm\sqrt{\frac{-b_2}{2b_4}},0\right) = \frac{b_2^2}{4b_4}. \tag{15}$$

Letting $h^{\pm} = h_0$, we obtain two possible bifurcation curves of (11) in the (b_2,b_4)-parametric plane as follows

$$L_1^{\pm}: b_2 = 0, b_4 \in \mathbb{R}^{\pm};\quad L_2^{\pm}: b_4 = 0, b_2 \in \mathbb{R}^{\pm}. \tag{16}$$

The above bifurcation curves partition the (b_2,b_4)-plane into four different subregions as follows:

$$A_1 = \{(b_2,b_4) \mid b_2 > 0, b_4 > 0\};\quad A_2 = \{(b_2,b_4) \mid b_2 < 0, b_4 > 0\};$$
$$A_3 = \{(b_2,b_4) \mid b_2 < 0, b_4 < 0\};\quad A_4 = \{(b_2,b_4) \mid b_2 > 0, b_4 < 0\}.$$

By applying the above results to do qualitative analysis, we obtain the following results.

Proposition 2.

1) When $(b_2,b_4) \in A_1 \cup L_1^+$, system (11) has a unique equilibrium point at O which is a saddle.

2) When $(b_2,b_4) \in A_2$, system (11) has three equilibrium points at O and $S^{\pm}$. O is a center while S^+ and S^- both are saddles. For $H(\phi,y) = h^{\pm}$ defined in (12), there are two heteroclinic orbits connected to saddles $S^{\pm}$, respectively. There is a family of periodic orbits defined by $H(\phi,y) = h$ for $h \in (0,h^{\pm})$.

3) When $(b_2,b_4) \in A_3 \cup L_1^-$, system (11) has a unique equilibrium point at O which is a center. There exists a family of periodic orbits defined by $H(\phi,y) = h$ for $h > 0$.

4) When $(b_2,b_4) \in A_4$, system (11) has three equilibria at O and $S^{\pm}$. O is a saddle and $S^{\pm}$ both are centers. There are two homoclinic orbits to saddle O for $H(\phi,y) = 0$. There exist two families of periodic orbits defined by $H(\phi,y) = h$ for $h \in (h^{\pm},0)$ and a family of large-scale periodic orbits defined by $H(\phi,y) = h$ for $h > 0$.

The phase portraits of (11) are shown in **Figure 1**.

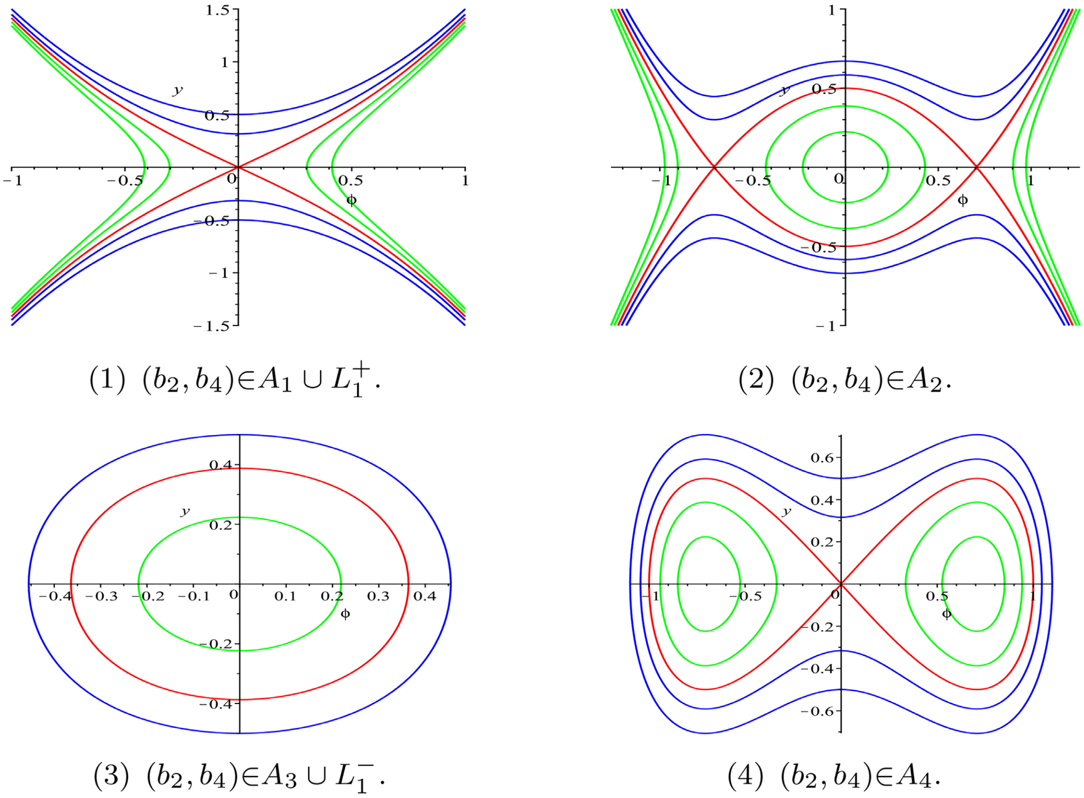


Figure 1. The phase portraits of system (11).

4. Exact Solutions of Generalized Boussinesq Equation

In this section, we will derive exact solutions of the generalized Boussinesq Equation (2). By using the transformation (7) and the Hamiltonian system (11) along with the Hamiltonian (12), we obtain the following results.

(1) When $(b_2, b_4) \in A_1$, *i.e.*, $b_2 > 0, b_4 > 0$ (see **Figure 1(1)**).

(1.1) For $b_0 < 0$, it follows from (12) that

$$y = \pm\sqrt{b_4\left(\phi^2 - \frac{\Delta - b_2}{2b_4}\right)\left(\phi^2 + \frac{\Delta + b_2}{2b_4}\right)}, \tag{17}$$

where $\Delta = \sqrt{b_2^2 - 4b_0b_4}$. From the first equation in (11), we have

$$\int_{\sqrt{\frac{\Delta - b_2}{2b_4}}}^{\phi} \frac{\mathrm{d}\phi}{\sqrt{\left(\phi^2 - \frac{\Delta - b_2}{2b_4}\right)\left(\phi^2 + \frac{\Delta + b_2}{2b_4}\right)}} = \pm\int_0^{\xi} \sqrt{b_4}\,\mathrm{d}\xi. \tag{18}$$

Thus we obtain two unbounded periodic solutions to (11)

$$\phi_1^{\pm}(\xi) = \pm\sqrt{\frac{\Delta - b_2}{2b_4}}\,\mathrm{nc}\left(\sqrt{\Delta}\xi, k_1\right), \tag{19}$$

where $k_1 = \sqrt{\frac{\Delta + b_2}{2\Delta}}$, while $\mathrm{nc}(x,k)$ and the following $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$, $\mathrm{dn}(x,k)$ are Jacobian elliptic functions with modulus *k*. The quotients and reciprocals of $\mathrm{sn}(x,k)$, $\mathrm{cn}(x,k)$ and $\mathrm{dn}(x,k)$ are given by [41]

$$\mathrm{nc}(x,k) \equiv \frac{1}{\mathrm{cn}(x,k)}, \mathrm{sc}(x,k) \equiv \frac{\mathrm{sn}(x,k)}{\mathrm{cn}(x,k)}, \mathrm{dc}(x,k) \equiv \frac{\mathrm{dn}(x,k)}{\mathrm{cn}(x,k)}.$$

(1.2) For $b_0 = 0$, it follows from (12) that

$$y = \pm\phi\sqrt{b_2 + b_4\phi^2}. \tag{20}$$

From the first equation in (11), we have

$$\int_{\phi}^{\infty} \frac{\mathrm{d}\phi}{\phi\sqrt{b_2 + b_4\phi^2}} = \pm\int_{\xi}^{0} \mathrm{d}\xi. \tag{21}$$

Hence we obtain two unbounded hyperbolic function solutions of (11)

$$\phi_2^{\pm}(\xi) = \pm\sqrt{\frac{b_2}{b_4}}\,\mathrm{csch}\left(\sqrt{b_2}\,\xi\right). \tag{22}$$

(1.3) Similarly, for $0 < b_0 < \frac{b_2^2}{4b_4}$, Equation (11) has two unbounded periodic solutions with two periods

$$\phi_3^{\pm}(\xi) = \pm\sqrt{\frac{b_2 - \Delta}{2b_4}}\,\mathrm{sc}\left(\sqrt{\frac{b_2 + \Delta}{2}}\xi, \frac{1}{k_1}\right) \tag{23}$$

(1.4) For $b_0 = \frac{b_2^2}{4b_4}$, Equation (11) exists two unbounded trigonometric function solutions

$$\phi_4^{\pm}(\xi) = \pm\sqrt{\frac{b_2}{2b_4}}\tan\left(\sqrt{\frac{b_2}{2}}\xi\right). \tag{24}$$

(1.5) For $b_0 > \frac{b_2^2}{4b_4}$, Equation (11) has two singular elliptic function solutions

$$\phi_5^{\pm}(\xi) = \pm\sqrt[4]{\frac{b_0}{b_4}}\sqrt{\frac{1 + \mathrm{cn}\left(2\sqrt[4]{b_0b_4}\xi, k_2\right)}{1 - \mathrm{cn}\left(2\sqrt[4]{b_0b_4}\xi, k_2\right)}}, \tag{25}$$

where $k_2 = \sqrt{\frac{2\sqrt{b_0b_4} - b_2}{4\sqrt{b_0b_4}}}$.

From the nonlinear transformation (7) and the parametric conditions (9), we achieve five families of explicit and exact solutions to the generalized Boussinesq Equation (2)

$$u_i(x,t) = \frac{c^2 - \alpha c - 1 - 4\gamma b_2}{2\beta} - \frac{6\gamma b_4}{\beta}\left(\phi_i^{\pm}(x - ct)\right)^2, \ 1 \le i \le 5. \tag{26}$$

Remark 1. $\phi_i^{\pm}(\xi)(1 \le i \le 5)$ are all singular unbounded solutions of (11) and the corresponding solutions $u_i(x,t)$ are also singular unbounded solutions of (2).

(2) When $(b_2, b_4) \in L_1^+$, *i.e.*, $b_2 = 0, b_4 > 0$ (see Figure 1(1)).

(2.1) For $b_0 < 0$, Equation (11) has two singular elliptic function solutions with two periods

$$\phi_6^{\pm}(\xi)=\pm\sqrt[4]{-\frac{b_0}{b_4}}\,\mathrm{nc}\left(\sqrt[4]{-4b_0b_4}\,\xi,\frac{\sqrt{2}}{2}\right). \tag{27}$$

(2.2) For $b_0=0$, Equation (11) has two singular unbounded irrational function solutions

$$\phi_7^{\pm}(\xi)=\pm\frac{1}{\sqrt{b_4\xi}}. \tag{28}$$

(2.3) For $b_0>0$, Equation (11) has two singular unbounded elliptic function solutions with two periods

$$\phi_8^{\pm}(\xi)=\pm\sqrt[4]{\frac{b_0}{b_4}}\sqrt{\frac{1+\mathrm{cn}\left(2\sqrt[4]{b_0b_4}\,\xi,\frac{\sqrt{2}}{2}\right)}{1-\mathrm{cn}\left(2\sqrt[4]{b_0b_4}\,\xi,\frac{\sqrt{2}}{2}\right)}}, \tag{29}$$

From (7) and (9), we obtain three families of explicit and exact solutions to the generalized Boussinesq Equation (2)

$$u_i(x,t)=\frac{c^2-\alpha c-1}{2\beta}-\frac{6\gamma b_4}{\beta}\left(\phi_i^{\pm}(x-ct)\right)^2,\ i=6,7,8. \tag{30}$$

(3) When $(b_2,b_4)\in A_2$, *i.e.*, $b_2<0,b_4>0$ (see **Figure 1(2)**).

(3.1) For $b_0<0$, Equation (11) has two singular unbounded elliptic function solutions with two periods

$$\phi_9^{\pm}(\xi)=\pm\sqrt{\frac{\Delta-b_2}{2b_4}}\mathrm{nc}\left(\sqrt{\Delta}\xi,k_3\right), \tag{31}$$

where $k_3=\sqrt{\frac{\Delta+b_2}{2\Delta}}$.

(3.2) For $b_0=0$, Equation (11) has two singular unbounded trigonometric function solutions.

$$\phi_{10}^{\pm}(\xi)=\pm\sqrt{-\frac{b_2}{b_4}}\csc\left(\sqrt{-b_2}\xi\right). \tag{32}$$

(3.3) For $0<b_0<\frac{b_2^2}{4b_4}$, Equation (11) has an analytical elliptic function solution with two periods

$$\phi_{11}(\xi)=\sqrt{\frac{\Delta+b_2}{-2b_4}}\,\mathrm{sn}\left(\sqrt{\frac{\Delta-b_2}{2}}\xi,k_4\right), \tag{33}$$

and two singular unbounded elliptic function solutions

$$\phi_{12}^{\pm}(\xi)=\pm\sqrt{\frac{\Delta-b_2}{2b_4}}\,\mathrm{dc}\left(\sqrt{\frac{\Delta-b_2}{2}}\xi,k_4\right), \tag{34}$$

where $k_4=\frac{2\sqrt{b_0b_4}}{\Delta-b_2}$.

(3.4) For $b_0 = \frac{b_2^2}{4b_4}$, Equation (11) has two analytical kink soliton solutions represented by hyperbolic functions

$$\phi_{13}^{\pm}(\xi) = \pm\sqrt{\frac{-b_2}{2b_4}} \tanh\left(\sqrt{-\frac{b_2}{2}}\xi\right), \tag{35}$$

and two singular unbounded solutions

$$\phi_{14}^{\pm}(\xi) = \pm\sqrt{\frac{-b_2}{2b_4}} \coth\left(\sqrt{-\frac{b_2}{2}}\xi\right). \tag{36}$$

(3.5) For $b_0 > \frac{b_2^2}{4b_4}$, Equation (11) has two singular unbounded periodic solutions

$$\phi_{15}^{\pm}(\xi) = \pm\sqrt[4]{\frac{b_0}{b_4}} \sqrt{\frac{1+\mathrm{cn}\left(2\sqrt[4]{b_0 b_4}\xi, k_5\right)}{1-\mathrm{cn}\left(2\sqrt[4]{b_0 b_4}\xi, k_5\right)}}, \tag{37}$$

where $k_5 = \sqrt{\frac{2\sqrt{b_0 b_4} - b_2}{4\sqrt{b_0 b_4}}}$.

From (7) and (9), we derive seven families of explicit and exact solutions of (2)

$$u_i(x,t) = \frac{c^2 - \alpha c - 1 - 4\gamma b_2}{2\beta} - \frac{6\gamma b_4}{\beta}\left(\phi_i^{\pm}(x-ct)\right)^2, \quad 9 \le i \le 15. \tag{38}$$

Remark 2. $\phi_{13}^{+}(\xi)$ and $\phi_{13}^{-}(\xi)$ are kink-shaped soliton solution and anti-kink-shaped soliton solution of (11), respectively, but the corresponding solution $u_{13}(x,t)$ is a bell-shaped soliton solution of (2).

(4) When $(b_2, b_4) \in A_3$, *i.e.*, $b_2 < 0, b_4 < 0$ (see **Figure 1(3)**).

(4.1) For $b_0 > 0$, Equation (11) has an analytical periodic solution represented by elliptic function

$$\phi_{16}(\xi) = \sqrt{\frac{\Delta + b_2}{-2b_4}} \mathrm{cn}\left(\sqrt{\Delta}\xi, k_6\right), \tag{39}$$

where $k_6 = \sqrt{\frac{\Delta + b_2}{2\Delta}}$. Thus we construct a family of analytical periodic solutions of (2)

$$u_{16}(x,t) = \frac{c^2 - \alpha c - 1 - 4\gamma b_2}{2\beta} + \frac{3\gamma}{\beta}(\Delta + b_2)\mathrm{cn}^2\left(\sqrt{\Delta}(x-ct), k_6\right). \tag{40}$$

(5) When $(b_2, b_4) \in L_1^-$, *i.e.*, $b_2 = 0, b_4 < 0$ (see **Figure 1(3)**).

(5.1) For $b_0 > 0$, Equation (11) exists an analytical periodic solution

$$\phi_{17}(\xi) = \sqrt[4]{-\frac{b_0}{b_4}} \mathrm{cn}\left(\sqrt[4]{-4b_0 b_4}\xi, \frac{\sqrt{2}}{2}\right). \tag{41}$$

Then we obtain a family of analytical periodic solutions of (2)

$$u_{17}(x,t)=\frac{c^2-\alpha c-1}{2\beta}+\frac{6\gamma}{\beta}\sqrt{-b_0b_4}\,\mathrm{cn}^2\left(\sqrt[4]{-4b_0b_4}(x-ct),\frac{\sqrt{2}}{2}\right). \tag{42}$$

(6) When $(b_2,b_4)\in A_4$, *i.e.*, $b_2>0,b_4<0$ (see **Figure 1(4)**).

(6.1) As $\frac{b_2^2}{4b_4}<b_0<0$, there are two analytical periodic solutions with double periods

$$\phi_{18}^{\pm}(\xi)=\pm\sqrt{\frac{\Delta+b_2}{-2b_4}}\,\mathrm{dn}\left(\sqrt{\frac{\Delta+b_2}{2}}\xi,\frac{1}{k_6}\right). \tag{43}$$

(6.2) For $b_0=0$, Equation (11) has two analytical soliton solutions with peak form and valley form, respectively

$$\phi_{19}^{\pm}(\xi)=\pm\sqrt{-\frac{b_2}{b_4}}\,\mathrm{sech}\left(\sqrt{b_2}\,\xi\right). \tag{44}$$

(6.3) As $b_0>0$, Equation (11) exists an analytical periodic solution with two periods

$$\phi_{20}(\xi)=\sqrt{\frac{\Delta+b_2}{-2b_4}}\,\mathrm{cn}\left(\sqrt{\Delta}\xi,k_6\right). \tag{45}$$

Hence the generalized Boussinesq Equation (2) has three families of corresponding explicit and exact solutions

$$u_i(x,t)=\frac{c^2-\alpha c-1-4\gamma b_2}{2\beta}-\frac{6\gamma b_4}{\beta}\left(\phi_i^{\pm}(x-ct)\right)^2,\ i=18,19,20. \tag{46}$$

5. Simulations and Discussions

Many analytical or singular exact solutions to (1 + 1)-dimensional generalized Boussinesq Equation (1) are obtained by using the improved sub-equation method. In this section, we take $u_{13}(x,t)$ as an example to discuss the impact of system parameters on wave propagation by selecting appropriate parameters.

From the constraint parameter conditions (9), we have

$$b_2=\frac{c^2-\alpha c-1-2\beta a_0}{4\gamma}. \tag{47}$$

Thus it follows from Equations (35) and (38) that

$$u_{13}(x,t)=a_0+\frac{3\left(c^2-\alpha c-1-2\beta a_0\right)}{4\beta}\tanh^2\left(\sqrt{\frac{c^2-\alpha c-1-2\beta a_0}{-8\gamma}}(x-ct)\right). \tag{48}$$

This solution is a bell-shaped soliton solution. We consider the case when $c^2-\alpha c-1-2\beta a_0<0$, $\beta>0$ and $\gamma>0$ in order to analyze the influence of system parameters on the height and width of wave. Then the wave height of (48) is

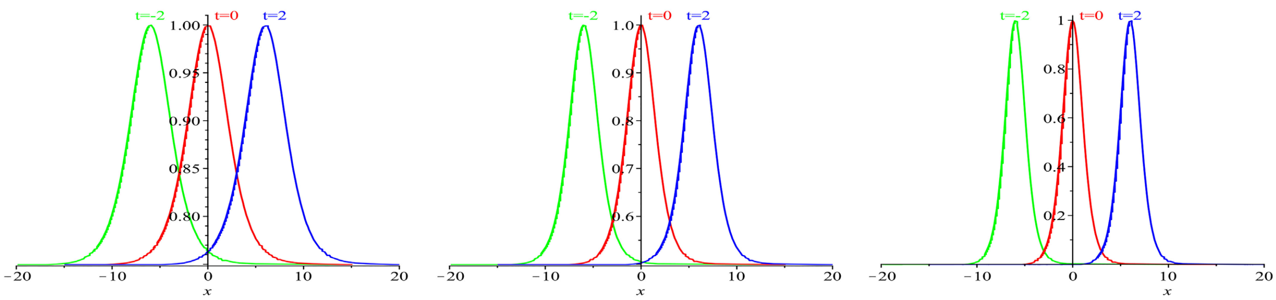
$$h=\frac{3}{4\beta}\left(\alpha c+1+2\beta a_0-c^2\right), \tag{49}$$

which is a function of α,β,a_0 and *c*. Its partial derivatives with respect to system parameters are

$$h_\alpha = \frac{3c}{4\beta},\ h_\beta = -\frac{3}{4\beta^2}\left(\alpha c + 1 - c^2\right),\ h_\gamma = 0. \tag{50}$$

As α increases, h becomes larger for $\beta c > 0$ and smaller for $\beta c < 0$. As β increases, h becomes larger for $\alpha c + 1 - c^2 < 0$ and smaller for $\alpha c + 1 - c^2 > 0$. The wave height has not been affected by the parameter γ since the expression for h does not contain γ. We can similarly consider the influence of system parameters on wave width. **Figures 2-4** show the effects of system parameters on the height and width of wave (48), respectively.

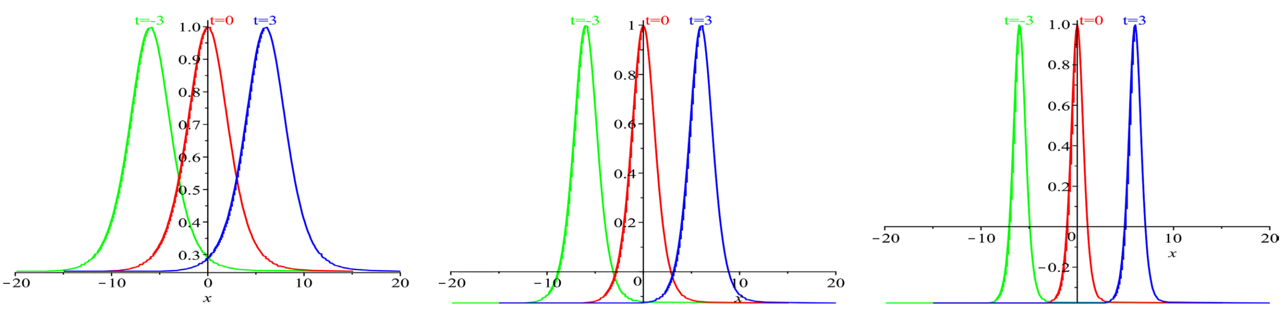
Figure 2 shows the behavior of solution $u_{13}(x,t)$ with parameters $a_0 = 1$, $c = 3$, $\gamma = 1$, $\beta = 3$ and different values of α, which presents an analytical bright soliton solution. Here we observe that the height of the wave becomes larger, the width of the wave becomes smaller, and the wave becomes steeper as α increases, which shows that the time-space dispersion parameter has a significant impact on the height and width of wave.



(1) $\alpha = 1$. (2) $\alpha = 4/3$. (3) $\alpha = 2$.

Figure 2. The impact of the time-space dispersion parameter on propagating of $u_{13}(x,t)$.

Figure 3 displays the behavior of solution $u_{13}(x,t)$ with parameters $a_0 = 1$, $c = 2$, $\gamma = 1$, $\alpha = 1$ and different values of β. As β grows larger, the height of the wave becomes larger while the width of the wave becomes smaller. The nonlinearity parameter also has a significant impact on the height and width of wave.

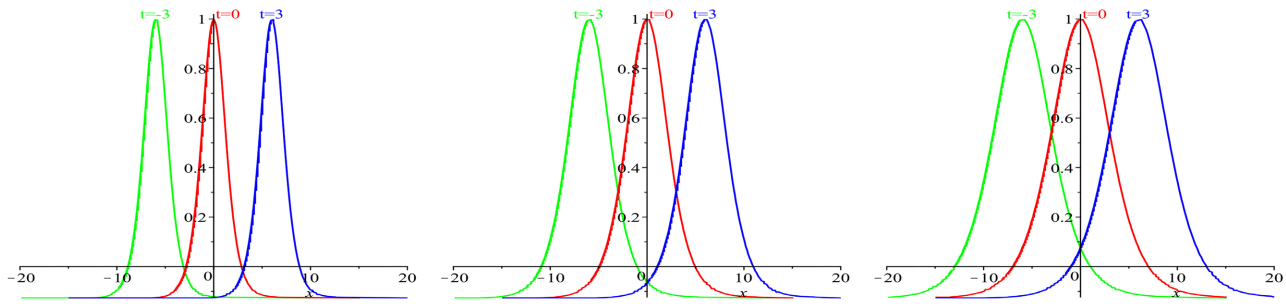


(1) $\beta = 1$. (2) $\beta = 2$. (3) $\beta = 6$.

Figure 3. The impact of the nonlinearity parameter on propagating of $u_{13}(x,t)$.

Figure 4 demonstrates the dynamics of solution $u_{13}(x,t)$ with constants

$a_0 = 1$, $c = 2$, $\alpha = 1$, $\beta = 2$ and different values of γ. As γ increases, the width of the wave increases while the height remains unchanged. This indicates that the characteristic speed γ only has an impact on the width of the wave, and has no effect on the height of the wave.



(1) $\gamma = 1$. (2) $\gamma = 3$. (3) $\gamma = 6$.

Figure 4. The impact of the characteristic speed on propagating of $u_{13}(x,t)$.

6. Conclusion

In this paper, we study the (1 + 1)-dimensional generalized Boussinesq equation with time-space dispersion term, and obtain many different types of exact solutions such as hyperbolic-form, trigonometric-form, rational-form and Jacobian elliptic function forms, which include analytical soliton and singular solitons, analytical periodic solutions and singular periodic solutions. In addition, we discuss the influence of system parameters on wave propagation. These results indicate that the improved sub-equation method is highly effective and can be used to solve other nonlinear problems.

Acknowledgements

This work is supported by the Science Technology Foundation of Guizhou Province, China (No. [2020]1Y001).

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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