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Binding Number and Fractional *k*-Factors of Graphs

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Abstract

In this paper, we consider the relationship between the binding number and the existence of fractional *k*-factors of graphs. The binding number of G is defined by Woodall as $bind(G)=\min\left\{\frac{|N_G(X)|}{|X|}:\varnothing\neq X\subseteq V(G)\right\}$. It is proved that a graph G has a fractional 1-factor if $bind(G)\geq 1$ and has a fractional *k*-factor if $bind(G)\geq k-\frac{1}{k}$. Furthermore, it is showed that both results are best possible in some sense.

Keywords

Binding Number, Fractional *k*-Factor, Fractional Matching, Independent Set, Covering Set

1. Introduction

All graphs considered in this paper are assumed to be infinite and simple. We refer the readers to [1] for the terminologies not defined here. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For $x\in V(G)$, the degree of x in G is denoted by $d_G(x)$. The minimum vertex degree of G is denoted by $\delta(G)$. For any $S\subseteq V(G)$, we denote by $N_G(S)$ the neighborhood set of S in G and we use $G[S]$ and $G-S$ to denote the subgraph of G induced by S and $V(G)-S$, respectively. A subset S of $V(G)$ is called an independent set (a covering set) of G if every edge of G is incident with at most (at least) one vertex of S.

Let g and f be two integer-valued functions defined on $V(G)$ with $g(x)\leq f(x)$ for any $x\in V(G)$. A spanning subgraph F of G is called a (g,f)-factor if $g(x)\leq d_F(x)\leq f(x)$ holds for any vertex $x\in V(G)$. A (g,f)

-factor is called an $[a,b]$-factor if $g(x)=a$ and $f(x)=b$ for any $x\in V(G)$. An $[a,b]$-factor is called a *k*-factor if $a=b=k$.

Let $h:E(G)\to[0,1]$ be a function, and let $k\geq 1$ be an integer. If $\sum_{e\ni x}h(e)=k$ holds for each vertex $x\in V(G)$, we call $G[F_h]$ a *fractional* *k*-factor of G with indicator function h where $F_h=\{e\in E(G)\mid h(e)>0\}$. A fractional 1-factor is also called a fractional perfect matching [2].

The binding number of G is defined by Woodall [3] as $bind(G)=\min\left\{\frac{|N_G(X)|}{|X|}:\varnothing\neq X\subseteq V(G)\right\}$. It is trivial by the definition that $bind(G)\geq c$ implies that for every subset $X\subseteq V(G)$, we have either $N_G(X)=V(G)$ or $|N_G(X)|\geq c|X|$. It is also obvious that if $bind(G)>1$, then G is connected. Many authors have investigated the relationship between binding number and the existence of factors in graphs. For more information, please refer to [4]-[6]. We begin with some known results.

Anderson gave a sufficient condition for the existence of 1-factors.

Theorem 1.1. ([7]) *If a graph G has even order and* $bind(G)\geq\frac{4}{3}$, *then G has a* 1*-factor.*

Woodall showed the relationship of the binding number and the existence of a Hamiltonian cycle in a graph. In [8] obtained the binding number condition for restricted matching extension in graphs.

Katerinis and Woodal [9] and Katerinis [10] found the minimum degree and binding number conditions for a graph to have *k*-factors. Zhou obtained the binding number condition for a graph to be ID-*k*-factor-critical [11].

[12] considered binding number and the existence of *f*-factors in graphs. The researchers discussed binding number and the existence of (g,f)-factors in graphs [13] and binding number and the existence of (g,f)-factors with prescribed properties in graphs [14]. The authors studied binding number and the existence of fractional (g,f)-factor-critical in graphs [15]. Recently, the researchers considered binding number conditions and various factors ([16]-[18]).

In this paper, we consider the relationship between the binding number and the existence of fractional *k*-factors in *G*. Our main results are the following two theorems.

Theorem 1.2. *Let G be a connected graph with* $|V(G)|\geq 2$, *then G has a fractional perfect matching if* $bind(G)\geq 1$.

Theorem 1.3. *Let* $k\geq 2$ *be an integer. A Graph G with* $|V(G)|\geq k+1$ *has a fractional k-factor if* $bind(G)\geq k-\frac{1}{k}$.

2. Preliminary Lemmas

Liu and Zhang gave a necessary and sufficient condition for a graph to have a fractional *k*-factor in [19].

Lemma 2.1. ([19]) *Let* $k \ge 1$ *be an integer. A graph G has a fractional k-factor if and only if for any subset S of* $V(G)$,

$$k|T| - d_{G-S}(T) \le k|S|$$

where $T = \{x \in V(G) - S \mid d_{G-S}(x) \le k-1\}$.

In particular, for $k = 1$, Scheinermman obtained the following result.

Lemma 2.2. *A graph G has a fractional perfect matching if and only if for any subset S of* $V(G)$,

$$i(G-S) \le |S|$$

where $i(G-S) = \{x \in V(G) - S \mid d_{G-S}(x) = 0\}$.

We obtained the following result in [20].

Lemma 2.3. ([20]) *Let G be a graph and* $H = G[T]$ *such that* $d_G(x) = k-1$ *for every* $x \in V(H)$ *and no component of H is isomorphic to* K_k *where* $T \subseteq V(G)$ *and* $k \ge 2$. *Then H has a maximal independent set I and a covering set* $C = V(G) - I$ *satisfying*

$$|V(H)| \le \left(k - \frac{1}{k+1}\right) i' + \sum_{j=0}^{k-2} (j+1) i''_j,\ |C| \le \left(k - 1 - \frac{1}{k+1}\right) i' + \sum_{j=0}^{k-2} j i''_j,$$

where $i' = |I'| = \left|\{x \mid x \in I, d_H(x) = k-1 = d_G(x)\}\right|$,
$i''_j = \left|\{x \mid x \in I'' = I - I', d_H(x) = j < d_G(x)\}\right|$.

Lemma 2.4. ([21]) *Let G be a graph and* $H = G[T]$ *such that* $\delta(H) \ge 1$ *and* $1 \le d_G(x) \le k-1$ *for every* $x \in V(H)$ *where* $T \subseteq V(G)$ *and* $k \ge 2$. *Let* $T_1, \cdots, T_{k-1}$ *be a partition of the vertices of H satisfying* $d_G(x) = j$ *for each* $x \in T_j$ *where we allow some* T_j *to be empty. If each component of H has a vertex of degree at most* $k-2$ *in G, then H has a maximal independent set I and a covering set* $C = V(H) - I$ *such that*

$$\sum_{j=1}^{k-1} (k-j) c_j \le \sum_{j=1}^{k-1} (k-2)(k-j) i_j$$

where $c_j = |C \cap T_j|$ and $i_j = |I \cap T_j|$ for $j = 1, \cdots, k-1$.

3. Proof of Theorems

Suppose that G satisfies the conditions in Theorem 1.2, but G has no fractional perfect matching. By Lemma 2.2, there exists a subset S of $V(G)$ such that $i(G-S) > |S|$. We choose X as the set of isolated vertices of $G-S$, that is, $|X| = i(G-S)$. Since G is connected, it follows that $S \ne \varnothing$, and $|N_G(X)| \subseteq S \ne V(G)$.

According to the definition of $bind(G)$, we obtain

$$bind(G) \le \frac{|N_G(X)|}{|X|} \le \frac{|S|}{i(G-S)} < 1, \text{ contradicting with } bind(G) \ge 1. \ \square$$

Proof of Theorem 1.3. Suppose that G satisfies the conditions in Theorem 1.3, but G has no fractional k-factors. From Lemma 2.1, there exists a subset S of $V(G)$ such that

$$k|T| - d_{G-S}(T) > k|S|, \tag{1}$$

where $T=\{x\in V(G)-S \mid d_{G-S}(x)\le k-1\}$.

Let m be the number of the components of $H'=G[T]$ which are isomorphic to K_k, we may assume that $C_1,C_2,\cdots,C_m$ are these components. Let $T_0=\{x\in V(H') \mid d_{G-S}(x)=0\}$ and $H=H'-mK_k-T_0$.

If $|V(H)|=0$, by (1) we get $k|T_0|+mk>k|S|$, that is, $|S|\le|T_0|+m-1$.

If $m=0$, $|S|<|T_0|$. Set $X=T_0$, then $|X|=|T_0|$ and $N_G(X)\subseteq S$, $N_G(X)\ne V(G)$. We obtain that $bind(G)\le\frac{|N_G(X)|}{|X|}\le\frac{|S|}{|T_0|}<1$, a contradiction.

If $m\ge1$, let $X=C_1\cup\cdots\cup C_{m-1}\cup\{x\}\cup T_0$ where x is an arbitrary vertex of C_m, then $|X|=(m-1)k+1+|T_0|$ and $N_G(X)\subseteq C_1\cup\cdots\cup C_{m-1}\cup(C_m-\{x\})\cup S$, $N_G(X)\ne V(G)$. This follows that

$$bind(G)\le\frac{|N_G(X)|}{|X|}\le\frac{mk-1+|S|}{(m-1)k+1+|T_0|}\le\frac{mk-1+m-1+|T_0|}{(m-1)k+1+|T_0|}.$$

When $k\ge2$,

$$\begin{aligned}&\left((m-1)k+1+|T_0|\right)\left(k-\frac{1}{k}\right)-\left(mk-1+m-1+|T_0|\right)\\&\ge\left((m-1)k+1\right)\left(k-\frac{1}{k}\right)-(mk-1+m-1)=(m-1)\left(k^2-k-2\right)+1-\frac{1}{k}>0.\end{aligned}$$

That is, $bind(G)<k-\frac{1}{k}$, a contradiction.

Now we consider that $|V(H)|>0$. Let $H=H_1\cup H_2$ where H_1 is the union of components of H which satisfies that $d_{G-S}(x)=k-1$ for any vertex $x\in V(H_1)$ and $H_2=H-H_1$. By Lemma 2.3, H_1 has a maximal independent set I_1 and the covering set $C_1=V(H_1)-I_1$ such that

$$|V(H_1)|\le\left(k-\frac{1}{k+1}\right)i'+\sum_{j=0}^{k-2}(j+1)i''_j,\ |C_1|\le\left(k-1-\frac{1}{k+1}\right)i'+\sum_{j=0}^{k-2}ji''_j,$$

where $i'=|I'|=\left|\{x \mid x\in I_1,d_{H_1}(x)=k-1=d_{G-S}(x)\}\right|$, $i''_j=\left|\{x \mid x\in I''=I_1-I'_1,d_{H_1}(x)=j<d_{G-S}(x)\}\right|$.

On the other hand, we may assume that $\delta(H_2)\ge1$. Since $\Delta(H_2)\le k-1$, let $T_j=\{x\in V(H_2)|_{G-S}(x)=j\}$ for $1\le j\le k-1$. By the definition of H_2 we know that there exists one vertex with degree at most $k-2$ in $G-S$ from each component of H_2. According to Lemma 2.4, H_2 has a maximal independent set I_2 and the covering set $C_2=V(H_2)-I_2$ such that

$$\sum_{j=1}^{k-1}(k-j)c_j\le\sum_{j=1}^{k-1}(k-2)(k-j)i_j \tag{2}$$

where $c_j=|C_2\cap T_j|$ and $i_j=|I_2\cap T_j|$ for $j=1,\cdots,k-1$.

Set $W=V(G)-S-T$ and $U=S\cup C_1\cup(N_G(I''_1)\cap W)\cup C_2\cup(N_G(I_2)\cap W)$. Then

$$|U|\le|S|+|C_1|+\sum_{j=0}^{k-2}(k-1-j)i''_j+\sum_{j=0}^{k-1}ji_j.$$

Let X' be the set of isolated vertices of $G-U$, then

$\left|X'\right| = i\left(G-U\right) \geq t_0 + i_1' + \sum_{j=0}^{k-2} i_j'' + \sum_{j=0}^{k-1} i_j$. Set $X = X' \cup C_1 \cup \cdots \cup C_m$, then $\left|X\right| \geq t_0 + i_1' + \sum_{j=0}^{k-2} i_j'' + \sum_{j=0}^{k-1} i_j + mk$ and $N_G\left(X\right) \subseteq U \cup C_1 \cup \cdots \cup C_m$, $N_G\left(X\right) \neq V\left(G\right)$. By the definition of $bind\left(G\right)$ we have $\left|U\right| + mk \geq \left|N_G\left(X\right)\right| \geq bind\left(G\right)\left|X\right|$. Therefore

$$\left|S\right| + \left|C_1\right| + \sum_{j=0}^{k-2}\left(k-1-j\right)i_j'' + \sum_{j=0}^{k-1} j i_j \geq bind\left(G\right)\left(mk + t_0 + i_1' + \sum_{j=0}^{k-2} i_j'' + \sum_{j=0}^{k-1} i_j\right) - mk \quad (3)$$

From (1) we have $k\left(t_0+m\right) + \left|V\left(H_1\right)\right| + \sum_{j=1}^{k-1}\left(k-j\right)i_j + \sum_{j=1}^{k-1}\left(k-j\right)c_j \geq k\left|S\right|$.

Combined with (3),

$$\begin{aligned} & k\left(t_0+m\right) + \left|V\left(H_1\right)\right| + k\left|C_1\right| + \sum_{j=0}^{k-2} k\left(k-1-j\right)i_j'' + \sum_{j=1}^{k-1}\left(k-j\right)c_j \\ & > kbind\left(G\right)\left(t_0 + mk + i_1' + \sum_{j=0}^{k-2} i_j''\right) + \sum_{j=1}^{k-1}\left(kbind\left(G\right) - kj - k + j\right)i_j - mk^2. \end{aligned}$$

By the notation of $bind\left(G\right)$, we have

$$\begin{aligned} & \left|V\left(H_1\right)\right| + k\left|C_1\right| + \sum_{j=0}^{k-2} k\left(k-1-j\right)i_j'' + \sum_{j=1}^{k-1}\left(k-j\right)c_j \\ & > kbind\left(G\right)\left(i_1' + \sum_{j=0}^{k-2} i_j''\right) + \sum_{j=1}^{k-1}\left(kbind\left(G\right) - kj - k + j\right)i_j. \end{aligned}$$

By Lemma 2.3, we get that

$$\begin{aligned} & \left|V\left(H_1\right)\right| + k\left|C_1\right| + \sum_{j=0}^{k-2} k\left(k-1-j\right)i_j'' \\ & \leq \left(k - \frac{1}{k+1} + k\left(k-1-\frac{1}{k+1}\right)\right)i_1' + \sum_{j=0}^{k-2}\left(j+1+kj\right)i_j'' + \sum_{j=0}^{k-2} k\left(k-1-j\right)i_j'' \\ & = \left(k^2-1\right)i_1' + \sum_{j=0}^{k-2}\left(k^2-k+j+1\right)i_j''. \end{aligned}$$

Combined with (2),

$$\begin{aligned} & \sum_{j=1}^{k-1}\left(k-2\right)\left(k-j\right)i_j + \left(k^2-1\right)i_1' + \sum_{j=0}^{k-2}\left(k^2-k+j+1\right)i_j'' \\ & > kbind\left(G\right)\left(i_1' + \sum_{j=0}^{k-2} i_j''\right) + \sum_{j=1}^{k-1}\left(kbind\left(G\right) - kj - k + j\right)i_j \\ & \geq \left(k^2-1\right)i_1' + kbind\left(G\right)\sum_{j=0}^{k-2} i_j'' + \sum_{j=1}^{k-1}\left(kbind\left(G\right) - kj - k + j\right)i_j. \end{aligned}$$

We have

$$\sum_{j=1}^{k-1}\left(k-2\right)\left(k-j\right)i_j + \sum_{j=0}^{k-2}\left(k^2-1\right)i_j'' > kbind\left(G\right)\sum_{j=0}^{k-2} i_j'' + \sum_{j=1}^{k-1}\left(kbind\left(G\right) - kj - k + j\right)i_j.$$

Thus at least one of the following two cases must hold.

Case 1. There exists at least one *j* satisfying $\left(k-2\right)\left(k-j\right) > kbind\left(G\right) - kj - k + j$. It follows that $bind\left(G\right) < \frac{k^2-k+j}{k} \leq k - \frac{1}{k}$ ($j \leq k-1$), a contradiction.

Case 2. $k^2 - 1 > kbind(G)$. In this case we have $bind(G) < k - \frac{1}{k}$, a contradiction.

The proof of Theorem 1.3 is complete. □

Remark. The result in Theorem 1.2 is sharp. To see this, consider the graph $G_1 = K_n \vee (n+1)K_1$ where n is an arbitrary positive integer. We can immediately obtain that $bind(G_1) = \frac{n}{n+1} < 1$ and $bind(G_1)$ is arbitrary close to 1 when n is enough. If we choose $S = V(K_n)$, then $i(G-S) = n+1 > |S|$. It follows that G_1 has no fractional perfect matching by Lemma 2.2.

To see Theorem 1.3 is also sharp, we construct the following graph G_2: If $k = 2$, let $V(G_2) = V(A) \cup V(B) \cup V(C)$ where $A = K_{(nk+1)(k-1)} = K_{2n+1}$, $B = (nk+1)K_1 = (2n+1)K_1$ and $C = K_{n(k-1)} = K_n$. Set other edges in G_2 are a perfect matching between A and B and all the pairs between B and C. This follows that $bind(G_2) = \frac{(nk+1)(k-1) + n(k-1)}{nk+1}(k=2)$. It is easy to see that $bind(G_2) < k - \frac{1}{k}$ and $bind(G_2)$ can be made arbitrary close to $k - \frac{1}{k}$ when n is large enough ($k = 2$). Let $S = C$, by Lemma 2.1 G_2 has no fractional k-factor ($k = 2$).

4. Conclusion

Therefore, we conclude that a graph G has a fractional 1-factor if $bind(G) \geq 1$ and has a fractional k-factor if $bind(G) \geq k - \frac{1}{k}$. Furthermore, it is showed that both results are best possible in some sense.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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