Journal of Applied Mathematics and Physics, 2024, 12, 2682-2690
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Split-Tetraquaternion Algebra and Applications

Grégoire Lutanda Panga

Department of Mathematics and Informatic, University of Lubumbashi, Lubumbashi, The Democratic Republic of the Congo
Email: greglutandapanga@gmail.com

How to cite this paper: Panga, G.L. (2024) Split-Tetraquaternion Algebra and Applications. *Journal of Applied Mathematics and Physics*, **12**, 2682-2690.
https://doi.org/10.4236/jamp.2024.127159

Received: May 28, 2024
Accepted: July 28, 2024
Published: July 31, 2024

Abstract

In this paper, from the spacetime algebra associated with the Minkowski space $\mathbb{R}^{3,1}$ by means of a change of signature, we describe a quaternionic representation of the split-tetraquaternion algebra which incorporates the Pauli algebra, the split-biquaternion algebra and the split-quaternion algebra, we relate these algebras to Clifford algebras and we show the emergence of the stabilized Poincaré-Heisenberg algebra from the split-tetraquaternion algebra. We list without going into details some of their applications in Physics and in Born geometry.

Keywords

Tetraquaternion Algebra, Split-Tetraquaternion Algebra, Split Quaternion Algebra, Clifford Algebra

1. Introduction

In the realm of Clifford algebra, we present the quaternionic representation of the split-tetraquaternion algebra which is isomorphic to the Clifford algebra of the Minkowski space $\mathbb{R}^{1,3}$.

It is well-known that matrix theory occupies the foreground in the representation of Clifford algebras (Geometric algebras). In [1], matrix representations of lower-dimensional geometric algebras are reviewed dealing with the following cases: the algebra of complex numbers $Cl_{0,1}$, the algebra of hyperbolic numbers $Cl_{1,0}$, the geometric algebra $Cl_{2,0}$ of the 2D-Euclidean space $\mathbb{R}^2$, the geometric algebra $Cl_{1,1}$ of the pseudo-Euclidean space $\mathbb{R}^{1,1}$, the geometric algebra of quaternions $Cl_{0,2}$, the geometric algebra $Cl_{3,0}$ of the 3D-Euclidean space $\mathbb{R}^3$. A general table of matrix representations of Clifford algebras $Cl_{p,q}$ is given in ([1] p. 15). The quaternionic formulation is a way to represent Clifford algebras without using matrix, it is the case of hyperquaternion algebras. Four classes of

Clifford algebras constitute the main object of the hyperquaternion algebras studied in [2]-[8] which are in particular $Cl_{2m+1,2m-1}$, $Cl_{2m+1,2m-2}$, $Cl_{2m-2,2m}$ and $Cl_{2m-2,2m-1}$, for $m \in \mathbb{N}_0$. In other words, some Clifford algebras $Cl_{p,q}$ with $p-q=-1,-2,3,2 \pmod 8$ are isomorphic to hyperquaternion algebras [7]. A change of the signature (p,q) of the Euclidean or pseudo-Euclidean spaces in (q,p) leads to the treatment of some Clifford algebras with $p-q=0,1,4,5 \pmod 8$ from the quaternionic point of view. In this paper, we are interested on the study of the split-tetraquaternion algebra and some of its subalgebras. In [8], the tetraquaternion algebra $\mathbb{H}\otimes\mathbb{H}$ generated by the basis vectors of the pseudo-Euclidean space $\mathbb{R}^{3,1}$ is extensively studied, in particular the treatment of multivector calculus, Lorentz group, classical electromagnetiusm, special theory of relativity, general theory of relativity within the tetraquaternion algebra over $\mathbb{R}$ has been developed.

When we change the signature of the Minkowski space $\mathbb{R}^{3,1}$ from (3, 1) to (1, 3) we derive the split-tetraquaternion algebra, denoted $Sp(\mathbb{H}\otimes\mathbb{H})$, which is the main object of the present paper. The algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ is isomorphic to the Clifford algebras $Cl_{1,3}$ and has a wide variety of interesting sub-algebras, including the complex and split-complex algebras, the quaternion and split-quaternion algebras, the biquaternion and split-biquaternion algebras that we will present in our discussions. These sub-algebras are useful in mathematical and physics applications.

This paper is organized as follows; in Section 2, we present the tetraquaternion algebra which will serve as a support for the construction and the development of the split-tetraquaternion algebra. The algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ and its sub-algebras are presented in Section 3 and constitute the main object of our study. In the Section 4, we relate the split-tetraquaternion algebra to the stabilized Poincaré-Heisenberg algebra simply by defining a Lie structure on $Sp(\mathbb{H}\otimes\mathbb{H})$ and we give some applications of the algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ and its subalgebras.

2. Preliminaries

This section briefly presents the tetraquaternion algebra $\mathbb{H}\otimes\mathbb{H}$ which is isomorphic the Clifford spacetime algebra $Cl_{3,1}$ of the Minkowski space $\mathbb{R}^{3,1}$. The algebra $\mathbb{H}\otimes\mathbb{H}$ has been widely used in physics, especially in the special theory of relativity, classical electromagnetism and the general theory of relativity, … [8].

The sixteen basis elements of this algebra, chosen here, are expressed as follows: scalar (1), vectors (j, kI, kJ, kK), bivectors (I, J, K, iI, iJ, iK), trivectors (k, jI, jJ, jK) and pseudoscalar (*i*). The table below gives the multivector structure of $\mathbb{H}\otimes\mathbb{H}$ where (*I*, *j*, *k*) and (*I*, *J*, *K*) are two differents quaternionic systems *i.e.* $i^2=j^2=k^2=ijk=-1$ and $I^2=J^2=K^2=IJK=-1$, see ([5] p. 6).

1	$I=e_{32}$	$J=e_{13}$	$K=e_{21}$
$i=e_{0123}$	$iI=e_{01}$	$iJ=e_{02}$	$iK=e_{03}$
$j=e_{0}$	$jI=e_{032}$	$iJ=e_{013}$	$iK=e_{021}$
$k=e_{123}$	$kI=e_{1}$	$kJ=e_{2}$	$kK=e_{3}$

3. Split-Tetraquaternion Algebra and Its Subalgebras

3.1. Split-Tetraquaternion Algebra $Sp(\mathbb{H}\otimes\mathbb{H})$

To derive the split-tetraquaternion algebra form the tetraquaternion algebra $\mathbb{H}\otimes\mathbb{H}$, we make a change of the signature of the Minkowski space $\mathbb{R}^{3,1}$ by multiplying by an unit imaginary i' the elements of basis (j,kI,kJ,kK) of $\mathbb{R}^{3,1}$. We obtain the four vectors $i'j=e_0$, $i'kI=e_1$, $i'kJ=e_2$ and $i'kK=e_3$ which constitute a basis of the Minkowski space $\mathbb{R}^{1,3}$. The algebra table below describes the multivector structure of the split-tetraquaternion algebra, denoted by $Sp(\mathbb{H}\otimes\mathbb{H})$.

1	$I=e_{23}$	$J=e_{31}$	$K=e_{12}$
$i=e_{0123}$	$iI=e_{10}$	$iJ=e_{20}$	$iK=e_{30}$
$i'j=e_0$	$i'jI=e_{023}$	$i'jJ=e_{031}$	$i'jK=e_{012}$
$i'k=e_{132}$	$i'kI=e_1$	$i'kJ=e_2$	$i'kK=e_3$

There are sixteen basis elements of the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ represented by one scalar (1), four vectors $(i'j,i'kI,i'kJ,i'kK)$, six bivectors (I,J,K,iI,iJ,iK), four trivectors $(i'k,i'jI,i'jJ,i'jK)$ and one pseudoscalar *i*. Among these unit basis elements, six square to 1 and ten square to −1.

On one hand, it is easy to establish that the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ is isomorphic to the Clifford spacetime algebra $Cl_{1,3}$. And on the other hand, it is obvious that there is no isomorphism between the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ and the tetraquaternion algebra $\mathbb{H}\otimes\mathbb{H}$.

If we denote by $S_{p_n}(\mathbb{H}\otimes\mathbb{H})$ the set of multivectors of grade *n* it follows that $S_{p_0}(\mathbb{H}\otimes\mathbb{H})$ is the field of real numbers, $S_{p_1}(\mathbb{H}\otimes\mathbb{H})$ is the Minkowski space $\mathbb{R}^{1,3}$, $S_{p_2}(\mathbb{H}\otimes\mathbb{H})$ is the multivector space of all bivectors of $Sp(\mathbb{H}\otimes\mathbb{H})$, $S_{p_3}(\mathbb{H}\otimes\mathbb{H})$ is the multivector space of all trivectors of $Sp(\mathbb{H}\otimes\mathbb{H})$ and $S_{p_4}(\mathbb{H}\otimes\mathbb{H})$ is the multivector space of all quadrivectors of $Sp(\mathbb{H}\otimes\mathbb{H})$. Taking in account this notation, the split-tetraquaternion can be written as follows:

$$Sp(\mathbb{H}\otimes\mathbb{H})=\mathbb{R}\oplus\mathbb{R}^{3,1}\oplus S_{p_2}(\mathbb{H}\otimes\mathbb{H})\oplus S_{p_3}(\mathbb{H}\otimes\mathbb{H})\oplus S_{p_4}(\mathbb{H}\otimes\mathbb{H}). \quad (1)$$

An arbitrary split-tetraquaternion can be written as follows:

$$\begin{aligned} q=q_s&+q_0i'j+q_1i'kI+q_2i'kJ+q_3i'kK+q_4I+q_5J+q_6K+q_7iI \\ &+q_8iJ+q_9iK+q_{10}i'jI+q_{11}i'jJ+q_{12}i'jK+q_{13}i'k+q_{14}i \end{aligned} \quad (2)$$

where $q_n\in\mathbb{R}$.

The application $*:Sp(\mathbb{H}\otimes\mathbb{H})\to Sp(\mathbb{H}\otimes\mathbb{H})$ defined as follows, for any multivector q of grade *k*, $q\mapsto q^*=(-1)^{k(k+1)/2}q$, is called the conjugation and the conjugate of q is

$$\begin{aligned} q^*=q_s&-q_0i'j-q_1i'kI-q_2i'kJ-q_3i'kK-q_4I-q_5J-q_6K-q_7iI \\ &-q_8iJ-q_9iK+q_{10}i'jI+q_{11}i'jJ+q_{12}i'jK+q_{13}i'k+q_{14}i \end{aligned} \quad (3)$$

The conjugation preserves the grade of the multivector *i.e.* if $q\in S_{p_n}(\mathbb{H}\otimes\mathbb{H})$ then $q^*\in S_{p_n}(\mathbb{H}\otimes\mathbb{H})$.

From the product qq^* we define the quadratic form

$$I : Sp(\mathbb{H}\otimes\mathbb{H}) \to \mathbb{R}, q \mapsto I(q) = qq^* = q_s^2 - q_0^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 - q_7^2 - q_8^2 - q_9^2 - q_{10}^2 - q_{11}^2 - q_{12}^2 + q_{13}^2 - q_{14}^2. \quad (4)$$

The norm of the split-tetraquaternion q, denoted N_q, is defined as follows

$$N(q) = \sqrt{I(q)} = \sqrt{qq^*}. \quad (5)$$

In the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$, there are three types of elements according to the sign of the quadratic form $I(q)$:

1) If $I(q) > 0$ then q is said to be a timelike split-tetraquaternion,

2) If $I(q) < 0$ then q is said to be a spacelike split-tetraquaternion,

3) If $I(q) = 0$ then q is said to be a lightlike split-tetraquaternion.

The quadratic form $I : Sp(\mathbb{H}\otimes\mathbb{H}) \to \mathbb{R}, q \mapsto I(q) = qq^*$ defined above is isotropic it means that there exists a nonzero split-tetraquaternion q such that $I(q) = 0$ therefore the algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ isn't a division algebra but it is a split algebra.

We recall that:

1) an algebra is said to be a division algebra if and only if ($I(q) = 0 \Rightarrow q = 0$),

2) an algebra A together with a quadratic form $I : A \to \mathbb{R}$ isotropic *i.e.* there exists a nonzero element $q \in A$ such that $I(q) = 0$ is said to be a split algebra.

It would also be desirable to recall the Frobenius theorem which states the following: "The only division algebras over the field of the real numbers are $\mathbb{R}, \mathbb{C}, \mathbb{H}$ and the algebra of octonions".

Note that a nonzero split-tetraquaternion q admits an inverse $q^{-1} = q^*/I(q)$ and it is obvious that the inverse exists only for timelike and spacelike split-tetraquaternions.

For a vector $q \in S_{p_1}(\mathbb{H}\otimes\mathbb{H}) = \mathbb{R}^{1,3}$ *i.e.* $q = q_0 i'j + q_1 i'kI + q_2 i'kJ + q_3 i'kK$, the inverse exists if and only if $-q_0^2 + q_1^2 + q_2^2 + q_3^2 \neq 0$. In this case, the inverse of vector q is

$$q^{-1} = -q_0 i'j - q_1 i'kI - q_2 i'kJ - q_3 i'kK / -q_0^2 + q_1^2 + q_2^2 + q_3^2. \quad (6)$$

If the vector q is isotropic *i.e.* $-q_0^2 + q_1^2 + q_2^2 + q_3^2 = 0$, we said that q belongs to the lightcone.

The inverse of a bivector $q = q_4 I + q_5 J + q_6 K + q_7 iI + q_8 iJ + q_9 iK$ exists if and only if $q_4^2 + q_5^2 + q_6^2 \neq q_7^2 + q_8^2 + q_9^2$.

3.2. Subalgebras of the Tetraquaternion Algebra

Here, we cannot give a complete list of all subalgebras of $Sp(\mathbb{H}\otimes\mathbb{H})$, we take in account the most remarkable of them in terms of applications in physics and Born geometry which are the biquaternion and split-biquaternion algebras, the quaternion and split-quaternion algebras, the algebra of complex numbers and the split-complex algebra.

3.2.1. Biquaternion Algebra $\mathbb{C}\otimes\mathbb{H}$

The biquaternion algebra $\mathbb{C}\otimes\mathbb{H}$ or Pauli algebra is a subalgebra of $Sp(\mathbb{H}\otimes\mathbb{H})$

generated by the three bivectors $iI = e_{10}$, $iJ = e_{20}$ and $iK = e_{30}$ of the split-tetra-quaternion algebra $Sp(\mathbb{H} \otimes \mathbb{H})$. If denote $\mathcal{E}_i = e_{i0}, i = 1,2,3$ the three generators of the biquaternion algebra $\mathbb{C} \otimes \mathbb{H}$, then $I = \mathcal{E}_{32}$, $J = \mathcal{E}_{13}$, $K = \mathcal{E}_{21}$ are bivectors and $i = \mathcal{E}_{133}$ is pseudoscalar. The table below gives a basis of this algebra.

1	$I = \mathcal{E}_{32}$	$J = \mathcal{E}_{13}$	$K = \mathcal{E}_{21}$
$i = \mathcal{E}_{123}$	$iI = \mathcal{E}_1$	$iJ = \mathcal{E}_2$	$iK = \mathcal{E}_3$

An arbibraty element q of $\mathbb{C} \otimes \mathbb{H}$ is written

$$q = q_s + q_4 I + q_5 J + q_6 K + q_7 iI + q_8 iJ + q_9 iK + q_{14} i \tag{7}$$

and its conjugate is

$$q^* = q_s - q_4 I - q_5 J - q_6 K - q_7 iI - q_8 iJ - q_9 iK + q_{14} i \tag{8}$$

In this case, the quadratic form is

$$I(q) = qq^* = q_s^2 + q_4^2 + q_5^2 + q_6^2 - q_7^2 - q_8^2 - q_9^2 - q_{14}^2. \tag{9}$$

As $I^2 = J^2 = K^2 = 1$, it follows that the biquaternion algebra is isomorphic the Clifford algebra $Cl_{3,0}$ of the Euclidean space $\mathbb{R}^3$.

3.2.2. Split-Biquaternion Algebra $Sp(\mathbb{C} \otimes \mathbb{H})$

The three vectors $i'kI = e_1$, $i'kJ = e_2$ and $i'kK = e_3$ of the split-tetraquaternion algebra $Sp(\mathbb{H} \otimes \mathbb{H})$ generate the split-biquaternion $Sp(\mathbb{C} \otimes \mathbb{H})$, $I = e_{23}$, $J = e_{31}$, $K = e_{12}$ are bivectors and $i'k = e_{132}$ is the pseudoscalar element.

From $(i'kI)^2 = (i'kJ)^2 = (i'kK)^2 = -1$, it follows that the split-biquaternion algebra over $\mathbb{R}$ is isomorphic the Clifford algebra $Cl_{0,3}$ of the pseudo-Euclidean space $\mathbb{R}^{0,3}$. It appears that the passage from the biquaternion algebra $\mathbb{C} \otimes \mathbb{H}$ to the split-biquaternion algebra $Sp(\mathbb{C} \otimes \mathbb{H})$ can be seen as a transformation of the signature from (3, 0) to (0, 3).

A complete table of a basis of the split-biquaternion algebra $Sp(\mathbb{C} \otimes \mathbb{H})$ is given below.

1	$I = e_{23}$	$J = e_{31}$	$K = e_{12}$
$i'k = e_{132}$	$i'kI = e_1$	$i'kJ = e_2$	$i'kK = e_3$

Let q be an element of $Sp(\mathbb{C} \otimes \mathbb{H})$.

$$q = q_s + q_1 i'kI + q_2 i'kJ + q_3 i'kK + q_4 I + q_5 J + q_6 K + q_{13} i'k \tag{10}$$

its conjugate is

$$q^* = q_s - q_1 i'kI - q_2 i'kJ - q_3 i'kK - q_4 I - q_5 J - q_6 K + q_{13} i'k \tag{11}$$

and

$$I(q) = qq^* = q_s^2 + q_1^2 + q_2^2 + q_3^2 + q_4^2 + q_5^2 + q_6^2 + q_{13}^2. \tag{12}$$

3.2.3. Quaternion Algebra $\mathbb{H}$

Two of the three bivectors $I = e_{23}$, $J = e_{31}$ and $K = e_{12}$ of the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ generate the quaternion algebra $\mathbb{H}$, we select the bivectors I and J and we denote $\mathcal{E}_1 = I$, $\mathcal{E}_2 = J$ the generators of the quaternion algebra $\mathbb{H}$. We remark that $K = IJ = \mathcal{E}_{12}$ is a bivector and $I^2 = J^2 = K^2 = IJK = -1$.

A quaternion can be written

$$q = q_s + q_4 I + q_5 J + q_6 K, \tag{13}$$

its conjugate is

$$q^* = q_s - q_4 I - q_5 J - q_6 K \tag{14}$$

and the quadratic form $I:\mathbb{H}\to\mathbb{R}, q\mapsto I(q) = qq^* = q_s^2 + q_4^2 + q_5^2 + q_6^2$.

We remark that the quadratic form $I:\mathbb{H}\to\mathbb{R}$ defined above is anisotropic *i.e.*

$I(q) = qq^* = q_s^2 + q_4^2 + q_5^2 + q_6^2 = 0 \Rightarrow q = 0$ it follows that the quaternion algebra $\mathbb{H}$ is a division algebra and nonzero quaternions are invertible.

3.2.4. Split-Quaternion Algebra $Sp\mathbb{H}$

It would be appropriate to point out the existence of two split-quaternion algebras isomorphic respectively to the Clifford algebras $Cl_{2,0}$ and $Cl_{1,1}$.

1) Split-quaternion algebra $Sp(\mathbb{H}) \cong Cl_{2,0}$

The bivectors $iJ = e_{10}, iJ = e_{20}$ of the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ can be considered as the generators of the split-quaternion $Sp(\mathbb{H})$, the product $(iJ)(iI) = K$ is the pseudoscalar denoted $K = \mathcal{E}_{21}$ when $\mathcal{E}_1 = iI$ and $\mathcal{E}_2 = iJ$.

As $(iI)^2 = (iJ)^2 = 1$, it is obvious that the split-quaternion algebra over $\mathbb{R}$ is isomorphic the Clifford algebra $Cl_{2,0}$ of the Euclidean space $\mathbb{R}^2$.

A split-quaternion is

$$q = q_s + q_6 K + q_7 iI + q_8 iJ \tag{15}$$

its conjugate is

$$q^* = q_s - q_6 K - q_7 iI - q_8 iJ \tag{16}$$

and the quadratic form $I:Sp(\mathbb{H})\to\mathbb{R}, q\mapsto I(q) = qq^* = q_s^2 + q_6^2 - q_7^2 - q_8^2$.

Hence, the split-quaternion $Sp(\mathbb{H})$ isn't a division algebra.

2) Split-quaternion algebra $Sp(\mathbb{H}) \cong Cl_{1,1}$

The vectors $i'j = e_0$ and $i'kI = e_1$ of the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ generate of the 4D-dimensional algebra, named the split-quaternion $Sp(\mathbb{H})$. The multivetor structure of the algebra $Sp(\mathbb{H})$ is given in the table below:

1		
$iI = e_{10}$	$i'j = e_0$	$i'kI = e_1$

The product $(i'kI)(i'j) = iI$ is a bivector, specially the pseudoscalar of the algebra. From $(i'j)^2 = 1$ and $(i'kI)^2 = -1$, it follows that the split-quaternion algebra over $\mathbb{R}$ is isomorphic the Clifford algebra $Cl_{1,1}$ of the pseudo-Euclidean space $\mathbb{R}^{1,1}$.

A split-quaternion is

$$q = q_s + q_0 i'j + q_1 i'kI + q_7 iI \tag{15}$$

its conjugate is

$$q^* = q_s - q_0 i'j - q_1 i'kI - q_7 iI \tag{16}$$

and the quadratic form $I : Sp(\mathbb{H}) \to \mathbb{R}, q \mapsto I(q) = qq^* = q_s^2 - q_0^2 + q_1^2 + q_7^2$.

It is obvious that the split-quaternion $Sp(\mathbb{H})$ isn't a division algebra.

3.2.5. Algebra of Complex Numbers $\mathbb{C}$ and Split-Complex Algebra $Sp(\mathbb{C})$

The bivectors $I = e_{23}$ of the split-tetraquaternion algebra $Sp(\mathbb{H} \otimes \mathbb{H})$ generates the algebra of complex numbers $\mathbb{C}$ which is isomorphic the Clifford algebra $Cl_{0,1}$ of the pseudo-Euclidean space $\mathbb{R}^{0,1}$. Similarly, the bivectors $iI = e_{01}$ of the split-tetraquaternion algebra $Sp(\mathbb{H} \otimes \mathbb{H})$ generates the split-complex algebra $Sp(\mathbb{C})$ which is isomorphic the Clifford algebra $Cl_{1,0}$ of the Euclidean space $\mathbb{R}^{1,0}$.

A split-complex number is

$$q = q_s + q_7 iI \tag{17}$$

its conjugate is

$$q^* = q_s - q_7 iI \tag{18}$$

and the quadratic form $I : Sp(\mathbb{C}) \to \mathbb{R}, q \mapsto I(q) = qq^* = q_s^2 - q_7^2$.

It is obvious that the split-complex algebra $Sp(\mathbb{C})$ isn't a division algebra.

4. Applications

4.1. Some Applications in Physics

As applications in physics, the special theory of relativity, the classical electromagnetism, the general theory of relativity and the quantum theory are developed in the tetraquaternion algebra $\mathbb{H} \otimes \mathbb{H}$ and in the biquaternion algebra $\mathbb{C} \otimes \mathbb{H}$ [8].

4.2. Stabilized Poincaré-Heisenberg Algebra

In order to obtain a structure of Lie algebra on the split-tetraquaternion algebra, we deform this product as follows: for any q, q' in $Sp(\mathbb{H} \otimes \mathbb{H})$, $[q, q'] = qq' - q'q$. The split-tetraquaternion algebra $Sp(\mathbb{H} \otimes \mathbb{H})$ endowed with the product $[,]$, denoted $(Sp(\mathbb{H} \otimes \mathbb{H}), [,])$ is called the stabilized Poincaré-Heinsenberg algebra [9].

Here, we calculate the product of some elements of the stabilized Poincaré-Heinsenberg algebra $(Sp(\mathbb{H} \otimes \mathbb{H}), [,])$.

1) Product of two vectors

A direct calculation of the non-trivial products of vector leads to:

$$[i'j, i'kI] = 0,\ [i'j, i'kJ] = 0,\ [i'j, i'kK] = 0$$

$$[i'kI, i'kJ] = 2K,\ [i'kI, i'kK] = -2J,\ [i'kJ, i'kK] = 2I.$$

From the above, it follows that the product of two vectors of $Sp(\mathbb{H}\otimes\mathbb{H})$ is either zero or is a bivector.

2) Product of two bivectors

$$[I,J]=2K,\ [I,K]=-2J,\ [J,K]=2I$$

$$[I,iJ]=2iK,\ [I,iK]=-2iJ,\ [J,iK]=2iI,\cdots$$

The split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ is a semi-simple algebra and therefore it is a stable algebra.

4.3. Applications in Born Geometry

Without going in the details of the double structures, we give an application of some subalgebras of the split-tetraquaternion in Born geometry. The table below show the correspond between some subalgebras of the split-tetraquaternion algebra $Sp(\mathbb{H}\otimes\mathbb{H})$ and the algebra of double structures, ([10] p. 17).

Algebra of double structure	Subalgebras of $Sp(\mathbb{H}\otimes\mathbb{H})$
Generalized hyperkahler	$Sp(\mathbb{C}\otimes\mathbb{H})$
Born	$Sp(\mathbb{H})$
Born and generalized Kahler	$Sp(\mathbb{C}\otimes\mathbb{H})$
Born and generalized hyperkahler	$Sp(\mathbb{H}\otimes\mathbb{H})$

5. Conclusion

The study of the split-tetraquaternion algebra and some of its sub-algebras shows that some classes of Clifford algebras $Cl_{p,q}$ with $p-q=0,1,5(\mathrm{mod}\,8)$ can be represented by using the quaternionic formulation without using matrix representation. The fact that the split-tetraquaternion algebra is a stable algebra allows to describe the well-known stabilized Poincaré-Heisenberg algebra which is a Lie algebra obtained by defining the commutators of the non-scalars basis elements. The stabilized Poincaré-Heisenberg algebra is useful in quantum mechanics and in relativity. We list the algebras of double structure which can be approach by mean of the sub-algebras of $Sp(\mathbb{H}\otimes\mathbb{H})$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Calvet, R.G. (2017) On Matrix Representations of Geometric (Clifford) Algebras. *Journal of Geometry and Symmetry in Physics*, **43**, 1-36. https://doi.org/10.7546/jgsp-43-2017-1-36

[2] Girard, P.R., Clarysse, P., Pujol, R., Goutte, R. and Delachartre, P. (2018) Hyperquaternions: A New Tool for Physics. *Advances in Applied Clifford Algebras*, **28**, Article No. 68. https://doi.org/10.1007/s00006-018-0881-8

[3] Girard, P.R., Clarysse, P., Pujol, R., Goutte, R. and Delachartre, P. (2019) Hyperquaternions: An Efficient Mathematical Formalism for Geometry. In: Nielsen, F. and Barbaresco, F., Eds., *Geometric Science of Information*, Springer, 116-125. https://doi.org/10.1007/978-3-030-26980-7_13

[4] Girard, P.R., Clarysse, P., Pujol, R., Wang, L. and Delachartre, P. (2015) Differential Geometry Revisited by Biquaternion Clifford Algebra. In: Boissonnat, J.-D., *et al.*, Eds., *Curves and Surfaces*, Springer International Publishing, 216-242. https://doi.org/10.1007/978-3-319-22804-4_17

[5] Girard, P.R., Clarysse, P., Pujol, R., Goutte, R. and Delachartre, P. (2021) Hyperquaternion Poincaré Groups. *Advances in Applied Clifford Algebras*, **31**, Article No. 15. https://doi.org/10.1007/s00006-021-01120-z

[6] Girard, P.R., Clarysse, P., Pujol, R., Goutte, R. and Delachartre, P. (2021) Hyperquaternion Conformal Groups. *Advances in Applied Clifford Algebras*, **31**, Article No. 56. https://doi.org/10.1007/s00006-021-01159-y

[7] Girard, P.R., Pujol, R., Clarysse, P. and Delachartre, P. (2023) Hyperquaternions and Physics. *SciPost Physics Proceedings*, **14**, Article No. 030. https://doi.org/10.21468/scipostphysproc.14.030

[8] Girard, P.R. (2007) Quaternions, Clifford Algebras and Relativistic Physics. Birkhauser.

[9] Gresnigt, N.G., Renaud, P.F. and Butler, P.H. (2007) The Stabilized Poincare-Heisenberg Algebra: A Clifford Algebra Viewpoint. *International Journal of Modern Physics D*, **16**, 1519-1529. https://doi.org/10.1142/s0218271807010857

[10] Kimura, T., Sasaki, S. and Shiozawa, K. (2022) Complex Structures, T-Duality and Worldsheet Instantons in Born Sigma Models. *Journal of High Energy Physics*, **2022**, Article No. 119. https://doi.org/10.1007/jhep06(2022)119