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The Strange Relationship between the Momentum of a Photon Emitted from an Electron and the Momentum Acquired by the Electron

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Abstract

In quantum mechanics, the energy of a hydrogen atom is minimized when the principal quantum number n is 1. However, the author has previously pointed out that the hydrogen atom has a state where $n=0$. An electron in the state where $n=0$ has zero rest mass energy. However, a hydrogen atom has an energy level even lower than the $n=0$ state. This is hard to accept from the standpoint of common sense. Thus, the author has previously pointed out that an electron at the energy level where $n=0$ has zero energy because the positive energy $m_{\text{e}}c^2$ and negative energy $-m_{\text{e}}c^2$ cancel each other out. This paper elucidates the strange relationship between the momentum of a photon emitted when a hydrogen atom is formed by an electron with such characteristics, and the momentum acquired by the electron.

Keywords

Einstein's Energy-Momentum Relationship, Energy-Momentum Relationship in a Hydrogen Atom, Momentum of a Photon, Momentum of a Electron, Negative Energy Specific to the Electron, Dark Matter

1. Introduction

In this paper, the relationship between the momentum of a photon emitted from an electron and the momentum acquired by the electron is examined by using a hydrogen atom as an example.

Now, let us consider the case where a single electron placed at a point in free space is attracted by the electrical attraction of a proton, and a hydrogen atom is formed. If the electron is attracted by the proton (nucleus of the hydrogen atom) and it enters the region of the hydrogen atom, then photon energy $h\nu$ will be emitted. At this time, the electron acquires kinetic energy *K*. These energy sources are necessary because the electron emits energy and acquires kinetic energy.

For the law of conservation of energy to hold, the following relation must hold between these energies.

$$V(r)+h\nu+K=0,\ V(r)<0. \tag{1}$$

Here, $V(r)$ is the potential energy of the electron. Potential energy is the energy source for the emitted photon energy and the kinetic energy acquired by the electron. According to classical quantum theory, the relationship between these energies is as follows.

$$h\nu=K=-\frac{1}{2}V(r). \tag{2}$$

It seems natural for the values of $h\nu$ and K to be equal. However, what would happen if this were not energy but momentum? Is the momentum of a photon emitted from an electron at rest equal to the momentum acquired by the electron?

This paper discusses this problem via two approaches. First, the problem is solved by finding the momentum of the electron from the energy-momentum relationship applicable to electrons in a hydrogen atom. Second, the problem is solved by placing the energy and momentum of the electron into correspondence with line segments of an ellipse.

2. Energy-Momentum Relationship Applicable to an Electron in a Hydrogen Atom

According to the special theory of relativity, the following relation holds between the energy and momentum of a body moving in free space [1].

$$\left(mc^2\right)^2=\left(m_0c^2\right)^2+c^2p^2. \tag{3}$$

Here, m_0c^2 is the rest mass energy of the body. And mc^2 is the relativistic energy.

Incidentally, Einstein and Sommerfeld defined the relativistic kinetic energy as follows [2].

$$K_{\text{re}}=mc^2-m_0c^2. \tag{4}$$

The "re" subscript of K_{re} stands for "relativistic."

Taking Formula (4) into account, Formula (3) can be rewritten as follows.

$$\begin{aligned} c^2p^2&=\left(mc^2-m_0c^2\right)\left(mc^2+m_0c^2\right)\\ &=K_{\text{re}}\left(mc^2+m_0c^2\right). \end{aligned} \tag{5}$$

From this, the following formula for relativistic kinetic energy can be derived.

$$K_{\text{re}}=\frac{p_{\text{re}}^2}{m+m_0}. \tag{6}$$

Here, the subscript "re" is attached to *p*, just as in K_{re}.

Incidentally, Einstein's energy-momentum relationship (3) holds when the energy absorbed by a body is all converted to kinetic energy of that body. However, an electron in an atom acquires kinetic energy through emission of energy. Therefore, Einstein's relationship (3) cannot be applied to an electron in an atom.

In classical quantum theory, the total mechanical energy of a hydrogen atom is defined as the sum of the kinetic energy and potential energy of the electron. That is,

$$E_n = K_n + V\left(r_n\right) = -K_n, \ E_n < 0. \tag{7}$$

Here, n is the principal quantum number.

If we let $E_{\mathrm{ph},n}$ (a different expression for $h\nu$) be the energy emitted when an electron outside an atom drops to an energy level whose principal quantum number is n inside of a hydrogen atom, then the relationship between $E_{\mathrm{ph},n}$ and other energy is as follows.

$$E_{\mathrm{ph},n} = h\nu = K_{\mathrm{re},n} = -E_{\mathrm{re},n}. \tag{8}$$

Here, the "ph" subscript of $E_{\mathrm{ph},n}$ stands for "photon." Also, the "re" subscript of $E_{\mathrm{re},n}$ stands for "relativistic." $E_{\mathrm{re},n}$ are the relativistic energy levels of a hydrogen atom.

The relationship between the rest mass energy of the electron $m_{\mathrm{e}}c^2$ and the photon energy of the electron $E_{\mathrm{ab},n}$ (or $m_n c^2$) or $E_{\mathrm{re},n}$ is as follows.

$$E_{\mathrm{ab},n} = m_n c^2 = m_{\mathrm{e}}c^2 + V\left(r_n\right) + K_{\mathrm{re},n}. \tag{9}$$

$$E_{\mathrm{ab},n} = m_{\mathrm{e}}c^2 + E_{\mathrm{re},n}. \tag{10}$$

$$E_{\mathrm{re},n} = -K_{\mathrm{re},n}. \tag{11}$$

$$E_{\mathrm{ab},n} + E_{\mathrm{ph},n} = m_{\mathrm{e}}c^2. \tag{12}$$

Here, $E_{\mathrm{ab},n}\left(=m_n c^2\right)$ is the sum of the residual part of the rest mass energy of the electron $\left(m_{\mathrm{e}}c^2 + V\left(r_n\right)\right)$ and the kinetic energy $K_{\mathrm{re},n}$. Also, $E_{\mathrm{re},n}$ corresponds to the reduction in rest mass energy of the electron. The "ab" subscript of $E_{\mathrm{ab},n}$ stands for "absolute."

Here, the relativistic kinetic energy of an electron inside a hydrogen atom is defined as follows by referring to Formulas (4) and (6).

$$K_{\mathrm{re},n} = m_{\mathrm{e}}c^2 - m_n c^2. \tag{13}$$

$$K_{\mathrm{re},n} = \frac{p_{\mathrm{e},n}^2}{m_{\mathrm{e}} + m_n}. \tag{14}$$

Here, $p_{\mathrm{e},n}$ is the relativistic momentum of the electron. (Due to the situation in this paper, $p_{\mathrm{re},n}$ is rewritten here as $p_{\mathrm{e},n}$.)

Linking the right sides of Formulas (13) and (14) with an equals sign and rearranging, the following relationship can be derived [3] [4].

$$\left(m_n c^2\right)^2 + c^2 p_{e,n}^2 = \left(m_e c^2\right)^2. \tag{15}$$

This energy-momentum relationship is applicable to an electron inside a hydrogen atom.

Next, if Formula (15) is solved for $m_n c^2$, then the following formula can be derived.

$$m_n c^2 = \frac{m_e c^2}{\left(1 + \frac{v_n^2}{c^2}\right)^{1/2}}. \tag{16}$$

To change Formula (16) into a formula of quantum theory, the discreteness of energy must be incorporated into Formula (16).

Previously, the author has shown that the following relationship holds for an electron in a hydrogen atom [5] [6].

$$\frac{v_n}{c} = \frac{\alpha}{n}. \tag{17}$$

Here, α is the following fine-structure constant.

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = 7.2973525693 \times 10^{-3}. \tag{18}$$

Using the relationship in Formula (17), Formula (16) can be written as follows.

$$m_n c^2 = m_e c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}. \tag{19}$$

Incidentally, there are positive and negative solutions to Einstein's relationship (3). In the same way, Formula (15) also has the following positive and negative solutions [7] [8].

$$E_{ab,n}^{+} = m_n c^2 = m_e c^2 + E_{re,n} = m_e c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}, \quad n = 0, 1, 2, \cdots. \tag{20}$$

$$E_{ab,n}^{-} = -m_n c^2 = -m_e c^2 \left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2}, \quad n = 0, 1, 2, \cdots. \tag{21}$$

Thus, the energy levels of a hydrogen atom $E_{re,n}$ are given by the following formula.

$$E_{re,n} = m_n c^2 - m_e c^2 = m_e c^2 \left[\left(\frac{n^2}{n^2 + \alpha^2}\right)^{1/2} - 1\right], \quad n = 0, 1, 2, \cdots. \tag{22}$$

Now, Formula (20) absolutely and relativistically describes the photon energy of an electron constituting a hydrogen atom. In contrast, Formula (21) indicates previously unknown energy levels.

It is strange that negative energy levels exist even though energy is described with an absolute scale. To resolve this contradiction, the author has previously predicted the existence of photons with negative energy (**Figure 1**) [9]-[11].

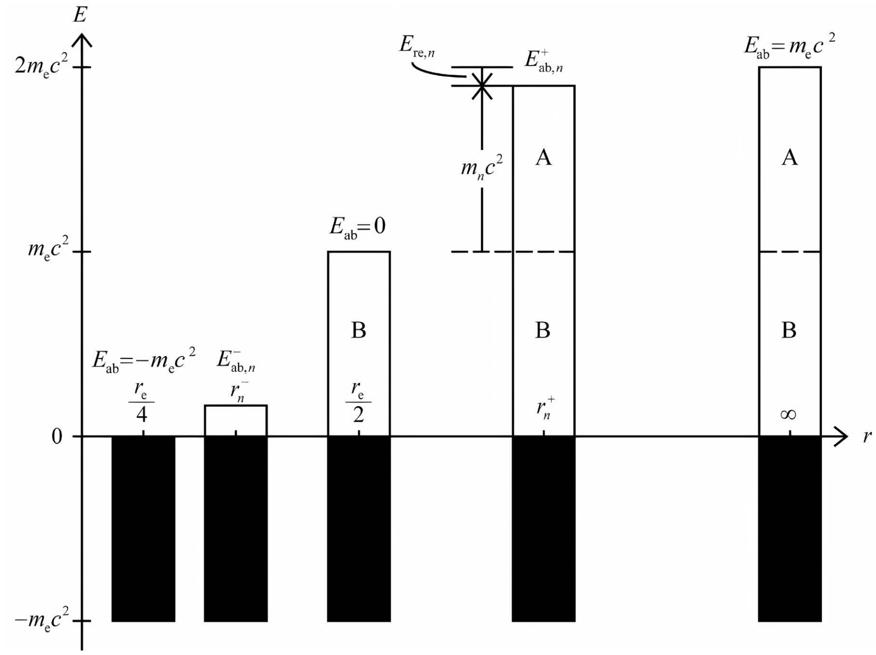

Figure 1. Photon energies of electrons in different states, and negative energy.

The photon energy of an electron corresponds to the white rectangle in the diagram (A + B). The sum of the residual part of the rest mass energy of the electron which decreased $\left(m_{\mathrm{e}}c^2 + V\left(r_n\right)\right)$ and the kinetic energy acquired by the electron $K_{\mathrm{re},n}$ corresponds to the $m_n c^2$ part. Energy A is an energy we understand well. This paper asserts the existence of the B part, but this is photon energy $m_{\mathrm{e}}c^2$ that is still not understood. Also, the negative energy specific to the electron $-m_{\mathrm{e}}c^2$ corresponds to the black rectangle. A is the only energy apprehended by modern physics. However, if the B energy and negative energy do not exist, then an electron cannot drop to the energy level in Formula (21). This figure shows that the original photon energy of an electron with rest mass energy $m_{\mathrm{e}}c^2$ is $2m_{\mathrm{e}}c^2$. (However, this figure is just a conceptual illustration. The r coordinate on the x-axis is not accurate.)

Incidentally, Daviau, C. has already discussed the cloud of photons of an electron. For details, please see that paper [12].

The author has previously pointed out that matter formed from a proton (hydrogen atom nucleus) and an electron at this ultra-low energy level (21) is the true nature of dark matter, a source of gravity whose true nature is currently unknown [13] [14]. The author has also given the name "dark hydrogen atoms" (DHA) to hydrogen atoms at this ultra-low energy level.

An electron with negative mass forming DHA exists near the atomic nucleus (proton) [15] [16].

Next, if the electron orbital radii corresponding to the energy levels in Formulas (20) and (21) are taken to be, respectively, r_n^+ and r_n^-,

$$r_n^+ = \frac{r_{\mathrm{e}}}{2}\frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2}-n}. \tag{23}$$

$$r_n^- = \frac{r_e}{2} \frac{\left(n^2+\alpha^2\right)^{1/2}}{\left(n^2+\alpha^2\right)^{1/2}+n}. \tag{24}$$

Formulas (23) and (24) can be written as follows [17].

$$r_n^+ = \frac{r_e}{2}\left[1+\frac{n}{\left(n^2+\alpha^2\right)^{1/2}-n}\right]. \tag{25}$$

$$r_n^- = \frac{r_e}{2}\left[1-\frac{n}{\left(n^2+\alpha^2\right)^{1/2}+n}\right]. \tag{26}$$

Here, r_e is the classical electron radius, defined as follows.

$$r_e = \frac{e^2}{4\pi\varepsilon_0 m_e c^2}. \tag{27}$$

Now, the following ratio is obtained from Formulas (23) and (24).

$$\frac{r_n^-}{r_n^+} = \frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}. \tag{28}$$

Here, if we set $n=1$,

$$\frac{r_1^-}{r_1^+} = \frac{\left(1+\alpha^2\right)^{1/2}-1}{\left(1+\alpha^2\right)^{1/2}+1} = 1.3312484168\times10^{-5} \approx \frac{1}{75120}. \tag{29}$$

Also, if the radius of the proton r_p is assumed to be $r_e/4$, then the ratio of r_p and the maximum radius of a DHA r_1^- is as follows.

$$\frac{r_1^-}{r_p} = \frac{r_e}{2}\frac{\left(1+\alpha^2\right)^{1/2}}{\left(1+\alpha^2\right)^{1/2}+1}\cdot\frac{4}{r_e} = 1.0000133124. \tag{30}$$

The following shows classical illustrations of an ordinary hydrogen atom and a DHA (**Figure 2**).

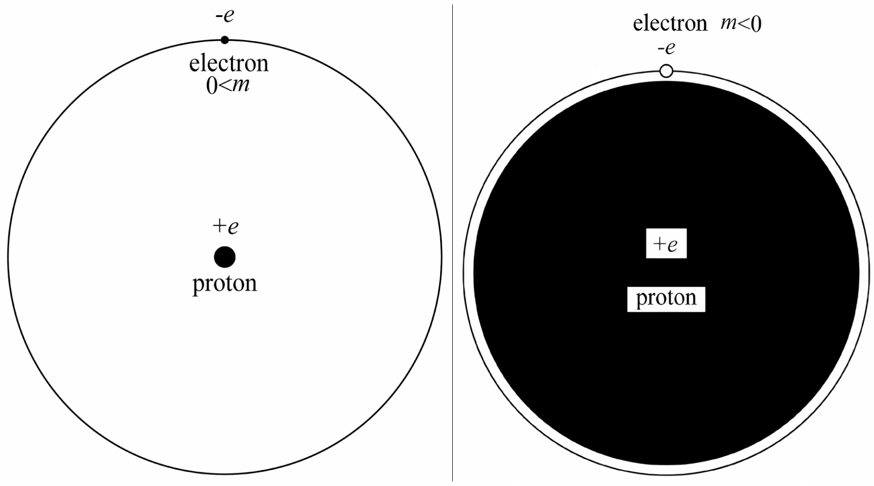

Figure 2. Classical illustrations of a hydrogen atom and a dark hydrogen atom (DHA).

The figure at left is a classical illustration of an ordinary hydrogen atom. The distance from the center of the atomic nucleus to the electron is r_n^+. In contrast, the figure at right is an illustration of a DHA at the ultra-low energy level. The distance from the center of the atomic nucleus to the electron is r_n^-. As is evident from Formula (30), an electron with negative mass which forms a DHA is present near the proton (black circle part). It can be predicted that a DHA is matter extremely similar to a neutron.

3. Physical Quantities of Electrons Revealed by Considering an Ellipse

Formula (15) can be derived through considerations using an ellipse. This problem has already been discussed [10]. However, in this section, we will advance the discussion further.

Let A and A′ be the points where the ellipse intersects the x-axis, and let B and B′ be the points where the ellipse intersects the y-axis. Also, let $2a$ be the length of the line segment $\overline{\mathrm{AA'}}$, $2b$ be the length of the line segment $\overline{\mathrm{BB'}}$, and $2f$ be the length of the line segment $\overline{\mathrm{FF'}}$ (**Figure 3**).

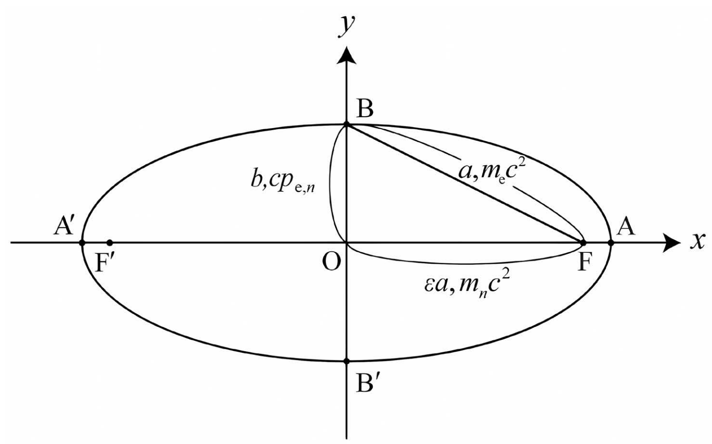

Figure 3. First, the energy $m_e c^2$ is taken to correspond to the line segment $\overline{\mathrm{OA}}$, and then Formula (35) is assumed. Formula (15) can be derived if the Pythagorean theorem is applied to the right triangle OBF.

The eccentricity of the ellipse in this case is defined as follows.

$$\varepsilon = \frac{f}{a}. \tag{31}$$

The eccentricity of the ellipse can also be expressed using the following formula.

$$\varepsilon = \left(1 - \frac{b^2}{a^2}\right)^{1/2}. \tag{32}$$

The following formula can be derived from Formula (32).

$$b = a\left(1 - \varepsilon^2\right)^{1/2}. \tag{33}$$

Here, the line segment $\overline{\mathrm{OA}}$ is placed into correspondence with the energy $m_{\mathrm{e}}c^2$. Let us express this as follows.

$$a = m_{\mathrm{e}}c^2. \tag{34}$$

Also, it is assumed that the shape of an ellipse that can describe the state of an electron is the case satisfying the following conditions.

$$\varepsilon = \left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}. \tag{35}$$

Taking Formulas (34) and (35) into account, b can be expressed with the following formula.

$$b = m_{\mathrm{e}}c^2\left(\frac{\alpha^2}{n^2+\alpha^2}\right)^{1/2}. \tag{36}$$

Also, if Formula (34) and Formula (35) are taken into account, then the f in Formula (31) is as follows.

$$f = a\varepsilon = m_{\mathrm{e}}c^2\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}. \tag{37}$$

Using Formulas (36) and (37),

$$f^2+b^2 = \left(\frac{n^2}{n^2+\alpha^2}\right)\left(m_{\mathrm{e}}c^2\right)^2 + \left(\frac{\alpha^2}{n^2+\alpha^2}\right)\left(m_{\mathrm{e}}c^2\right)^2 = \left(m_{\mathrm{e}}c^2\right)^2. \tag{38}$$

Here, $\mathrm{BF} = a$ so the following relationship holds.

$$\overline{\mathrm{OA}} = \overline{\mathrm{BF}}. \tag{39}$$

Here, if Formula (19) is also taken into consideration, then Formula (36) can be expressed as follows.

$$b^2 = \left(\frac{\alpha^2}{n^2+\alpha^2}\right)m_{\mathrm{e}}^2c^2\cdot c^2 = \left(\frac{\alpha^2}{n^2+\alpha^2}\right)m_n^2\left(\frac{n^2+\alpha^2}{n^2}\right)c^2\cdot c^2. \tag{40}$$

Furthermore, if the relationship in Formula (17) is used for c in Formula (40), the result is as follows.

$$b^2 = \frac{\alpha^2}{n^2}\cdot m_n^2\frac{n^2v_n^2}{\alpha^2}c^2 = c^2\left(m_nv_n\right)^2 = c^2p_n^2. \tag{41}$$

Substituting this result for Formula (41) into Formula (38),

$$f^2+b^2 = \left(m_nc^2\right)^2 + c^2p_n^2 = \left(m_{\mathrm{e}}c^2\right)^2. \tag{42}$$

When $m_{\mathrm{e}}c^2$ is taken to correspond to the line segment $\overline{\mathrm{OA}}$, and Formula (32) is assumed, then Formula (15) can be derived from the right triangle OBF.

Incidentally, the author has previously derived the following relation in ref. [14].

$$\frac{r_n^-}{r_n^+} = \frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n} = \frac{m_{\mathrm{e}}-m_n}{m_{\mathrm{e}}+m_n}. \tag{43}$$

However, when considered using an ellipse, Formula (43) can be expanded as follows.

$$\frac{\overline{\mathrm{A'F'}}}{\overline{\mathrm{A'F}}}=\frac{a-f}{a+f}=\frac{1-\varepsilon}{1+\varepsilon}=\frac{r_n^-}{r_n^+}=\frac{\left(n^2+\alpha^2\right)^{1/2}-n}{\left(n^2+\alpha^2\right)^{1/2}+n}=\frac{m_\mathrm{e}-m_n}{m_\mathrm{e}+m_n}. \tag{44}$$

Also, the following relation can be obtained from **Figure 3**.

$$\frac{\overline{\mathrm{OF}}}{\overline{\mathrm{OA}}}=\varepsilon=\frac{m_n}{m_\mathrm{e}}=\left(\frac{n^2}{n^2+\alpha^2}\right)^{1/2}. \tag{45}$$

Here, the relationship between energy and r in the ellipse is represented in the following diagram (**Figure 4**).

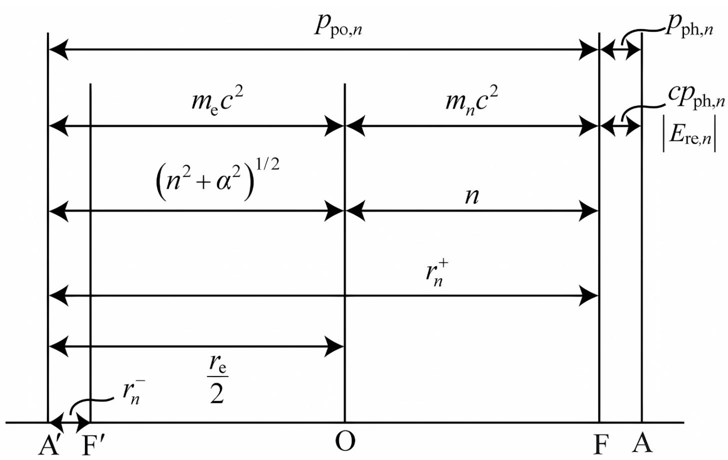

Figure 4. Caution is necessary because the lengths of the line segments in this diagram do not give the absolute value of physical quantities. The figure also includes $p_{\mathrm{po},n}$ defined with Formula (46).

To prepare for the following section, here we introduce the momentum $p_{\mathrm{po},n}$ corresponding to the line segment A′F. $p_{\mathrm{po},n}$ is defined as follows as the latent momentum of the photon of an electron at the m_nc^2 energy level.

$$p_{\mathrm{po},n}=\left(m_\mathrm{e}+m_n\right)c. \tag{46}$$

Here the subscript "po" stands for "potential".

4. Relationship of the Momentum of a Photon Emitted by an Electron and the Momentum Acquired by the Electron

This section derives the relationship between the momentum of a photon emitted $p_{\mathrm{ph},n}$ and the momentum acquired by an electron that was at rest $p_{\mathrm{ph},n}$ when the electron that was at rest is taken into a hydrogen atom. According to Maxwell, if the momentum of a photon is represented as $p_{\mathrm{ph},n}$, then the photon energy $E_{\mathrm{ph},n}$ is given by the following formula.

$$E_{\mathrm{ph},n}=h\nu=cp_{\mathrm{ph},n}. \tag{47}$$

Also, there is the following relationship between $p_{\mathrm{ph},n}$ and the electron's kinetic energy $K_{\mathrm{re},n}$.

$$cp_{\mathrm{ph},n}=K_{\mathrm{re},n}=m_{\mathrm{e}}c^{2}-m_{n}c^{2}. \tag{48}$$

From Formula (48), the photon's momentum is as follows.

$$p_{\mathrm{ph},n}=m_{\mathrm{e}}c-m_{n}c. \tag{49}$$

Next, the relativistic momentum of the electron $p_{\mathrm{e},n}$ is derived.

First, the following formula is obtained from Formula (15).

$$p_{\mathrm{e},n}^{2}=\left(m_{\mathrm{e}}c-m_{n}c\right)\left(m_{\mathrm{e}}c+m_{n}c\right). \tag{50}$$

Hence $p_{\mathrm{e},n}$ is:

$$p_{\mathrm{e},n}=\left(m_{\mathrm{e}}c-m_{n}c\right)^{1/2}\left(m_{\mathrm{e}}c+m_{n}c\right)^{1/2}. \tag{51}$$

Also, Formula (49) can be written as follows.

$$p_{\mathrm{ph},n}=\left(m_{\mathrm{e}}c-m_{n}c\right)^{1/2}\left(m_{\mathrm{e}}c-m_{n}c\right)^{1/2}. \tag{52}$$

Here, taking the ratio of Formulas (51) and (52) yields the following.

$$\frac{p_{\mathrm{ph},n}}{p_{\mathrm{e},n}}=\left(\frac{m_{\mathrm{e}}-m_{n}}{m_{\mathrm{e}}+m_{n}}\right)^{1/2}. \tag{53}$$

Formula (53) can be written as follows using the relationship in Formula (43).

$$\frac{p_{\mathrm{ph},n}}{p_{\mathrm{e},n}}=\left[\frac{\left(n^{2}+\alpha^{2}\right)^{1/2}-n}{\left(n^{2}+\alpha^{2}\right)^{1/2}+n}\right]^{1/2}. \tag{54}$$

Now, taking into account Formulas (46) and (49), Formula (50) can be written as follows.

$$p_{\mathrm{e},n}^{2}=p_{\mathrm{po},n}p_{\mathrm{ph},n}. \tag{55}$$

Also, Formula (55) can be written as follows, taking into account Formula (28).

$$\frac{p_{\mathrm{e},n}}{p_{\mathrm{po},n}}=\frac{p_{\mathrm{ph},n}}{p_{\mathrm{e},n}}=\left(\frac{r_{n}^{-}}{r_{n}^{+}}\right)^{1/2}. \tag{56}$$

Here, $p_{\mathrm{ph},n}$ is the momentum of a photon emitted when an electron with rest mass energy $m_{\mathrm{e}}c^{2}$ is taken into a hydrogen atom. $p_{\mathrm{e},n}$ is the momentum acquired by the electron. Also, as is evident from **Figure 1** and **Figure 4**, the photon energy of an electron in a state where the principal quantum number is n is $\left(m_{\mathrm{e}}+m_{n}\right)c^{2}$ not $m_{n}c^{2}$. $p_{\mathrm{po},n}$ is the momentum defined by also taking into account the electron's latent photon.

Although the importance is unclear, a previously unknown formula for energy levels in a hydrogen atom can be derived from Formulas (48) and (55).

That is,

$$E_{\mathrm{re},n}=-cp_{\mathrm{ph},n}=-c\cdot\frac{p_{\mathrm{e},n}^{2}}{p_{\mathrm{po},n}}. \tag{57}$$

In addition, Butto, N. has also discussed electron spin when discussing momentum of the electron. However, electron spin is not incorporated into the formula derived in this paper. Therefore, Formula (55) is important, but it may not be the final formula [18].

5. Discussion

The following law of energy conservation holds for an electron in a hydrogen atom.

$$m_e c^2 = m_n c^2 + c p_{ph,n}. \tag{58}$$

Next, let us rewrite Formula (58) into a formula including not photon momentum $p_{ph.n}$ but electron momentum $p_{e.n}$.

Here, Formula (58) becomes as follows when Formula (57) is applied.

$$m_e c^2 = m_n c^2 + c \cdot \frac{p_{e,n}^2}{p_{po,n}}. \tag{59}$$

Also, Formula (59) turns into the following when the definition of $p_{po.n}$ in Formula (46) is used.

$$m_e c^2 = m_n c^2 + c \cdot \frac{p_{e,n}^2}{(m_e + m_n)c.}. \tag{60}$$

Hence, Formula (60) can be written as follows.

$$m_e c^2 = m_n c^2 + \frac{c^2 p_{e,n}^2}{(m_e + m_n)c^2}. \tag{61}$$

Rearranging, the following formula can be derived.

$$\left(m_n c^2\right)^2 + c^2 p_{e,n}^2 = \left(m_e c^2\right)^2. \tag{62}$$

Formulas (59) and (62) stand on the same footing. Therefore, the photon energy when an electron with rest mass energy of $m_e c^2$ is at rest is actually $2m_e c^2$ not $m_e c^2$, and thus Formula (62) holds.

6. Conclusions

The emitted photon energy $h\nu$ and momentum $p_{ph,n}$ when an electron at rest is taken into a hydrogen atom are given by the following formulas.

$$h\nu = E_{ph,n} = m_e c^2 - m_n c^2. \tag{63}$$

$$p_{ph,n} = m_e c - m_n c. \tag{64}$$

In contrast, the kinetic energy and momentum acquired by the electron are given by the following formulas.

$$K_{re,n} = -E_{re,n} = m_e c^2 - m_n c^2. \tag{65}$$

$$p_{e,n} = \left[\left(m_e c\right)^2 - \left(m_n c\right)^2\right]^{1/2}. \tag{66}$$

The photon energy emitted by the electron and kinetic energy acquired by the electron are equal. However, the momentum of the photon emitted by the electron and the momentum acquired by the electron are ordinarily not the same.

There is the following relationship between $p_{\mathrm{ph},n}$ and $p_{\mathrm{e},n}$.

$$\frac{p_{\mathrm{ph},n}}{p_{\mathrm{e},n}} = \left(\frac{m_{\mathrm{e}} - m_n}{m_{\mathrm{e}} + m_n}\right)^{1/2}. \tag{67}$$

$$\frac{p_{\mathrm{ph},n}}{p_{\mathrm{e},n}} = \left[\frac{\left(n^2+\alpha^2\right)^{1/2} - n}{\left(n^2+\alpha^2\right)^{1/2} + n}\right]^{1/2}. \tag{68}$$

$p_{\mathrm{ph}} = p_{\mathrm{e}}$ holds only when $n = 0$.

Finally let us check the correlation of the physical quantities in the formulas derived in this paper with the line segments of the ellipse.

First, in the ellipse in **Figure 3**,

$$\overline{\mathrm{OF}}^2 + \overline{\mathrm{OB}}^2 = \overline{\mathrm{BF}}^2. \tag{69}$$

This provides an energy-momentum relationship applicable to an electron in a hydrogen atom. That is,

$$\left(m_n c^2\right)^2 + c^2 p_{\mathrm{e},n}^2 = \left(m_{\mathrm{e}} c^2\right)^2. \tag{70}$$

The following relationship still holds in the ellipse.

$$\overline{\mathrm{OB}}^2 = \overline{\mathrm{FA}} \cdot \overline{\mathrm{A'F}}. \tag{71}$$

This gives the relationship between the three types of momentum treated as a problem in this paper. That is,

$$p_{\mathrm{e},n}^2 = p_{\mathrm{ph},n} p_{\mathrm{po},n}, \quad p_{\mathrm{po},n} = \left(m_{\mathrm{e}} + m_n\right)c. \tag{72}$$

An unusual formula like Formula (67) holds because, as explained in **Figure 1** and **Figure 4**, the photon energy of an electron at an energy level $E_{\mathrm{ab},n} = m_n c^2$ is actually $\left(m_{\mathrm{e}} + m_n\right)c^2$ not $m_n c^2$. However, the $m_{\mathrm{e}} c^2$ in $\left(m_{\mathrm{e}} + m_n\right)c^2$ is cancelled out by the negative energy specific to the electron $-m_{\mathrm{e}} c^2$, and the energy of the electron is understood to be $m_n c^2$.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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