

Influence of Waveguide Properties on Wave Prototypes Likely to Accompany the Dynamics of Four-Wave Mixing in Optical Fibers

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Abstract

In this article, we study the impacts of nonlinearity and dispersion on signals likely to propagate in the context of the dynamics of four-wave mixing. Thus, we use an indirect resolution technique based on the use of the iB -function to first decouple the nonlinear partial differential equations that govern the propagation dynamics in this case, and subsequently solve them to propose some prototype solutions. These analytical solutions have been obtained; we check the impact of nonlinearity and dispersion. The interest of this work lies not only in the resolution of the partial differential equations that govern the dynamics of wave propagation in this case since these equations are not at all easy to integrate analytically and their analytical solutions are very rare, in other words, we propose analytically the solutions of the nonlinear coupled partial differential equations which govern the dynamics of four-wave mixing in optical fibers. Beyond the physical interest of this work, there is also an appreciable mathematical interest.

Keywords

Optical Fiber, Four Waves Mixing, Implicit Bogning Function, Coupled Nonlinear Partial Differential Equations, Nonlinear Coefficient, Dispersive Coefficient, Waveguide Properties

1. Introduction

Optical fiber, which today presents itself as the essential instrument of new information technologies, had its first beginnings in the 19th century. Today, the book that best situates readers about the history of its discovery is a book published in the USA in 1999 [1]. Its first manufacturing means transmission support was made in 1920 [2]-[4]. However, research on optical fibers began 30 years later, that is to say exactly in 1950 [5]-[8]. Gradually with the observations of K.C. Kao and G.A. Hockham [9], work on optical fibers took off. Since then, much work has been done, but what generally attract our attention in the context of this article are the phenomena that accompany a wave or a signal during its propagation in the optical fiber. Of all these phenomena that accompany a signal during its propagation in the optical fiber, we retain the effects linked to dispersion and polarization of the dispersion mode [10]-[23], Raman effects [24], self-steepening and shock formation [25]-[29], cross-phase modulation (XPM) [30]-[37], self-phase modulation (SPM) [38]-[50]. It should be noted that the first studies of nonlinear effects began after the 1970s [51]-[58]. Taking these effects into account, when modeling the equations that govern the propagation dynamics in optical fiber, brings additional modifications each time.

However, it should be noted that at this time, the solutions that authors proposed to interpret the evolution of signals in optical fibers were based mainly on numerical simulations [59]-[74]. Of all these methods, one of the most used was the finite difference method [75]-[79]. However, it should be noted that numerical methods only really began to integrate the non-linear character of this propagation medium around 1992 [80]-[85]. Thus, the development of multiple digital techniques has favored the studies of other phenomena such as modulational instabilities [86]-[94] and dispersion management [95]-[98]. The analytical techniques most used at this time were disruptive [99]-[106], and it was that they which in the majority of cases made it possible to approach the nonlinear partial differential equations (NPDEs), resulted from the different models. The combination of these numerical and analytical methods also favored the advent of solitons in optical fibers [107]-[109].

Of all these effects which accompany the propagation of the signal in the optical fiber and which will be at the center of our study in this article is the four-wave mixing (FWM) effect. It is an effect who's taking into account leads to the obtaining of four nonlinear and coupled partial differential equations whose analytical resolution is not always easy until today. The first solutions proposed were digital. In this work, we want to use the indirect mathematical technique based on a careful choice of the analytical sequence solution to construct solutions and therefore, the framework is formed by the iB-function (implicit Bogning function) to provide solutions to coupled nonlinear partial differential equations where effects of FWM were taken into account during modeling in optical fibers. It is important to note that the coupled nonlinear partial differential equations established by previous authors, until these days have not yet had real ex-

act solutions. The attempts at resolutions have always been based on digital programming, hence the relevance of this article. In addition, we study the impacts of dispersion and nonlinear coefficients on the solutions obtained. Thus, this manuscript is organized as follows: in Section 2 we will first return to the propagation equation in optical fiber, then recall the origin of four-wave mixing. In Section 3, we will see what happens for ultrafast FWM. Section 4 proposes the new technique for solving such equations as well as the choice of the necessary ansatz. In Section 5, we determine the main coefficient range equation; in Section 6, we make an inventory of possible solutions; in Section 7, we construct some solutions. The profiles of the solutions obtained as well as some associated comments are given in Section 8. Finally, we end our article with a conclusion.

2. Propagation Equation in Optical Fibers

The propagation dynamics in optical fibers, like all electromagnetic phenomena, are governed by Maxwell's equations [110]-[117].

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (1)$$

$$\nabla \wedge \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}, \quad (2)$$

$$\nabla \cdot \mathbf{D} = \rho_f, \quad (3)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (4)$$

where \mathbf{E} is the electrical field, \mathbf{H} the magnetic field vector, \mathbf{D} the electric flux density, \mathbf{j} the magnetic flux density, $\phi(x_j, t)$ the current charge density vector, ρ_f the source of electromagnetic charge and ∇ the Nabla's vector. In optical fibers, there are no free charges and $\mathbf{j} = \mathbf{0}$, $\rho_f = 0$. The flux densities \mathbf{D} and \mathbf{B} arise in response to the electric and magnetic fields and propagation inside the medium such that

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad (5)$$

and

$$\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}, \quad (6)$$

where ε_0 is the vacuum permittivity, μ_0 the vacuum permeability, \mathbf{P} the induced electric polarization, \mathbf{M} the induced magnetic polarization. For non-magnetic mediums such as optical fibers $\mathbf{M} = \mathbf{0}$.

Eliminating \mathbf{B} and \mathbf{D} in favor of \mathbf{E} and \mathbf{P} in Maxwell equations leads to

$$\nabla \wedge \nabla \wedge \mathbf{E} = -\frac{1}{C^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} - \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2}. \quad (7)$$

The induced polarization is generally considered such as it has the linear part and the nonlinear part and is given by

$$\mathbf{P} = \mathbf{P}_L + \mathbf{P}_{NL}, \quad (8)$$

where \mathbf{P}_L indicates the linear part of \mathbf{P} and \mathbf{P}_{NL} its nonlinear part. Know-

ing that

$\nabla \wedge \nabla \wedge \mathbf{E} \equiv \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$, Equation (7) becomes

$$\nabla^2 \mathbf{E} - \frac{1}{C^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}_L}{\partial t^2} + \mu_0 \frac{\partial^2 \mathbf{P}_{NL}}{\partial t^2}. \tag{9}$$

This equation is generally the one that governs propagation in optical fiber. But this equation must from time to time undergo modifications depending on the excitation modes in the fiber, the type of fiber, the transmission mode, the type of coupling, the non-linearity and the precise connections.

In this work, we want to study the dynamics and the effect of simultaneous propagation of four waves (FWM) in the optical fiber through the modeling of the differential equations with partial drifts that govern such dynamics and analytically propose the solutions. Thus, our approach will be directly oriented towards the theory of mixing four waves of different frequencies.

3. Origin of Four-Wave Mixing

The origin of the parametric process is based on the response of the electrons linked to the material to an applied optical field, more precisely to the polarization induced by the medium which is not linear. It contains a non-linear part such that its intensity is governed by the non-linear susceptibility [9]-[13]. The parametric processes can be classified as second-order susceptibility $\chi^{(2)}$ or third-order susceptibility $\chi^{(3)}$ vanishes in the isotropic medium in the case of dipole approximation that is the case in the optical fibers that transform main constituent silica. In practice, these processes do not occur because of quadrupole and magnetic dipole effects, but with relatively low conversion efficiency. The third-order parametric processes involve in general nonlinear interaction among four optical waves and include phenomena such as third-harmonic generation and parametric amplification. The study of FWM is very important in optical fibers because it can be very efficient for generating new waves [110]-[130].

Because second-order susceptibility is not taken into account in optical fiber, the third-order polarization vector is often given as follows

$$\mathbf{P}_{NL} = \varepsilon_0 \chi^{(3)} : \mathbf{EEE}, \tag{10}$$

where ε_0 is vacuum permittivity, \mathbf{P}_{NL} the induced nonlinear polarization vector and \mathbf{E} the electric field vector.

If we consider four optical waves oscillating at respective frequencies ω_1 , ω_2 , ω_3 and ω_4 , polarizer along the same unit vector axis \mathbf{e}_x , the total electric field can be written as

$$\mathbf{E} = \frac{1}{2} \mathbf{e}_x \sum_{j=1}^4 E_j \exp[i(k_j z - \omega_j t)] + C.C., \tag{11}$$

where the propagation constant $k_j = n_j \omega_j / C$ is such that, n_j is the refractive index. Under these conditions, taking into account (11) in (10) gives

$$P_{NL} = \frac{1}{2} e_x \sum_{j=1}^4 P_j \exp[i(k_j z - \omega_j t)] + C.C. \quad (12)$$

Thus, substitute Equations (11) and (12) into Equation (9) while considering a similar expression for the linear part P_L , while also neglecting the dependence of the field relative to time while assuming the conditions of quasi-continuity of the waves (quasi-CW conditions), the spatial dependence of the coordinates of the electric field is given by

$$E_j(\mathbf{r}) = F_j(x, y) A_j(z), \quad (13)$$

where $F_j(x, y)$ is the spatial distribution of the fiber in which the j^{th} propagates in the fiber [19]. Under these conditions, the evolution of the amplitudes A_j inside the multimode fiber is governed by the coupled equations

$$\frac{dA_1}{dz} = \frac{i\tilde{n}_2\omega_1}{C} \left[\left(f_{11} |A_1|^2 + 2 \sum_{k \neq 1} f_{1k} |A_k|^2 \right) A_1 + 2f_{1234} A_2^* A_3 A_4 e^{i\Delta k z} \right], \quad (14)$$

$$\frac{dA_2}{dz} = \frac{i\tilde{n}_2\omega_2}{C} \left[\left(f_{22} |A_2|^2 + 2 \sum_{k \neq 2} f_{2k} |A_k|^2 \right) A_2 + 2f_{2134} A_1^* A_3 A_4 e^{i\Delta k z} \right], \quad (15)$$

$$\frac{dA_3}{dz} = \frac{i\tilde{n}_2\omega_3}{C} \left[\left(f_{33} |A_3|^2 + 2 \sum_{k \neq 3} f_{3k} |A_k|^2 \right) A_3 + 2f_{3412} A_1 A_2 A_4^* e^{-i\Delta k z} \right], \quad (16)$$

and

$$\frac{dA_4}{dz} = \frac{i\tilde{n}_2\omega_4}{C} \left[\left(f_{44} |A_4|^2 + 2 \sum_{k \neq 4} f_{4k} |A_k|^2 \right) A_4 + 2f_{4312} A_1 A_2 A_3^* e^{-i\Delta k z} \right], \quad (17)$$

where the wave-vector mismatch Δk is given by

$$\Delta k = (\tilde{n}_3\omega_3 + \tilde{n}_4\omega_4 - \tilde{n}_1\omega_1 - \tilde{n}_2\omega_2)/C. \quad (18)$$

The refractive indices \tilde{n}_1 and \tilde{n}_4 stand for the effective indices of the fiber modes. Notice that \tilde{n}_1 and \tilde{n}_2 can differ from each other when the pump waves A_1 and A_2 propagate in different fiber modes even if they are degenerate in frequencies. The overlap integral f_{ijkl} is defined by [19]

$$f_{ijkl} = \frac{\langle F_i^* F_j^* F_k F_l \rangle}{\left[\langle |F_i|^2 \rangle \langle |F_j|^2 \rangle \langle |F_k|^2 \rangle \langle |F_l|^2 \rangle \right]^{1/2}}, \quad (19)$$

where angle brackets denote integration over the transverse coordinates x and y . In deriving Equations (14) to (18), only nearly phase-matched terms are kept and we neglect frequency dependence $\chi^{(3)}$. The parameter \tilde{n}_2 is the nonlinear parameter.

4. Ultrafast FWM

The effects of both GVD and fiber losses can be included following and allowing $A_j(z)$ in Equations (14) to (17) to be a slow carrying of time. If polarization effects are neglected assuming that all four waves are polarized along a principal axis of a birefringent fiber, the inclusion of GVD effects in Equations (14) to (17)

makes it such that the derivative $\frac{dA_j}{dz}$ can be replaced by

$$\frac{dA_j}{dz} \rightarrow \frac{\partial A_j}{\partial z} + \beta_{1j} \frac{\partial A_j}{\partial t} + \frac{i}{2} \beta_{2j} \frac{\partial^2 A_j}{\partial t^2} + \frac{1}{2} \alpha_j A_j. \tag{20}$$

The resulting four coupled nonlinear Schrödinger equations (NLS) describe the FWM of picoseconds optical pulses and include the effects of group velocity dispersion (GVD), self-phase modulation (SPM), and cross-phase modulation (XPM). Generally, the prediction of solutions of the resulting coupled nonlinear partial differential equations is given numerically because they are very difficult to solve analytically. Assuming that a single pump of power P_0 is incident at $z=0$, the signal and idler fields are found to satisfy the following set of two coupled nonlinear Schrödinger equations [110]

$$\begin{aligned} \frac{\partial A_3}{\partial z} + \beta_{13} \frac{\partial A_3}{\partial t} + \frac{i}{2} \beta_{23} \frac{\partial^2 A_3}{\partial t^2} + \frac{1}{2} \alpha_3 A_3 \\ = i\gamma \left[|A_3|^2 + 2|A_4|^2 + 2P_0 \right] A_3 + i\gamma P_0 A_4^* e^{-i\theta}, \end{aligned} \tag{21}$$

and

$$\begin{aligned} \frac{\partial A_4}{\partial z} + \beta_{14} \frac{\partial A_4}{\partial t} + \frac{i}{2} \beta_{24} \frac{\partial^2 A_4}{\partial t^2} + \frac{1}{2} \alpha_4 A_4 \\ = i\gamma \left[|A_4|^2 + 2|A_3|^2 + 2P_0 \right] A_4 + i\gamma P_0 A_3^* e^{i\theta}, \end{aligned} \tag{22}$$

where $\theta = \Delta k + 2\gamma P_0 z$ is the net phase mismatch.

However, it should be noted that the equations established above may undergo further modifications if certain phenomena are taken into account.

The equations obtained above are generally very difficult to solve analytically. But as the aim of this work is to provide analytical solutions to these equations, we felt that it was necessary to do the above work of recall on the theory of FWM to better explain to our readers the origin of the amplitude couplings.

To avoid confusion of indices in the resolution of Equations (21) and (22) we make the correspondences $U \leftarrow A_3$, $V \leftarrow A_4$, $\beta_1 \leftarrow \beta_{13}$, $\beta_2 \leftarrow \frac{1}{2} \beta_{23}$, $\lambda \leftarrow \frac{1}{2} \alpha_3$, $\beta_3 \leftarrow \beta_{14}$, $\beta_4 \leftarrow \frac{1}{2} \beta_{24}$, $\theta \equiv 0[2k\pi], k \in \mathbb{Z}$ and $\lambda' \leftarrow \alpha_4/2$ the equations ready to solve become

$$\frac{\partial U}{\partial z} + \beta_1 \frac{\partial U}{\partial t} + i\beta_2 \frac{\partial^2 U}{\partial t^2} + \lambda U = i\gamma \left[|U|^2 + 2|V|^2 + 2P_0 \right] U + i\gamma P_0 V^*, \tag{23}$$

and

$$\frac{\partial V}{\partial z} + \beta_3 \frac{\partial V}{\partial t} + i\beta_4 \frac{\partial^2 V}{\partial t^2} + \lambda' V = i\gamma \left[|V|^2 + 2|U|^2 + 2P_0 \right] V + i\gamma P_0 U^*. \tag{24}$$

Since then, very little serious work has been done in the analytical research of solutions to nonlinear partial differential equations with partial derivatives which integrate the effects of FWM on the modeling of propagation dynamics in optical fiber. In the remainder of this work, we will concentrate on the analytical search for solutions to the coupled Equations (23) and (24).

5. Method of Resolution

The resolution method is an inverse method based essentially on the use of the ansatz of the generalized wave function, constructed based on iB-functions [131]-[138]. The principle consists of substituting the search for the solution of the solved equations with the search for the indices and the parameter of the iB-function. The iB-function is defined in the following subsection.

5.1. iB-Functions

iB-functions are functions with multiple properties that find numerous applications in solving problems in nonlinear physics and Mathematics for nonlinear physics and Mathematics in general. They come in two forms; a so-called main form and a secondary form.

The main form is defined by

$$J_{n,m} \left(\sum_{i=0}^p \alpha_i x_i \right) = \sinh^m \left(\sum_{i=0}^p \alpha_i x_i \right) / \cosh^n \left(\sum_{i=0}^p \alpha_i x_i \right), \quad (25)$$

where $J_{n,m} \left(\sum_{i=0}^p \alpha_i x_i \right)$ represents the implicit form of the function,

$\sinh^m \left(\sum_{i=0}^p \alpha_i x_i \right) / \cosh^n \left(\sum_{i=0}^p \alpha_i x_i \right)$, the explicit form of the function, α_i

($i = 0, 1, 2, \dots, p$) represents the parameters associated with the independent variables x_i ($i = 0, 1, 2, \dots, p$), and the pair $(n, m) \in \mathbb{R}^2$ indicates the power of the function. More precisely, n is the power of $\cosh \left(\sum_{i=0}^p \alpha_i x_i \right)$ and m the power of $\sinh \left(\sum_{i=0}^p \alpha_i x_i \right)$. This function as defined in (1) is also called the iB-function of several variables and any derivative operation engaged in this case is partial.

The secondary form is given by

$$T_{n,m} \left(\sum_{i=1}^p \alpha_i x_i \right) = \frac{\sin^m \left(\sum_{i=1}^p \alpha_i x_i \right)}{\cos^n \left(\sum_{i=1}^p \alpha_i x_i \right)}, \quad (26)$$

where $T_{n,m} \left(\sum_{i=1}^p \alpha_i x_i \right)$ represents the implicit form of the secondary form,

$\sin^m \left(\sum_{i=1}^p \alpha_i x_i \right) / \cos^n \left(\sum_{i=1}^p \alpha_i x_i \right)$ the explicit secondary form, α_i ($i = 0, 1, 2, \dots, p$)

represents the parameters associated with the independent variables x_i ($i = 0, 1, 2, \dots, p$), and the pair $(n, m) \in \mathbb{R}^2$ indicates the power of the function.

More precisely, n is the power of $\cos \left(\sum_{i=0}^p \alpha_i x_i \right)$ and m the power of

$\sin \left(\sum_{i=0}^p \alpha_i x_i \right)$. This function as defined in relation (1) is also called the

iB-function of several variables and any derivative operation engaged in this case is partial.

The main form and the secondary form are linked for any number by formulas

$$J_{n,m}(ix) = i^m T_{n,m}(x), \tag{27}$$

and

$$T_{n,m}(ix) = i^m J_{n,m}(x). \tag{28}$$

For Equations (27) and (28) i represents the imaginary number such that $i^2 = -1$.

These functions can be defined in dimensions 1, 2, 3, 4, etc. depending on the studies we want to carry out. In the rest of our demonstrations, we must use iB-functions with a single independent variable like

$$J_{n,m}(\alpha x) = \sinh^m(\alpha x) / \cosh^n(\alpha x), \tag{29}$$

and

$$T_{n,m}(\alpha x) = \sin^m(\alpha x) / \cos^n(\alpha x). \tag{30}$$

Simple forms, that is to say, which can no longer undergo transformations, are given by $J_{n,0}$, $J_{n,1}$, $T_{n,0}$ and $T_{n,1}$ when n is positive.

5.2. Principle of Utilization

The principle of the solution consists of identifying and choosing the ansatz solution to construct according to the iB-function. Thus, once the form of the ansatz solution to be constructed is chosen, it is injected into the equation to be solved, and subsequently the effective selection of the solutions past through the determination of the indices and the parameter of the iB-functions envisaged from the start. However it is important to point out that good management of the calculations involved in this case requires good mastery of the properties of these functions. We give in the following lines some useful properties.

5.3. Common Transformation Properties

- 1) $J_{n,m}^p = J_{np,mp}$
- 2) $J_{n,m} \cdot J_{n',m'} = J_{n+n',m+m'}$
- 3) $\frac{J_{n,m}}{J_{n',m'}} = J_{n-n',m-m'}$
- 4) $\frac{1}{J_{n,m}} = J_{-n,-m}$
- 5) $\frac{dJ_{n,m}(\alpha x)}{dx} = m\alpha J_{n-1,m-1}(\alpha x) - n\alpha J_{n+1,m+1}(\alpha x)$,
- 6) $J_{2n,2n} = J_{2,2}^n = (1 - J_{2,0})^n$,
- 7) $J_{-2n,-2n} = J_{2,2}^{-n} = (1 - J_{2,0})^{-n} = \frac{1}{(1 - J_{2,0})^n}$,

- 8) $J_{n,n} = J_{2n,2n}^{\frac{1}{2}} = (1 - J_{2,0})^{\frac{n}{2}}$,
- 9) $J_{n,n}^2 = J_{2,2}^n$,
- 10) $J_{n,n}^p = J_{p,p}^n$,
- 11) $J_{n,m}^p = J_{np,mp}$, $p \in R$.
- 12) $J_{0,2n} = (J_{-2,0} - 1)^n \Rightarrow J_{0,-2n} = (J_{-2,0} - 1)^{-n}$,
- 13) $J_{-2n,0} = J_{-2,0}^n = (J_{0,2} + 1)^n \Rightarrow J_{2n,0} = (J_{0,2} + 1)^{-n}$,
- 14) $T_{0,0}(x) = 1, x \neq 0$
- 15) $T_{2,2}(x) = T_{2,0}(x) - 1$,
- 16) $T_{4,4}(x) = (T_{2,0} - 1)^2 = T_{4,0} - 2T_{2,0} + 1$
- 17) $T_{2n,2n}(x) = (T_{2,0} - 1)^n$,
- 18) $\forall p, n, m \in R, T_{n,m}^p = T_{np,mp}$,
- 19) $\forall p, n \in R, T_{n,n}^p = T_{1,1}^{np} = T_{p,p}^n$,
- 20) $\forall n, m, n', m' \in R, T_{n,m} \cdot T_{n',m'} = T_{n+n', m+m'}$,
- 21) $\frac{dT_{n,m}(\alpha x)}{dx} = m\alpha T_{n-1,m-1}(\alpha x) + n\alpha T_{n+1,m+1}(\alpha x)$,
- 22) $J_{n,m}(-x) = (-1)^m J_{n,m}(x)$,

In the properties above, n, m, p, n', m' and x are elements of the set of reals and α an element of the complex body.

6. Main Equations of Ranges of Coefficients

We propose to construct the solutions of Equations (23) and (24) in the forms

$$U = \psi e^{-i(kz - \omega t)}, \tag{31}$$

and

$$V = \phi e^{i(kz - \omega t)}. \tag{32}$$

Thus, substituting Equations (31) and (32) into Equations (23) and (24) gives

$$\begin{aligned} & \frac{\partial \psi}{\partial z} + (\beta_1 - 2\omega\beta_2) \frac{\partial \psi}{\partial t} + i\beta_2 \frac{\partial^2 \psi}{\partial t^2} \\ & + \left[\lambda + i(-k + \omega\beta_1 - \beta_2\omega^2) - i\gamma(|\psi|^2 + 2|\phi|^2 + 2P_0) \right] \psi - i\gamma P_0 \phi^* = 0, \end{aligned} \tag{33}$$

and

$$\begin{aligned} & \frac{\partial \phi}{\partial z} + (\beta_3 + 2\omega\beta_4) \frac{\partial \phi}{\partial t} + i\beta_4 \frac{\partial^2 \phi}{\partial t^2} \\ & + \left[\lambda' + i(k - \omega\beta_3 - \beta_4\omega^2) - i\gamma(|\phi|^2 + 2|\psi|^2 + 2P_0) \right] \phi - i\gamma P_0 \psi^* = 0. \end{aligned} \tag{34}$$

The problem is reduced to solving Equations (33) and (34). To this end, we propose to seek solutions in the forms

$$\psi = aJ_{n,m}(\alpha z - \alpha_0 t), \tag{35}$$

and

$$\phi = bJ_{n',m'}(\alpha z - \alpha_0 t). \tag{36}$$

where, n, m, n', m' are real but a, b, α and α_0 can be real or complex. To further simplify the manipulation of terms, we introduce the change of variable $\xi = \alpha z - \alpha_0 t$. Thus, Equations (33) and (34) are transformed as follows

$$\begin{aligned} &(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)\frac{\partial\psi}{\partial\xi} + i\beta_2\alpha_0^2\frac{\partial^2\psi}{\partial\xi^2} \\ &+ \left[\lambda + i(-k + \omega\beta_1 - \beta_2\omega^2) - i\gamma(|\psi|^2 + 2|\phi|^2 + 2P_0) \right] \psi - i\gamma P_0\phi^* = 0, \end{aligned} \tag{37}$$

and

$$\begin{aligned} &(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4)\frac{\partial\phi}{\partial\xi} + i\alpha_0^2\beta_4\frac{\partial^2\phi}{\partial\xi^2} \\ &+ \left[\lambda' + i(k - \omega\beta_3 - \beta_4\omega^2) - i\gamma(|\phi|^2 + 2|\psi|^2 + 2P_0) \right] \phi - i\gamma P_0\psi^* = 0. \end{aligned} \tag{38}$$

Equations (37) and (38) in the variable change condition become respectively

$$\psi = aJ_{n,m}(\xi), \tag{39}$$

and

$$\phi = bJ_{n',m'}(\xi). \tag{40}$$

Inserting Equations (39) and (40) into Equations (37) and (38) while keeping in mind that are real, gives respectively

$$\begin{aligned} &am[\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2]J_{n-1,m-1} + i\beta_2\alpha_0^2am(m-1)J_{n-2,m-2} \\ &+ \left\{ -ia\beta_2\alpha_0^2[m(n-1) + n(m+1)] + a[\lambda + i(-k + \omega\beta_1 - \beta_2\omega^2)] - 2i\gamma P_0a \right\} J_{n,m} \\ &- an(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{n+1,m+1} + i\beta_2\alpha_0^2an(n+1)J_{n+2,m+2} \\ &- i\gamma|a|^2 aJ_{3n,3m} - 2i\gamma|b|^2 aJ_{2n'+n,2m'+m} - i\gamma P_0b^* J_{n',m'} = 0, \end{aligned} \tag{41}$$

and

$$\begin{aligned} &bm'[\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4]J_{n'-1,m'-1} + i\beta_4\alpha_0^2bm'(m'-1)J_{n'-2,m'-2} \\ &+ \left\{ -ib\beta_4\alpha_0^2[m'(n'-1) + n'(m'+1)] + b[\lambda' + i(k - \omega\beta_3 - \beta_4\omega^2)] - 2i\gamma P_0b \right\} J_{n',m'} \\ &- bn'(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4)J_{n'+1,m'+1} + i\beta_4\alpha_0^2bn'(n'+1)J_{n'+2,m'+2} \\ &- i\gamma|b|^2 bJ_{3n',3m'} - 2i\gamma|a|^2 bJ_{2n+n',2m+m'} - i\gamma P_0a^* J_{n,m} = 0, \end{aligned} \tag{42}$$

Equations (39) and (40) are coupled equations of ranges of coefficients in (n, m) and (n', m') .

7. Study of the Possibilities of Solving Equations (41) and (42)

Equations (41) and (42) in the case where there is no possibility of grouping of terms exclusively admit solutions if and only if each term of these equations is zero. Under these conditions, the solutions obtained are trivial. But to be reassured we must study the case where certain terms of Equations (41) and (42) can

be grouped together.

Pairs (n, m) and (n', m') for Which Certain Terms of Equations (41) and (42) Group Together

The search for pairs for which certain terms of Equations (41) and (42) group together gives fixed couples $(n, m) = (-1/2, -1/2); (1/2, 1/2); (-1, -1); (1, 1); (0, 0)$ and $(n', m') = (-1/2, -1/2); (1/2, 1/2); (-1, -1); (1, 1); (0, 0)$. Apart from the fixed pairs, we obtain constraint relations which link the pairs $(n' = n - 2, m' = m - 2)$, $(n' = n - 1, m' = m - 1)$, $(n' = n, m' = m)$, $(n' = n + 1, m' = m + 1)$, $(n' = n + 2, m' = m + 2)$, $(n' = 3n, m' = 3m)$ and $(n = -3n', m = -3m')$.

The constraint relations between the pairs (n, m) and (n', m') allow to decouple Equations (41) and (42). These relationships offer good alternatives regarding the resolution of these equations because they establish direct links between couples (n, m) and (n', m') .

7.2. Decoupling of Equations (41) and (42) and Resolution

➤ **Study of case** $(n' = n - 1, m' = m - 1) \Rightarrow (n = n' + 1, m = m' + 1)$

Equations (41) and (42) respectively lead using the above relations to the following decoupled range equations

$$\begin{aligned} & \left[-i\gamma P_0 b^* + am(\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2) \right] J_{n-1, m-1} + i\beta_2 \alpha_0^2 am(m-1) J_{n-2, m-2} \\ & + \left\{ -ia\beta_2 \alpha_0^2 [m(n-1) + n(m+1)] + a \left[\lambda + i(-k + \omega \beta_1 - \beta_2 \omega^2) \right] - 2i\gamma P_0 a \right\} J_{n, m} \\ & - an(\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2) J_{n+1, m+1} + i\beta_2 \alpha_0^2 an(n+1) J_{n+2, m+2} \\ & - i\gamma |a|^2 a J_{3n, 3m} - 2i\gamma |b|^2 a J_{3n-2, 3m-2} = 0, \end{aligned} \tag{43}$$

and

$$\begin{aligned} & bm'(\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4) J_{n'-1, m'-1} + i\beta_4 \alpha_0^2 bm'(m'-1) J_{n'-2, m'-2} \\ & + \left\{ -ib\beta_4 \alpha_0^2 [m'(n'-1) + n'(m'+1)] + b \left[\lambda' + i(k - \omega \beta_3 - \beta_4 \omega^2) \right] - 2i\gamma P_0 b \right\} J_{n', m'} \\ & - \left[bn'(\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4) + i\gamma P_0 a^* \right] J_{n'+1, m'+1} + i\beta_4 \alpha_0^2 bn'(n+1) J_{n'+2, m'+2} \\ & - i\gamma |b|^2 b J_{3n', 3m'} - 2i\gamma |a|^2 b J_{3n'+2, 3m'+2} = 0. \end{aligned} \tag{44}$$

Equations (43) and (44) are range equations decoupled by the constraint relations $(n' = n - 1, m' = m - 1)$. Equation (43) depends exclusively on pairs (n, m) and Equation (44) exclusively on pairs (n', m') . Thus, the preponderant pairs (n, m) for which certain terms of Equation (43) group together are $(1/2, 1/2); (0, 0)$. Similarly, the leading pairs (n', m') for which certain terms of Equation (44) group together are $(-1/2, -1/2); (-1, -1)$. We note that the case involves and brings us back to solving a single initial nonlinear differential equation. Our concern is to have two non-trivial solutions, we subsequently search for the solutions for the case $(n, m) = (1/2, 1/2) \Rightarrow (n', m') = (-1/2, -1/2)$. Equations (41) and (42) give respectively

$$\begin{aligned} & \left[-i\gamma P_0 b^* + \frac{a}{2}(\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_3) - 2i\gamma |b|^2 a \right] J_{\frac{-1}{2}, \frac{-1}{2}} - i \frac{\beta_2 \alpha_0^2 a}{4} J_{\frac{-3}{2}, \frac{-3}{2}} \\ & + \left\{ \frac{-ia\beta_2 \alpha_0^2}{4} + a \left[\lambda + i(-k + \omega \beta_1 - \beta_2 \omega^2) \right] - 2i\gamma P_0 a \right\} J_{\frac{1}{2}, \frac{1}{2}} \\ & - \left[i\gamma |a|^2 a + \frac{a}{2}(\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2) \right] J_{\frac{3}{2}, \frac{3}{2}} + \frac{3}{4} i \beta_2 \alpha_0^2 a J_{\frac{5}{2}, \frac{5}{2}} = 0, \end{aligned} \tag{45}$$

and

$$\begin{aligned} & - \left[\frac{b}{2}(\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4) + i\gamma |b|^2 b \right] J_{\frac{-3}{2}, \frac{-3}{2}} + \frac{3i\beta_4 \alpha_0^2 b}{4} J_{\frac{-5}{2}, \frac{-5}{2}} \\ & + \left\{ \frac{-ib\beta_4 \alpha_0^2}{4} + b \left[\lambda' + i(k - \omega \beta_3 - \beta_4 \omega^2) \right] - 2i\gamma P_0 b \right\} J_{\frac{-1}{2}, \frac{-1}{2}} \\ & - \left[-\frac{b}{2}(\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4) + i\gamma P_0 a^* + 2i\gamma |a|^2 b \right] J_{\frac{1}{2}, \frac{1}{2}} - \frac{i\alpha_0^2 \beta_4 b}{4} J_{\frac{3}{2}, \frac{3}{2}} = 0, \end{aligned} \tag{46}$$

Equations (45) and (46) are verified if and only if for $a \neq 0, b \neq 0$, we have the following relations

$$\beta_2 = \beta_4 = 0, \tag{47}$$

$$\frac{a}{2}(\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2) - i(\gamma P_0 b^* + 2\gamma |b|^2 a) = 0, \tag{48}$$

$$\lambda = i(2\gamma P_0 + k - \omega \beta_1), \tag{49}$$

$$|a|^2 = \frac{i}{2\gamma}(\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2), \tag{50}$$

$$\lambda' = i(2\gamma P_0 - k + \omega \beta_3), \tag{51}$$

$$\frac{-1}{2}(\alpha - \alpha_0 \beta_3) + i\gamma P_0 a^* + 2i\gamma |a|^2 b = 0, \tag{52}$$

and

$$|b|^2 = \frac{i}{2\gamma}(\alpha - \alpha_0 \beta_3). \tag{53}$$

To continue solving Equations (48) to (53), we set $\eta = i(\alpha - \alpha_0 \beta_1)$ and $\eta' = i(\alpha - \alpha_0 \beta_3)$. We obtain from Equation (50) and (53) respectively

$$|a| = \sqrt{\frac{\eta}{2\gamma}}, \gamma\eta > 0, \tag{54}$$

and

$$|b| = \sqrt{\frac{\eta'}{2\gamma}}, \gamma\eta' > 0. \tag{55}$$

Inserting relations (54) and (55) into Equation (48) initially gives

$$\frac{\eta a}{2} + \gamma P_0 b^* + 2\gamma |b|^2 a = 0, \tag{56}$$

and secondly

$$\eta \left| \frac{\eta}{2} + \eta' \right|^2 = \gamma^2 P_0^2 \eta'. \tag{57}$$

The constraint relation (55) can still be written

$$\eta(\eta + 2\eta')^2 - 4\gamma^2 P_0^2 \eta' = 0. \tag{58}$$

Analogously, inserting relations (54) and (55) into Equation (52) initially gives

$$\frac{\eta' b}{2} + \gamma P_0 a^* + 2\gamma |a|^2 b = 0, \tag{59}$$

and secondly

$$\eta' \left| \frac{\eta'}{2} + \eta \right|^2 = \gamma^2 P_0^2 \eta, \tag{60}$$

or

$$(\eta' + 2\eta)^2 \eta' - 4\gamma^2 P_0^2 \eta = 0. \tag{61}$$

Eliminating P_0^2 between the constraint relations (58) and (61) gives us the principal constraint relation

$$(\eta^2 - \eta'^2)(\eta^2 + 4\eta'\eta + \eta'^2) = 0. \tag{62}$$

The analysis of Equation (62) shows that the most suitable constraint between η and η' is given by

$$\eta^2 - \eta'^2 = 0. \tag{63}$$

We obtain from relation (63) the following equalities

$$\eta = \eta', \tag{64}$$

and

$$\eta = -\eta'. \tag{65}$$

We obtain respectively from (64) and (65) the following relations

$$\beta_3 = \frac{\beta_1}{2\omega + 1}, \tag{66}$$

and

$$\alpha = \frac{\alpha_0}{2} (\beta_1 + 2\omega\beta_3 - \beta_3). \tag{67}$$

By combining Equations (66) and (67) we obtain

$$\frac{\alpha}{\alpha_0} = \frac{2\beta_1\omega}{2\omega + 1}. \tag{68}$$

The analysis of the terms of η and η' shows that our approach is logical if α and α_0 are pure imaginaries. We also obtained $\lambda = i(2\gamma P_0 + k - \omega\beta_1)$ and $\lambda' = i(2\gamma P_0 - k + \omega\beta_3)$; this would simply mean that the dissipation coefficients must also be pure imaginary. With all the constraints above being admitted we have

$$|a| = \sqrt{\frac{\eta}{2\gamma}}, \gamma \eta > 0, \Rightarrow \exists \theta \in R / a = \sqrt{\frac{\eta}{2\gamma}} \exp i\theta \Rightarrow \psi = \sqrt{\frac{\eta}{2\gamma}} J_{\frac{1}{2}, \frac{1}{2}}(\xi) \exp i\theta, \tag{69}$$

and

$$|b| = \sqrt{\frac{\eta'}{2\gamma}}, \gamma\eta' > 0 \Rightarrow \exists \theta' \in R / b = \sqrt{\frac{\eta'}{2\gamma}} \exp i\theta' \Rightarrow \psi = \sqrt{\frac{\eta'}{2\gamma}} J_{\frac{-1}{2}, \frac{-1}{2}}(\xi) \exp i\theta', \quad (70)$$

Returning to the initial variables, the solutions in this case are given by

$$U = \sqrt{\frac{\eta'}{2\gamma}} J_{\frac{1}{2}, \frac{1}{2}}(\xi) \exp i(kz - \omega t + \theta), \quad (71)$$

and

$$V = \sqrt{\frac{\eta'}{2\gamma}} J_{\frac{-1}{2}, \frac{-1}{2}}(\xi) \exp -i(kz - \omega t - \theta'), \quad (72)$$

As we demonstrated above, α and α_0 are pure imaginaries and under these conditions ξ is also pure imaginary such as $\xi = i\xi_0 / \xi_0 \in R$. Thus the solutions (71) and (72) can then be expressed in terms of the corresponding secondary iB-functions assuming that the quantities under the square root are positive

$$\begin{aligned} U &= \sqrt{i} \sqrt{\frac{\eta'}{2\gamma}} T_{\frac{1}{2}, \frac{1}{2}}(\xi_0) \exp i(kz - \omega t + \theta) \\ &= \sqrt{\frac{\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_3}{2\gamma}} \tan^{\frac{1}{2}}(\alpha z - \alpha_0 t) \exp i(kz - \omega t + \theta), \end{aligned} \quad (73)$$

and

$$\begin{aligned} V &= \frac{1}{\sqrt{i}} \sqrt{\frac{\eta'}{2\gamma}} T_{\frac{-1}{2}, \frac{-1}{2}}(\xi_0) \exp -i(kz - \omega t - \theta') \\ &= \sqrt{\frac{\alpha - \alpha_0\beta_3}{2\gamma}} \cotan^{\frac{1}{2}}(\alpha z - \alpha_0 t) \exp -i(kz - \omega t - \theta'). \end{aligned} \quad (74)$$

► **Study of case** $(n' = n + 1, m' = m + 1) \Rightarrow (n = n' - 1, m = m' - 1)$

By inserting $(n' = n + 1, m' = m + 1)$ and $(n = n' - 1, m = m' - 1)$ respectively into Equations (41) and (42) respectively leads to the following decoupled range equations

$$\begin{aligned} &am[\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2] J_{n-1, m-1} + i\beta_2\alpha_0^2 am(m-1) J_{n-2, m-2} \\ &+ \left\{ -ia\beta_2\alpha_0^2 [m(n-1) + n(m+1)] + a[\lambda + i(-k + \omega\beta_1 - \beta_2\omega^2)] - 2i\gamma P_0 a \right\} J_{n, m} \\ &- [an(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2) + i\gamma P_0 b^*] J_{n+1, m+1} \\ &i\beta_2\alpha_0^2 an(n+1) J_{n+2, m+2} - i\gamma |a|^2 a J_{3n, 3m} - 2i\gamma |b|^2 a J_{3n+2, 3m+2} = 0, \end{aligned} \quad (75)$$

and

$$\begin{aligned} &[bm'(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4) - i\gamma P_0 a^*] J_{n'-1, m'-1} + i\alpha_0^2 \beta_4 bm'(m'-1) J_{n'-2, m'-2} \\ &+ \left\{ -i\alpha_0^2 \beta_4 b [m'(n'-1) + n'(m'+1)] + [\lambda' + i(k - \omega\beta_3 - \beta_4\omega^2)] b - 2i\gamma P_0 b \right\} J_{n', m'} \\ &- n'(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4) b J_{n'+1, m'+1} + i\alpha_0^2 \beta_4 bn'(n'+1) J_{n'+2, m'+2} \\ &- i\gamma |b|^2 b J_{3n', 3m'} - 2i\gamma |a|^2 b J_{3n'-2, 3m'-2} = 0. \end{aligned} \quad (76)$$

Equations (75) and (76) are range equations decoupled by the constraint relations $(n' = n + 1, m' = m + 1)$. As in the previous case, Equation (75) depends exclusively on the pairs (n, m) , and Equation (76) exclusively on the pairs (n', m') . Thus, the leading pairs (n, m) for which certain terms of Equation (75) group together are $(-1/2, -1/2); (-1, -1); (0, 0)$. Likewise, the leading pairs (n', m') for which certain terms of Equation (76) group together are $(1/2, 1/2); (1, 1); (0, 0)$. We note that the case involving $(n, m) \in \{(-1, -1); (0, 0)\}$ and $(n', m') \in \{(1, 1); (0, 0)\}$ brings us back to solving a single initial nonlinear differential equation. Our concern is to have two non-trivial solutions, we search for the solutions for the case $(n, m) = (-1/2, -1/2) \Rightarrow (n', m') = (1/2, 1/2)$. Equations (75) and (76) respectively give

$$\begin{aligned} & \left[-i\gamma|a|^2 a - \frac{a}{2}(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2) \right] J_{\frac{-3}{2}, \frac{-3}{2}} + \frac{3}{4}i\beta_2\alpha_0^2 J_{\frac{-5}{2}, \frac{-5}{2}} \\ & + \left[\lambda a + i \left(\frac{-a\beta_2\alpha_0^2}{2} - ka + \omega\beta_1 a - \beta_2\omega^2 a - 2\gamma P_0 a \right) \right] J_{\frac{-1}{2}, \frac{-1}{2}} \quad (77) \\ & + \left[\frac{a}{2}(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2) - i\gamma P_0 b^* - 2i\gamma|b|^2 a \right] J_{\frac{1}{2}, \frac{1}{2}} - i \frac{\beta_2\alpha_0^2 a}{4} J_{\frac{3}{2}, \frac{3}{2}} = 0, \end{aligned}$$

and

$$\begin{aligned} & \left[\frac{b}{2}(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4) - i\gamma P_0 a^* - 2i\gamma|a|^2 b \right] J_{\frac{-1}{2}, \frac{-1}{2}} - \frac{i\alpha_0^2\beta_4 b}{4} J_{\frac{-3}{2}, \frac{-3}{2}} \\ & + \left\{ \lambda' b + i \left(\frac{-\alpha_0^2\beta_4 b}{2} + kb - \omega\beta_3 b - \beta_4\omega^2 b - 2\gamma P_0 b \right) \right\} J_{\frac{1}{2}, \frac{1}{2}} \quad (78) \\ & - \left[\frac{1}{2}(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4) + i\gamma|b|^2 b \right] J_{\frac{3}{2}, \frac{3}{2}} + \frac{3i\alpha_0^2\beta_4 a}{4} J_{\frac{5}{2}, \frac{5}{2}} = 0. \end{aligned}$$

Solving Equations (77) and (78) allows to obtain, as in the previous case, the constraint relations $\frac{\alpha}{\alpha_0} = \frac{2\beta_1\omega}{2\omega + 1}$, $\lambda = i(2\gamma P_0 + k - \omega\beta_1)$; and

$\lambda' = i(2\gamma P_0 - k + \omega\beta_3)$. We therefore obtain the following solutions

$$U = \sqrt{\frac{\eta}{2\gamma}} J_{\frac{-1}{2}, \frac{-1}{2}}(\xi) \exp i(kz - \omega t + \theta), \quad (78)$$

and

$$V = \sqrt{\frac{\eta'}{2\gamma}} J_{\frac{1}{2}, \frac{1}{2}}(\xi) \exp -i(kz - \omega t - \theta'). \quad (79)$$

In the case where ξ is pure imaginary i.e. $\xi = i\xi_0 / \xi_0 \in R$, we deduce from solutions (78) and (79)

$$\begin{aligned} U &= \frac{1}{\sqrt{i}} \sqrt{\frac{\eta}{2\gamma}} T_{\frac{-1}{2}, \frac{-1}{2}}(\xi_0) \exp i(kz - \omega t + \theta) \\ &= \sqrt{\frac{\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_3}{2\gamma}} \cotan^{\frac{1}{2}}(\alpha z - \alpha_0 t) \exp i(kz - \omega t + \theta), \quad (80) \end{aligned}$$

and

$$\begin{aligned}
 V &= \sqrt{i} \sqrt{\frac{\eta'}{2\gamma}} T_{1,1}^{\frac{1}{2}}(\xi_0) \exp-i(kz - \omega t - \theta') \\
 &= \sqrt{\frac{\alpha - \alpha_0 \beta_3}{2\gamma}} \tan^{\frac{1}{2}}(\alpha z - \alpha_0 t) \exp-i(kz - \omega t - \theta'),
 \end{aligned}
 \tag{81}$$

For the two above studied cases, $(n' = n - 1, m' = m - 1)$ and $(n' = n + 1, m' = m + 1)$ these solutions are verified for the very low dispersion.

➤ **Study of case** $(n' = n - 2, m' = m - 2) \Rightarrow (n = n' + 2, m = m' + 2)$

The insertion of $(n' = n - 2, m' = m - 2)$ and $(n = n' + 2, m = m' + 2)$ respectively in Equations (41) and (42) lead respectively to the following decoupled range equations

$$\begin{aligned}
 &am[\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2] J_{n-1, m-1} + i[\beta_2 \alpha_0^2 am(m-1) - \gamma P_0 b^*] J_{n-2, m-2} \\
 &+ \{-ia\beta_2 \alpha_0^2 (m(n-1) + n(m+1)) + a[\lambda + i(-k + \omega \beta_1 - \beta_2 \omega^2)] - 2i\gamma P_0 a\} J_{n, m} \\
 &- an[\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2] J_{n+1, m+1} + i\beta_2 \alpha_0^2 an(n+1) J_{n+2, m+2} \\
 &- i\gamma |a|^2 a J_{3n, 3m} - 2i\gamma |b|^2 a J_{3n-4, 3m-4} = 0,
 \end{aligned}
 \tag{82}$$

and

$$\begin{aligned}
 &bm'[\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4] J_{n'-1, m'-1} + i\alpha_0^2 \beta_4 bm'(m'-1) J_{n'-2, m'-2} \\
 &+ \{-i\alpha_0^2 \beta_4 b[m'(n'-1) + n'(m'+1)] + [\lambda' + i(k - \omega \beta_3 - \beta_4 \omega^2)] b - 2i\gamma P_0 b\} J_{n', m'} \\
 &- n'(\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4) J_{n'+1, m'+1} + i[\alpha_0^2 \beta_4 an'(n'+1) - \gamma P_0 a^*] J_{n'+2, m'+2} \\
 &- i\gamma |b|^2 b J_{3n', 3m'} - 2i\gamma |a|^2 b J_{3n'+4, 3m'+4} = 0.
 \end{aligned}
 \tag{83}$$

The dominant pair which favors a large number of groupings of terms in Equation (82) is $(1, 1)$ while the pair which favors a large number of groupings of terms in Equation (83) is $(-1, -1)$. For $(n, m) = (1, 1)$, Equation (82) becomes

$$\begin{aligned}
 &a(\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2) - i(\gamma P_0 b^* + 2\gamma |b|^2 a) J_{-1, -1} \\
 &+ [-2ia\beta_2 \alpha_0^2 + \lambda a + i(-k + \omega \beta_1 - \beta_2 \omega^2)] a - 2i\gamma P_0 a J_{1, 1} \\
 &- a[\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2] J_{2, 2} + i(2\beta_2 \alpha_0^2 a - \gamma |a|^2 a) J_{3, 3} = 0.
 \end{aligned}
 \tag{84}$$

By introducing $(n', m') = (-1, -1)$ into Equation (83), we obtain

$$\begin{aligned}
 &\{-2i\alpha_0^2 \beta_4 b + \lambda' b + i(k - \omega \beta_3 - \beta_4 \omega^2) b - 2i\gamma P_0 b\} J_{-1, -1} \\
 &b(\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4) - i(\gamma P_0 a^* + 2\gamma |a|^2 b) J_{1, 1} \\
 &+ i[2\alpha_0^2 \beta_4 b - \gamma |b|^2 b] J_{-3, -3} - b(\alpha - \alpha_0 \beta_3 - 2\alpha_0 \omega \beta_4) J_{-2, -2} = 0.
 \end{aligned}
 \tag{85}$$

Equations (84) and (85) are verified for $a \neq 0$ and $b \neq 0$ if we have the following sets of equations

$$\alpha - \alpha_0 \beta_1 + 2\alpha_0 \omega \beta_2 = 0, \tag{86}$$

$$\gamma P_0 b^* + 2\gamma |b|^2 a = 0, \tag{87}$$

$$\lambda + i[k + \omega\beta_1 - \beta_2\omega^2 - 2\gamma P_0 - 2\beta_2\alpha_0^2] = 0, \tag{88}$$

$$2\beta_2\alpha_0^2 - \gamma|a|^2 = 0, \tag{89}$$

$$\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4 = 0, \tag{90}$$

$$\gamma P_0 a^* + 2\gamma|a|^2 b = 0, \tag{91}$$

$$\lambda' + i[k - \omega\beta_3 - \beta_4\omega^2 - 2\gamma P_0 - 2\beta_4\alpha_0^2] = 0, \tag{92}$$

$$2\alpha_0^2\beta_4 - \gamma|b|^2 b = 0. \tag{93}$$

The combination of Equations (86) and (90) allows us to obtain

$$\omega = \frac{\beta_1 - \beta_3}{2(\beta_2 + \beta_4)}, \beta_2 \neq -\beta_4. \tag{94}$$

Equations (87) and (91) give us respectively

$$\lambda = -i[k + \omega\beta_1 - \beta_2\omega^2 - 2\gamma P_0 - 2\beta_2\alpha_0^2], \tag{95}$$

and

$$\lambda' = -i[k - \omega\beta_3 - \beta_4\omega^2 - 2\gamma P_0 - 2\beta_4\alpha_0^2]. \tag{96}$$

Equations (95) and (96) require under these conditions that the dissipation coefficients be pure imaginary. Solving Equations (89) and (93) gives

$$|a| = \alpha_0 \sqrt{\frac{2\beta_2}{\gamma}}, \beta_2\gamma > 0, \Rightarrow \exists \theta \in R / a = \alpha_0 \sqrt{\frac{2\beta_2}{\gamma}} \exp i\theta, \tag{97}$$

and

$$|b| = \alpha_0 \sqrt{\frac{2\beta_4}{\gamma}}, \beta_4\gamma > 0, \Rightarrow \exists \theta' \in R / a = \alpha_0 \sqrt{\frac{2\beta_4}{\gamma}} \exp i\theta', \tag{98}$$

Inserting (97) and (98) respectively in (87) and (91) gives the constraint relation on the polarization intensity P_0 as

$$|P_0| = \frac{4\alpha_0^2}{\gamma} \sqrt{\beta_2\beta_4}, \beta_2\beta_4 > 0. \tag{99}$$

Solutions in this case are given by

$$\begin{aligned} \psi(\xi) &= \alpha_0 \sqrt{\frac{2\beta_2}{\gamma}} J_{1,1}(\xi) \exp i\theta, \\ \Rightarrow U(z,t) &= \alpha_0 \sqrt{\frac{2\beta_2}{\gamma}} J_{1,1}(\alpha z - \alpha_0 t) \exp i(kz - \omega t + \theta), \end{aligned} \tag{100}$$

and

$$\begin{aligned} \phi(\xi) &= \alpha_0 \sqrt{\frac{2\beta_4}{\gamma}} J_{-1,-1}(\xi) \exp i\theta', \\ \Rightarrow V(z,t) &= \alpha_0 \sqrt{\frac{2\beta_4}{\gamma}} J_{-1,-1}(\alpha z - \alpha_0 t) \exp -i(kz - \omega t - \theta'), \end{aligned} \tag{101}$$

These solutions verify Equations (33) and (34) without any modification of

their initial structures.

➤ **Study of case** $(n' = n + 2, m' = m + 2) \Rightarrow (n = n' - 2, m = m' - 2)$

The insertion of $(n' = n + 2, m' = m + 2)$ and $(n = n' - 2, m = m' - 2)$ respectively in Equations (41) and (42) lead respectively to the following decoupled range equations

$$\begin{aligned}
 & am(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{n-1,m-1} + i\beta_2\alpha_0^2am(m-1)J_{n-2,m-2} \\
 & + \left\{ -ia\beta_2\alpha_0^2[m(n-1) + n(m+1)] + a[\lambda + i(-k + \omega\beta_1 - \beta_2\omega^2)] - 2i\gamma P_0a \right\} J_{n,m} \\
 & - an(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{n+1,m+1} + i[\beta_2\alpha_0^2an(n+1) - \gamma P_0b^*]J_{n+2,m+2} \\
 & - i\gamma|a|^2 aJ_{3n,3m} - 2i\gamma|b|^2 aJ_{3n+4,3m+4} = 0,
 \end{aligned} \tag{102}$$

and

$$\begin{aligned}
 & bm'[\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4]J_{n'-1,m'-1} + i[\alpha_0^2\beta_4bm'(m'-1) - \gamma P_0a^*]J_{n'-2,m'-2} \\
 & + \left\{ -i\alpha_0^2\beta_4b[m'(n'-1) + n'(m'+1)] + [\lambda' + i(k - \omega\beta_3 - \beta_4\omega^2)]b - 2i\gamma P_0b \right\} J_{n',m'} \\
 & - n'(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4)J_{n'+1,m'+1} + i\alpha_0^2\beta_4an'(n'+1)J_{n'+2,m'+2} \\
 & - i\gamma|b|^2 bJ_{3n',3m'} - 2i\gamma|a|^2 bJ_{3n'-4,3m'-4} = 0.
 \end{aligned} \tag{103}$$

The dominant pair of indices that favors the large number of groupings of terms in Equation (102) is $(n, m) = (-1, -1)$. Likewise, the dominant couple which favors a large number of groupings of terms in Equation (103) is $(n', m') = (1, 1)$. Substituting the indices $(n, m) = (-1, -1)$ into Equation (102) and the indices $(n', m') = (1, 1)$ into Equation (103) gives respectively

$$\begin{aligned}
 & a(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2) - a(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{-2,-2} \\
 & + i(2\beta_2\alpha_0^2a - \gamma|a|^2 a)J_{-3,-3} \\
 & + [\lambda a + i(-ka + \omega\beta_1a - \beta_2\omega^2a - 2a\beta_2\alpha_0^2 - 2\gamma P_0a)]J_{-1,-1} \\
 & + i(\gamma P_0b^* + 2\gamma|b|^2 a)J_{1,1} = 0,
 \end{aligned} \tag{104}$$

and

$$\begin{aligned}
 & b(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4) - i(\gamma P_0a^* + 2\gamma|a|^2 b)J_{-1,-1} \\
 & + [\lambda'b + i(k - \omega\beta_3 - \beta_4\omega^2 - 2\gamma P_0 - 2\alpha_0^2\beta_4)b]J_{1,1} \\
 & - (\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4)J_{2,2} + i(2\alpha_0^2\beta_4 - \gamma|b|^2 b)J_{3,3} = 0.
 \end{aligned} \tag{105}$$

The analysis of Equations (104) and (105) leads to the same equations obtained in the case $(n' = n - 2, m' = m - 2) \Rightarrow (n = n' + 2, m = m' + 2)$ treated above. We also obtain the same constraints that the parameters of the equations must obey. Thus the solutions obtained in this case are given by

$$\begin{aligned}
 \psi(\xi) &= \alpha_0 \sqrt{\frac{2\beta_2}{\gamma}} J_{-1,-1}(\xi) \exp i\theta, \\
 \Rightarrow U(z, t) &= \alpha_0 \sqrt{\frac{2\beta_2}{\gamma}} J_{-1,-1}(\alpha z - \alpha_0 t) \exp i(kz - \omega t + \theta),
 \end{aligned} \tag{106}$$

and

$$\begin{aligned} \phi(\xi) &= \alpha_0 \sqrt{\frac{2\beta_4}{\gamma}} J_{1,1}(\xi) \exp i\theta', \\ \Rightarrow V(z,t) &= \alpha_0 \sqrt{\frac{2\beta_4}{\gamma}} J_{1,1}(\alpha z - \alpha_0 t) \exp -i(kz - \omega t - \theta'), \end{aligned} \tag{107}$$

➤ **Study of case** $n = n'$

For $n = n'$, Equations (39) and (40) respectively lead to the following decoupled range equations

$$\begin{aligned} &am(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{n-1,m-1} + i\beta_2\alpha_0^2am(m-1)J_{n-2,m-2} \\ &+ \left\{ -ia\beta_2\alpha_0^2[m(n-1) + n(m+1)] + a[\lambda + i(-k + \omega\beta_1 - \beta_2\omega^2)] - 2i\gamma P_0a - i\gamma P_0b^* \right\} J_{n,m} \\ &- an(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{n+1,m+1} + i\beta_2\alpha_0^2an(n+1)J_{n+2,m+2} \\ &- i(\gamma|a|^2a + 2\gamma|b|^2a)J_{3n,3m} = 0, \end{aligned} \tag{108}$$

and

$$\begin{aligned} &bm'(\alpha - \alpha_0\beta_3 + 2\alpha_0\omega\beta_4)J_{n'-1,m'-1} + i\beta_4\alpha_0^2bm'(m'-1)J_{n'-2,m'-2} \\ &+ \left\{ -ib\beta_4\alpha_0^2[m'(n'-1) + n'(m'+1)] + b[\lambda' + i(k + \omega\beta_3 - \beta_4\omega^2)] - 2i\gamma P_0b - i\gamma P_0a^* \right\} J_{n',m'} \\ &- bn'(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4)J_{n'+1,m'+1} + i\beta_4\alpha_0^2bn'(n'+1)J_{n'+2,m'+2} \\ &- i(\gamma|b|^2b + 2\gamma|a|^2b)J_{3n,3m} = 0, \end{aligned} \tag{109}$$

The values of the pairs for which certain terms of Equations (108) and (109) group together without modifying the structures of Equations (33) and (34) are $(n, m) = (n', m') = (1, 1)$ and $(n, m) = (n', m') = (-1, -1)$.

From Equations (108) and (109) we obtain respectively

$$\begin{aligned} &a(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{0,0} \\ &+ \left\{ -2ia\beta_2\alpha_0^2a + a[\lambda + i(-k + \omega\beta_1 - \beta_2\omega^2)] - 2i\gamma P_0a - i\gamma P_0b^* \right\} J_{1,1} \\ &- a(\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2)J_{2,2} + i[2\beta_2\alpha_0^2a - (\gamma|a|^2a + 2\gamma|b|^2a)]J_{3,3} = 0, \end{aligned} \tag{110}$$

and

$$\begin{aligned} &b(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4)J_{0,0} \\ &+ \left\{ -2ib\beta_4\alpha_0^2b + b[\lambda' + i(k + \omega\beta_3 - \beta_4\omega^2)] - 2i\gamma P_0b - i\gamma P_0a^* \right\} J_{1,1} \\ &- b(\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4)J_{2,2} + i[2\beta_4\alpha_0^2b - (\gamma|b|^2b + 2\gamma|a|^2b)]J_{3,3} = 0. \end{aligned} \tag{111}$$

Equations (110) and (111) are verified if we have the following equations

$$\alpha - \alpha_0\beta_1 + 2\alpha_0\omega\beta_2 = 0, \tag{112}$$

$$\lambda a + ia(k + \omega\beta_1 - \beta_2\omega^2) - 2ia\beta_2\alpha_0^2 - 2i\gamma P_0a - i\gamma P_0b^* = 0, \tag{113}$$

$$\gamma|a|^2 + 2\gamma|b|^2 = 2\beta_2\alpha_0^2, a \neq 0, \tag{114}$$

$$\alpha - \alpha_0\beta_3 - 2\alpha_0\omega\beta_4 = 0, \tag{115}$$

$$\lambda'b + ib(k - \omega\beta_3 - \beta_4\omega^2) - 2ib\beta_4\alpha_0^2 - 2i\gamma P_0b - i\gamma P_0a^* = 0, \tag{116}$$

and

$$\gamma|b|^2 + 2\gamma|a|^2 = 2\beta_4\alpha_0^2, b \neq 0. \tag{117}$$

Equations (112) and (115) impose to write

$$\omega = \frac{\beta_1 - \beta_3}{2(\beta_2 + \beta_4)}, \beta_2 \neq -\beta_4. \tag{118}$$

Solving Equations (114) and (117) allows to obtain

$$|a| = \alpha_0 \sqrt{\frac{2(2\beta_4 - \beta_2)}{3\gamma}} \Rightarrow \exists \theta \in R / a = \alpha_0 \sqrt{\frac{2(2\beta_4 - \beta_2)}{3\gamma}} \exp i\theta, \tag{119}$$

and

$$|b| = \alpha_0 \sqrt{\frac{2(2\beta_2 - \beta_4)}{3\gamma}} \Rightarrow \exists \theta' \in R / a = \alpha_0 \sqrt{\frac{2(2\beta_2 - \beta_4)}{3\gamma}} \exp i\theta'. \tag{120}$$

The solutions subject to the constraints imposed by Equations (113) and (116) are given as follows

$$\begin{aligned} \psi(\xi) &= \alpha_0 \sqrt{\frac{2(2\beta_4 - \beta_2)}{3\gamma}} J_{1,1}(\xi) \exp i\theta, \\ \Rightarrow U(z,t) &= \alpha_0 \sqrt{\frac{2(2\beta_4 - \beta_2)}{3\gamma}} J_{1,1}(\alpha z - \alpha_0 t) \exp i(kz - \omega t + \theta), \end{aligned} \tag{121}$$

and

$$\begin{aligned} \phi(\xi) &= \alpha_0 \sqrt{\frac{2(2\beta_2 - \beta_4)}{3\gamma}} J_{1,1}(\xi) \exp i\theta', \\ \Rightarrow V(z,t) &= \alpha_0 \sqrt{\frac{2(2\beta_2 - \beta_4)}{3\gamma}} J_{1,1}(\alpha z - \alpha_0 t) \exp -i(kz - \omega t - \theta'), \end{aligned} \tag{122}$$

An approach similar to the previous one in this case $(n,m) = (n',m') = (-1,-1)$ allows to have the solutions

$$\begin{aligned} \psi(\xi) &= \alpha_0 \sqrt{\frac{2(2\beta_4 - \beta_2)}{3\gamma}} J_{-1,-1}(\xi) \exp i\theta, \\ \Rightarrow U(z,t) &= \alpha_0 \sqrt{\frac{2(2\beta_4 - \beta_2)}{3\gamma}} J_{-1,-1}(\alpha z - \alpha_0 t) \exp i(kz - \omega t + \theta), \end{aligned} \tag{123}$$

and

$$\begin{aligned} \phi(\xi) &= \alpha_0 \sqrt{\frac{2(2\beta_2 - \beta_4)}{3\gamma}} J_{-1,-1}(\xi) \exp i\theta', \\ \Rightarrow V(z,t) &= \alpha_0 \sqrt{\frac{2(2\beta_2 - \beta_4)}{3\gamma}} J_{-1,-1}(\alpha z - \alpha_0 t) \exp -i(kz - \omega t - \theta'), \end{aligned} \tag{124}$$

Our primary intention was not only to find some signals likely to propagate in optical fibers and subjected to the effects of FWM, but also to verify the impact

of FWM and other properties of the fiber on a wave or signal taken individually. This is how the solutions obtained do not depend on the factors characterizing the FWM, we verified that the dispersion and nonlinearity coefficients have a considerable impact on the intensity of the signals and even the angular frequency of the signals. **Figure 1** shows the variation of intensities as a function of the nonlinearity coefficient. We notice that the intensity gradually decreases for large values of the nonlinearity coefficient. **Figure 2** shows the variations in intensities as a function of the dispersion coefficients. The intensity in this case increases for larger values of the dispersion coefficients. **Figure 3** shows the variations of the angular frequency as a function of the coefficients of temporal variation and dispersion.

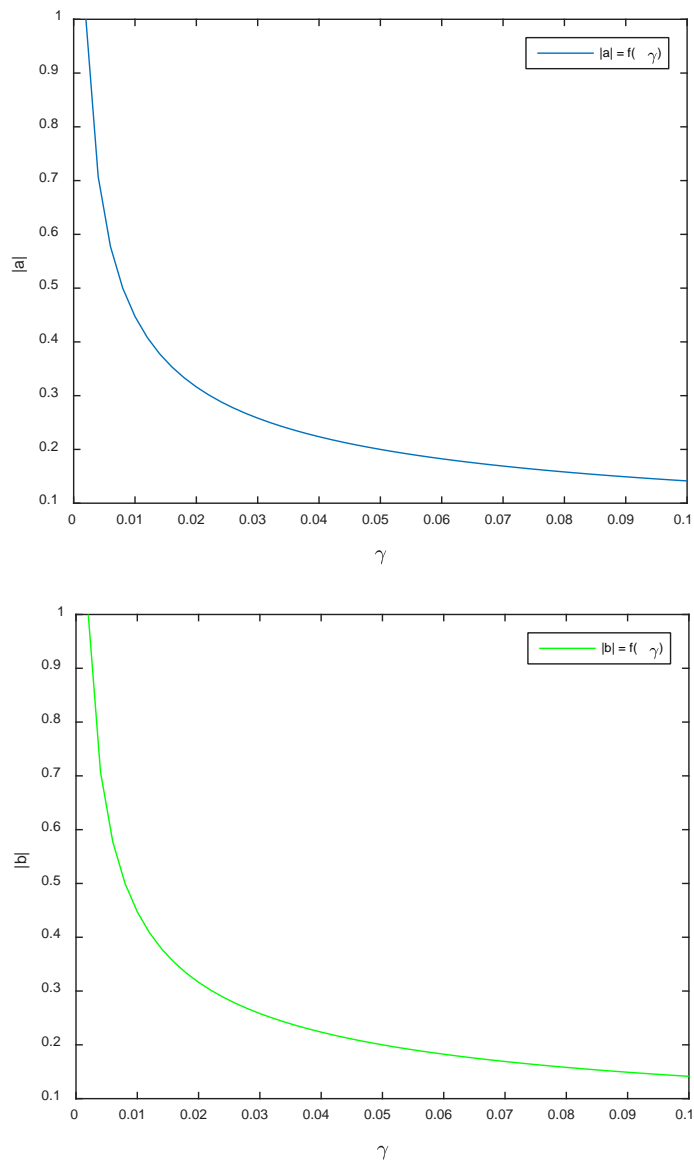


Figure 1. Variation of intensities $|a|$ and $|b|$ as a function of the nonlinearity γ coefficient for $\alpha_0 = 0.1$, $\beta_2 = 0.1$ and $\beta_4 = 0.1$.

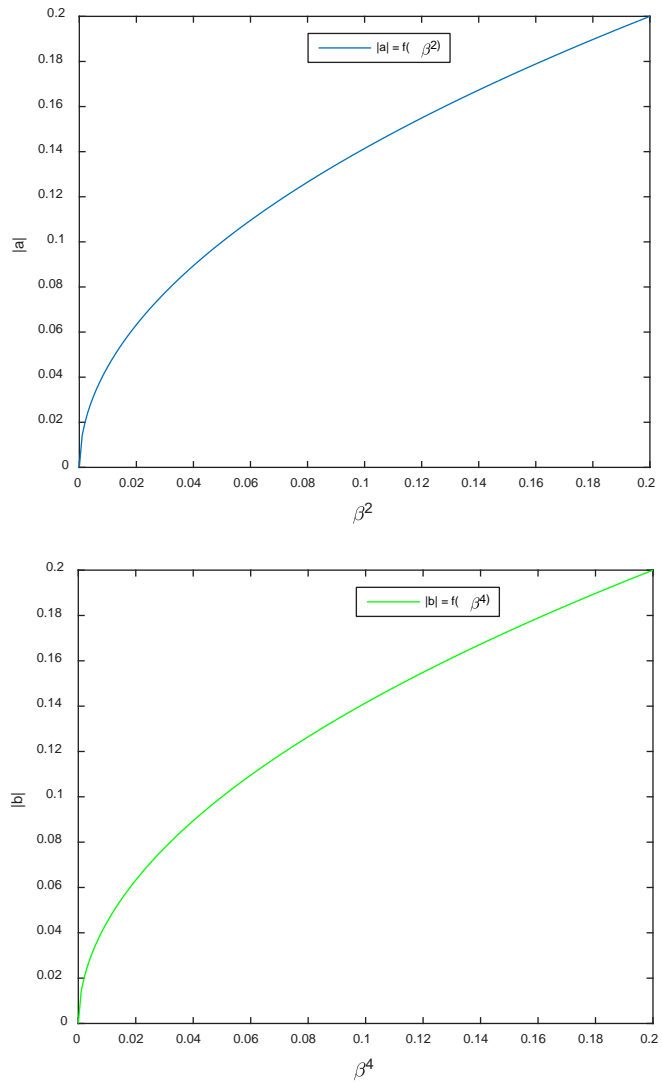
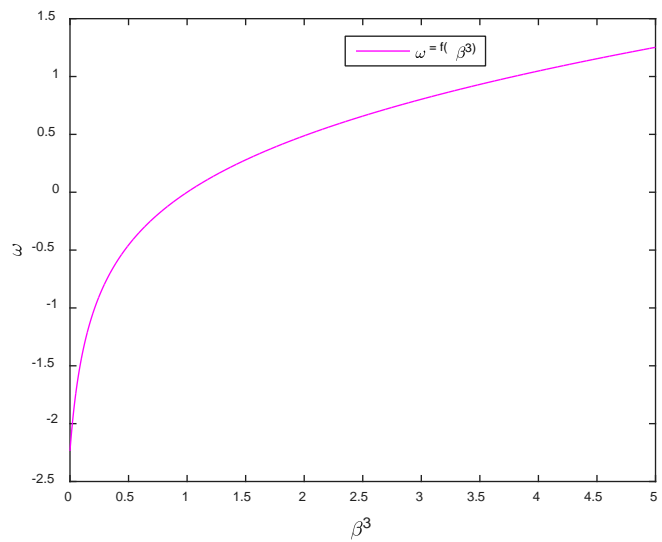


Figure 2. Variation of intensities and respectively according to the dispersion coefficients β_2 and β_4 for $\alpha_0 = 0.1$ and $\gamma = 0.1$.



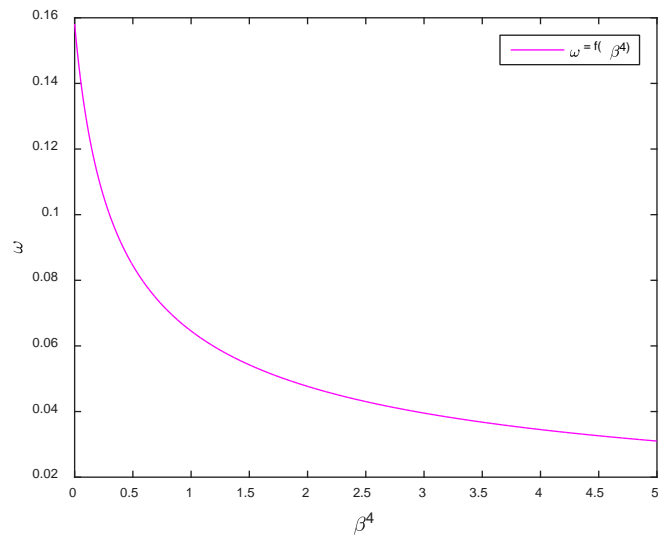


Figure 3. Variation of the angular frequency as a function of the temporal variation coefficient β_3 and the dispersion coefficient β_4 for $\beta_1 = 0.1$ and $\beta_3 = 0.2$.

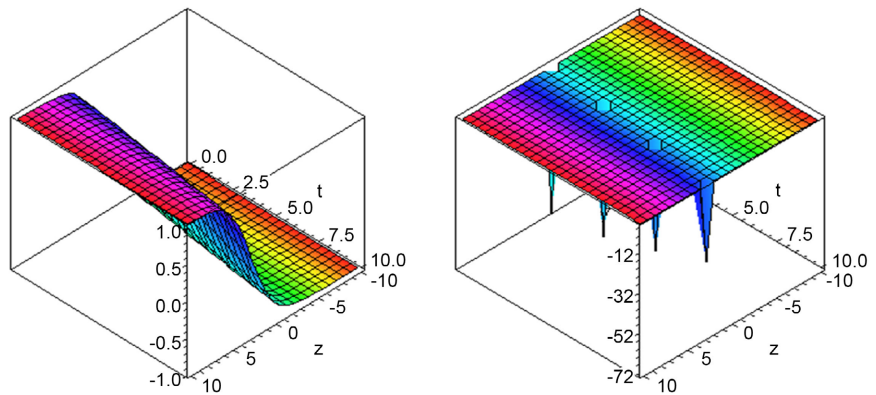


Figure 4. Variation of the signal envelope given respectively by Equation (99) and Equation (100) for $\alpha = 0.4$, $\alpha_0 = 0.1$, $\gamma = 0.1$, $\beta_2 = 0.2$ and $\beta_4 = 0.2$. The curve on the left comes from Equation (99) while the one on the right comes from Equation (101).

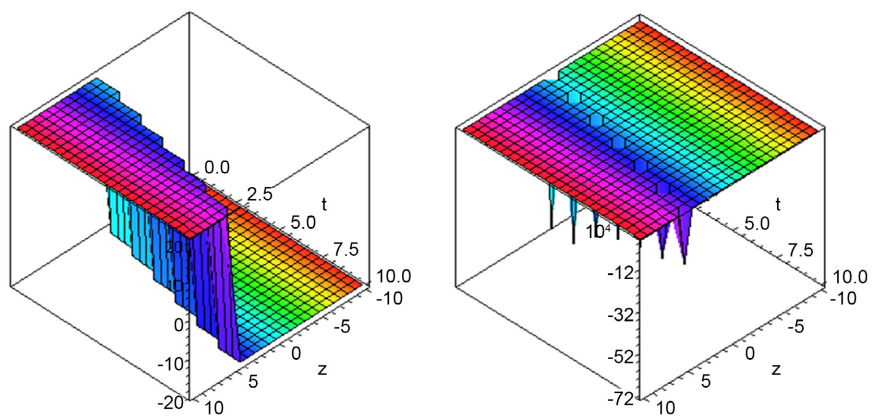


Figure 5. Variation of the signal envelope given respectively by Equation (99) and Equation (100) for $\alpha = 20$, $\alpha_0 = 10$, $\gamma = 0.1$, $\beta_2 = 0.2$ and $\beta_4 = 0.2$. The curve on the left comes from Equation (100) while the one on the right comes from Equation (101).

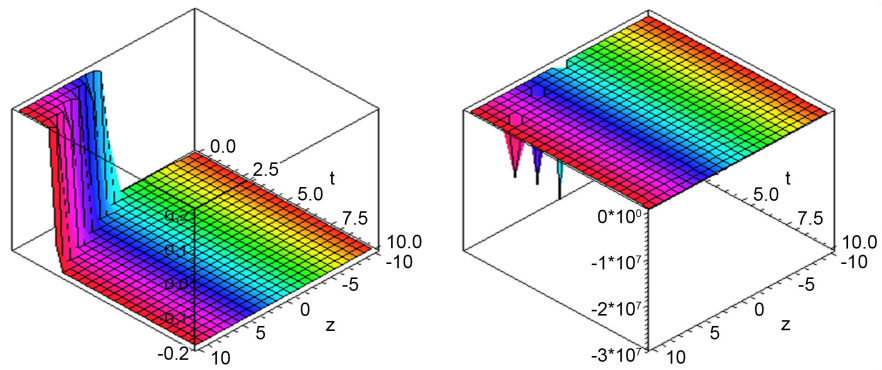


Figure 6. Variation of the signal envelope given respectively by Equation (99) and Equation (100) for $\alpha = 2$, $\alpha_0 = 10$, $\gamma = 0.1$, $\beta_2 = 0.2$ and $\beta_4 = 0.2$. The curve on the left comes from Equation (100) while the one on the right comes from Equation (101).

8. Some Signal Profiles and Comments

Figures 4-6 present the intensities of the envelopes of the signals resulting from solutions (99) and (100). For these cases represented, the profiles obtained are from the large family of kink type solitary waves.

9. Conclusions

This work aimed to determine the types of waves likely to be at the center of FWM experimentation in optical fibers. To this end, we have reviewed the bibliography relating to optical fiber in general and above all revisited the stages of modeling the partial differential equations which govern the propagation dynamics in single-mode, birefringent optical fibers and to end with the case where the effects of FWM are considered. It was therefore a question for us to provide exact solutions to the coupled nonlinear partial differential equations which summarize the propagation dynamics in this case. These equations being visibly very complicated to consider using direct integration in their resolution, we chose the indirect method, based on the use of iB-functions. This method has already proven its effectiveness in solving certain uncoupled and strongly nonlinear equations.

But this work was also a question of verifying its functionality in the resolution of nonlinear and coupled partial differential equations. Thus, we used this approach successfully because it allowed us to obtain solutions with the main particularity of obtaining the relationships uniting the indices of the iB-functions $((n' = n - 2, m' = m - 2), (n' = n - 1, m' = m - 1), (n' = n, m' = m), (n' = n + 1, m' = m + 1), (n' = n + 2, m' = m + 2), (n' = 3n, m' = 3m)$ and $(n = -3n', m = -3m')$ which allowed us to decouple the range equations to finally find pairs of indices leading to solutions in particular $((n, m) = (-1/2, -1/2)$ and $(n', m') = (1/2, 1/2); (n, m) = (1/2, 1/2)$ and $(n', m') = (-1/2, -1/2); (n, m) = (-1, -1)$ and $(n', m') = (1, 1); (n, m) = (1, 1)$ and $(n', m') = (-1, -1); (n, m) = (n', m') = (1, 1)$ and finally $(n, m) = (n', m') = (-1, -1)$.

One of our concerns was also to know if the effect of the FWM had a particu-

lar impact on one of the propagating signals (solutions), but we noted that only the properties of the fiber (dispersion coefficient, non-linearity coefficient) had a direct impact on the intensity of the wave and that the effect of the FWM was of less importance on the solutions concerning the analytical results obtained. We have also demonstrated that the angular frequency closely depends on the dispersion and temporal variation coefficients of the propagation medium which is here the optical fiber.

To return to the solutions, those obtained for $(n, m) = (-1/2, -1/2)$ and $(n', m') = (1/2, 1/2)$ as well as the solutions obtained for $(n, m) = (1/2, 1/2)$ and $(n', m') = (-1/2, -1/2)$ favor the modification of the initial structure of the fiber because are possible when $\beta_2 = \beta_4 = 0$. Under these conditions, the medium is very weakly dispersive. The solutions obtained in the other cases are without any modification of the structure or properties of the fiber. Our satisfaction with this work also comes from the fact that the method used allowed to find exact analytical solutions to the equations that govern the dynamics of FWM in optical fibers. Many attempts at numerical resolutions have been made but very few exact solutions have been proposed to date. This work further attempts to make a significant contribution to the understanding of propagation phenomena in optical fiber.

Data Availability Statement

All data that support the findings of this study are included within the article

Credit Authorship Contribution Statement

J.R. Bogning: Conception of the project, execution of the project, writing, Printing, Investigation, Computation and Numerical study, Interpretation of results.

M.N. Zambo Abou'ou: Writing, Printing, Investigation, Computation, Verification of results

C. R. Ngouo Tchinda: Writing, Printing, Investigation, Computation, Verification of results

M. Fomekong and others: Writing, Printing, Investigation, Computation, Verification of results

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The authors declare no conflicts of interest regarding the publication of this paper.

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