

A New Kind of Pre-Mean-Type Mappings and Its Gauss Iteration

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Abstract

A function which is reflexive is called by pre-mean, a more generalized definition of a mean. In this paper, we define a new pre-mean and study its properties, and then using the given invariant curve we consider the problem of convergence of Gauss iteration of a kind of pre-mean type mappings generated by the exponential and logarithmic functions.

Keywords

Pre-Mean-Type Mapping, Invariant Equation, Invariant Curves

1. Introduction

Mean value is an important fundamental concept in mathematics. For thousands of years, many forms of mean have been proposed, such as the arithmetic mean

$A: \mathbb{R}^p \rightarrow \mathbb{R}$:

$$A(x_1, \dots, x_p) = \frac{x_1 + \dots + x_p}{p},$$

the geometric mean $G: (0, +\infty)^p \rightarrow (0, \infty)$:

$$G(x_1, \dots, x_p) = \sqrt[p]{x_1 \cdots x_p},$$

the harmonize mean $H: (0, +\infty)^p \rightarrow (0, \infty)$:

$$H(x_1, \dots, x_p) = \frac{p}{\frac{1}{x_1} + \dots + \frac{1}{x_p}}.$$

Although there are various mean forms, they all meet the following definitions ([1]):

Let $I \subset \mathbb{R}$ be an open interval. If the function $M: I^p \rightarrow I$ satisfies

$$\min\{x_1, \dots, x_p\} \leq M(x, y) \leq \max\{x_1, \dots, x_p\}, \quad x_1, \dots, x_p \in I, \quad (1)$$

then M is called a mean on the interval I . When the above inequalities strictly hold, M is called strict mean.

Theorem 1. ([2] [3]) *If the mean M and N of interval I are continuous, that is, any $x, y \in I$, $x \neq y$, we have*

$$\max(M(x, y), N(x, y)) - \min(M(x, y), N(x, y)) < \max(x, y) - \min(x, y),$$

then

1) For any $n \in \mathbb{N}$, Gauss iteration $(M, N)^n$ of the mapping (M, N) is the mean type map on I^2 .

2) There is a continuous mean $K : I^2 \rightarrow I$, so that $\left((M, N)^n\right)_{n=0}^{\infty}$ converges to (K, K) .

3) K is (M, N) -invariant mean, and is the unique (M, N) -invariant mean.

4) If (M, N) is a strict mean mapping, that is, M and N are a strict mean, then K is also a strict mean.

5) If M and N are (strictly) increasing for each variable, then K is also (strictly) increasing for each variable.

The above theorem provides the relationship between invariant equations and convergence of iteration of mean-type mappings. By studying the problem of invariant equations, we can study the dynamic behavior of mean-type mappings. There are many works to study kinds of invariant equations with different means ([4]-[8]). For the pre-mean-type mapping, the results have been still few. Next we will introduce some definitions for the pre-mean.

Obviously, it can be seen from the inequalities (1) that any mean satisfies the reflexivity, *i.e.*

$$M(x, \dots, x) = x, \quad x \in I.$$

However, not all the functions that satisfy the reflexivity are means. In other words, there are functions in the interval I that satisfy the reflexivity but not be means. In 2006, Matkowski ([9]) gave the definition of *pre-mean* as follows

If the function $M : I^p \rightarrow I$ satisfies reflexivity, M is called a pre-mean.

This indicates that a mean must be a pre-mean, but a pre-mean is not necessarily a mean, and the pre-mean is a generalized function than the mean. For example, the function $M : \mathbb{R}^2 \rightarrow \mathbb{R}$ is defined as

$$M(x, y) = \ln \left(\frac{\frac{1}{3}e^{2x} + \frac{2}{3}e^{2y}}{\frac{1}{3}e^x + \frac{2}{3}e^y} \right), \quad x, y \in \mathbb{R},$$

which satisfies reflexivity, but $M\left(\ln \frac{1}{2}, 0\right) = \ln \frac{27}{24} > \max\left\{\ln \frac{1}{2}, 0\right\}$ does not satisfy the definition of mean, so this function is a pre-mean but not a mean. But if a pre-mean is (strictly) increasing with respect to each variable, then it must be a (strict) mean ([9] [10]).

In this paper, we will consider the convergence of iterates of pre-mean type

mappings of the form $(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})$ with $\lambda, \mu \in (0,1)$, $p, q \in \mathbb{R}$, $p \neq q$, where $E_{\lambda,\mu}^{[p,q]}: \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfies

$$E_{\lambda,\mu}^{[p,q]}(x, y) = \ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\mu e^{qx} + (1-\mu)e^{qy}} \right)^{\frac{1}{p-q}}, \quad x, y \in \mathbb{R}.$$

2. Some Definitions and Auxiliary Results

Lemma 1 Suppose $p, q \in \mathbb{R}$, $p \neq q$ and $\lambda, \mu \in (0,1)$, then

- 1) $E_{\lambda,\mu}^{[p,q]}$ is the pre-mean.
- 2) $E_{\lambda,\mu}^{[p,q]}$ is the mean if and only if $\lambda = \mu$ or $pq \leq 0$.

Proof. 1) If $\forall x \in \mathbb{R}$, make $x = y$, then $E_{\lambda,\mu}^{[p,q]}(x, x) = \ln \left(\frac{e^{px}}{e^{qx}} \right)^{\frac{1}{p-q}} = x$, so it is a pre-mean.

2) Firstly, for the case $\lambda = \mu$, $E_{\lambda,\mu}^{[p,q]}(x, x) = \ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\lambda e^{qx} + (1-\lambda)e^{qy}} \right)^{\frac{1}{p-q}}$. Let $x < y$, $p > q$, then $(p-q)x < (p-q)y$ and

$$e^{px+qy} < e^{qx+py}.$$

It can be obtained after doing a simple calculation on the above formula

$$(\lambda e^{px} + (1-\lambda)e^{py})e^{qy} < (\lambda e^{qx} + (1-\lambda)e^{qy})e^{py},$$

that is

$$\ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\lambda e^{qx} + (1-\lambda)e^{qy}} \right)^{\frac{1}{p-q}} < y.$$

Similarly, in the same way, we get $\ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\lambda e^{qx} + (1-\lambda)e^{qy}} \right)^{\frac{1}{p-q}} > x$, i.e.

$$x < \ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\lambda e^{qx} + (1-\lambda)e^{qy}} \right)^{\frac{1}{p-q}} < y.$$

For the case of $p < q$, it can be proved similarly. From the definition of the mean value, the $\ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\lambda e^{qx} + (1-\lambda)e^{qy}} \right)^{\frac{1}{p-q}} \in (x, y)$, $E_{\lambda,\mu}^{[p,q]}$ is a mean at $\lambda = \mu$.

Secondly, for the case $pq \leq 0$, it is obvious that the function $E_{\lambda,\mu}^{[p,q]}$ is increasing with respect to each of the variables, so it is a mean. The proof of the ‘‘if’’ part has been finished. To prove the ‘‘only if’’ result, assume that $\lambda = \mu$ or $pq \leq 0$ and consider two possible cases.

Case $p > 0$ and $q > 0$.

If $E_{\lambda,\mu}^{[p,q]}$ were a mean, there is

$$E_{\lambda,\mu}^{[p,q]}(\ln(x), 0) < 0 \quad \text{and} \quad E_{\lambda,\mu}^{[p,q]}(0, \ln(x)) < 0, \quad x \in (0,1)$$

that is

$$\left(\frac{\lambda x^p + (1-\lambda)}{\mu x^q + (1-\mu)}\right)^{\frac{1}{p-q}} < 1 \quad \text{and} \quad \left(\frac{\lambda + (1-\lambda)x^p}{\mu + (1-\mu)x^q}\right)^{\frac{1}{p-q}} < 1, \quad x \in (0,1).$$

Therefore, if $p - q > 0$, then

$$\frac{\lambda x^p + (1-\lambda)}{\mu x^q + (1-\mu)} < 1 \quad \text{and} \quad \frac{\lambda + (1-\lambda)x^p}{\mu + (1-\mu)x^q} < 1, \quad x \in (0,1).$$

Let $x \rightarrow 0$, then

$$\frac{\lambda}{1-\mu} \leq 1 \quad \text{and} \quad \frac{\lambda}{\mu} \leq 1. \tag{2}$$

If $p - q < 0$, then

$$\frac{\lambda x^p + (1-\lambda)}{\mu x^q + (1-\mu)} > 1 \quad \text{and} \quad \frac{\lambda + (1-\lambda)x^p}{\mu + (1-\mu)x^q} > 1, \quad x \in (0,1).$$

Let $x \rightarrow 0$, then

$$\frac{\lambda}{1-\mu} \geq 1 \quad \text{and} \quad \frac{\lambda}{\mu} \geq 1. \tag{3}$$

So, if and only if $\lambda = \mu$, Equations (2) and (3) hold.

Case $p < 0$ and $q < 0$.

Since

$$\begin{aligned} & E_{\lambda,\mu}^{[p,q]}(x,y) + E_{1-\lambda,1-\mu}^{[-p,-q]}(x,y) \\ &= \ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\mu e^{qx} + (1-\mu)e^{qy}} \cdot \frac{(1-\mu)e^{-qx} + \mu e^{-qy}}{(1-\lambda)e^{-px} + \lambda e^{-py}} \right)^{\frac{1}{p-q}} \\ &= \ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{py}}{\mu e^{qx} + (1-\mu)e^{qy}} \cdot \frac{e^{p(x+y)}}{e^{q(x+y)}} \cdot \frac{(1-\mu)e^{qy} + \mu e^{qx}}{(1-\lambda)e^{py} + \lambda e^{px}} \right)^{\frac{1}{p-q}} \\ &= \frac{1}{p-q} \ln e^{(p-q)(x+y)} = x + y, \end{aligned}$$

then $E_{1-\lambda,1-\mu}^{[-p,-q]}(x,y) = x + y - E_{\lambda,\mu}^{[p,q]}(x,y)$. Therefore, if $E_{\lambda,\mu}^{[p,q]}$ were a mean, that is, for all $x < y$ we have

$$x \leq E_{\lambda,\mu}^{[p,q]}(x,y) \leq y,$$

then by $E_{1-\lambda,1-\mu}^{[-p,-q]}(x,y) = x + y - E_{\lambda,\mu}^{[p,q]}(x,y)$, we have

$$x \leq E_{1-\lambda,1-\mu}^{[-p,-q]}(x,y) \leq y,$$

that is, $E_{1-\lambda,1-\mu}^{[-p,-q]}$ would be a mean, which contradicts the previous case. \square

Lemma 2 Let $p, q \in \mathbb{R}$, $p \neq q$, $\lambda, \mu \in (0,1)$, then the arithmetic mean A is invariant with respect to the pre-mean type mapping $(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})$, that is

$$A \circ (E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]}) = A.$$

Proof. For all $x, y \in \mathbb{R}$, we have

$$\begin{aligned}
 & A \circ \left(E_{\lambda, \mu}^{[p, q]}, E_{1-\lambda, 1-\mu}^{[-p, -q]} \right) (x, y) \\
 &= \frac{\ln \left(\frac{\lambda e^{px} + (1-\lambda) e^{py}}{\mu e^{qx} + (1-\mu) e^{qy}} \cdot \frac{(1-\mu) e^{-qx} + \mu e^{-qy}}{(1-\lambda) e^{-px} + \mu e^{-py}} \right)^{\frac{1}{p-q}}}{2} \\
 &= \frac{\ln \left(\frac{\lambda e^{px} + (1-\lambda) e^{py}}{\mu e^{qx} + (1-\mu) e^{qy}} \cdot \frac{e^{p(x+y)}}{e^{q(x+y)}} \cdot \frac{(1-\mu) e^{qy} + \mu e^{qx}}{(1-\lambda) e^{py} + \lambda e^{px}} \right)^{\frac{1}{p-q}}}{2} \\
 &= \frac{1}{2(p-q)} \ln e^{(p-q)(x+y)} = \frac{x+y}{2} = A(x, y).
 \end{aligned}$$

□

To consider the convergence of iterates of the pre-mean but not mean mappings, we now give the definition of invariant curves.

Let $M, N : I^2 \rightarrow I$, $K : I^2 \rightarrow \mathbb{R}$ be some functions, let $J \subset I$, $f : J \rightarrow I$ be a function on J . If there is

$$f(M(x, f(x))) = N(x, f(x)), \quad x \in J,$$

we say that the graph of the function f is an *invariant curve with respect to the map* (M, N) , briefly, (M, N) -invariant curve ([2]).

To explore the connection between the pre-mean map and its invariant curve, Lemma 3 is introduced.

Lemma 3 Let $p, q \in \mathbb{R}$, $p \neq q$, $\lambda, \mu \in (0, 1)$, then

(1) For every $a \in \mathbb{R}$, the graph of the function $f(x) = 2a - x, x \in \mathbb{R}$, that is,

$$H_a = \{(x, 2a - x) \mid x \in \mathbb{R}\},$$

is $(E_{\lambda, \mu}^{[p, q]}, E_{1-\lambda, 1-\mu}^{[-p, -q]})$ -invariant curve, and in particular $(E_{\lambda, \mu}^{[p, q]}, E_{1-\lambda, 1-\mu}^{[-p, -q]})(H_a) \subset H_a$.

(2) The point (a, a) is the only fixed point of the mapping $(E_{\lambda, \mu}^{[p, q]}, E_{1-\lambda, 1-\mu}^{[-p, -q]})$ in the set H_a .

(3) The family of sets $\{H_a : a > 0\}$ forms a partition of \mathbb{R}^2 , that is $H_a \cap H_b = \emptyset$ for all $a, b \in \mathbb{R}$ and $a \neq b$, and $\bigcup_{a \in \mathbb{R}} H_a = \mathbb{R}^2$.

Proof. Fix $a \in \mathbb{R}$, we have

$$A(x, f(x)) = \frac{x + f(x)}{2} = a,$$

and using Lemma 2, we get

$$A(E_{\lambda, \mu}^{[p, q]}(x, f(x)), E_{1-\lambda, 1-\mu}^{[-p, -q]}(x, f(x))) = A(x, f(x)) = a,$$

that is

$$\begin{aligned}
 & \frac{E_{1-\lambda, 1-\mu}^{[-p, -q]}(x, f(x)) + E_{\lambda, \mu}^{[p, q]}(x, f(x))}{2} = \frac{x + f(x)}{2} = a, \\
 & E_{1-\lambda, 1-\mu}^{[-p, -q]}(x, f(x)) = 2a - E_{\lambda, \mu}^{[p, q]}(x, f(x)),
 \end{aligned}$$

therefore,

$$f(E_{\lambda, \mu}^{[p, q]}(x, f(x))) = E_{1-\lambda, 1-\mu}^{[-p, -q]}(x, f(x)),$$

According to the definition of invariant curves, the function $f(x) = 2a - x$

is the invariant curve of $(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})$. Therefore, the first result is correct and the other two parts are obvious. \square

3. Convergence of Iteration of the Pre-Mean Mappings

Using Theorem 1 and Lemma 2, we can get the following:

Theorem 2 *If $\lambda = \mu$ or $pq \leq 0$, the iterative sequence*

$$\left((E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})^n \right)_{n=0}^{\infty} \text{ is pointwise convergent on } \mathbb{R}^2 \text{ and}$$

$$\lim_{n \rightarrow \infty} (E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})^n(x, y) = \left(\frac{x+y}{2}, \frac{x+y}{2} \right), \quad x, y \in \mathbb{R}.$$

It is difficult to consider the convergence problem for the case $\lambda \neq \mu$ and $pq > 0$. Lemma 3 tells us that, H_a is the invariant curve of $(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})$. We will study the convergence problem of the restrictions of this map to the invariant sets $H_a, a \in \mathbb{R}$.

Theorem 3 *Let $p, q \in \mathbb{R}, p \neq q, \lambda, \mu \in (0, 1)$. For every $a \in \mathbb{R}$ and $n \in \mathbb{N}$, we have for all $x \in \mathbb{R}$ it holds*

$$(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})^n(x, 2a-x) = \left((f_{[\lambda,\mu];a}^{[p,q]})^n(x), 2a - (f_{[\lambda,\mu];a}^{[p,q]})^n(x) \right), \quad (4)$$

where the function $f_{[\lambda,\mu];a}^{[p,q]}$ is defined as

$$f_{[\lambda,\mu];a}^{[p,q]}(x) = E_{\lambda,\mu}^{[p,q]}(x, 2a-x), \quad x \in \mathbb{R},$$

that is

$$f_{[\lambda,\mu];a}^{[p,q]}(x) = \ln \left(\frac{\lambda e^{px} + (1-\lambda)e^{p(2a-x)}}{\mu e^{qx} + (1-\mu)e^{q(2a-x)}} \right)^{\frac{1}{p-q}}, \quad x \in \mathbb{R}.$$

Proof. For the convenience of writing, write $E = E_{\lambda,\mu}^{[p,q]}$, $f_a = f_{[\lambda,\mu];a}^{[p,q]}$. Because

$$E_{\lambda,\mu}^{[p,q]}(x, y) + E_{1-\lambda,1-\mu}^{[-p,-q]}(x, y) = x + y,$$

there is,

$$(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})(x, y) = (E(x, y), x + y - E(x, y)), \quad x, y \in \mathbb{R}.$$

When $n = 1$, the Equation (4) is correct, in fact

$$\begin{aligned} (E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})(x, 2a-x) &= (E(x, 2a-x), 2a - E(x, 2a-x)) \\ &= (f_a(x), 2a - f_a(x)). \end{aligned}$$

Suppose (4) holds for some $n = k \in \mathbb{N}$, that is

$$(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})^k(x, 2a-x) = (f_a^k(x), 2a - f_a^k(x)),$$

then for all $x \in \mathbb{R}$

$$\begin{aligned} &(E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})^{k+1}(x, 2a-x) \\ &= (E_{\lambda,\mu}^{[p,q]}, E_{1-\lambda,1-\mu}^{[-p,-q]})^k(E(x, 2a-x), 2a - E(x, 2a-x)) \\ &= (f_a^k(E(x, 2a-x)), 2a - f_a^k(E(x, 2a-x))) \\ &= (f_a^k(f_a(x)), 2a - f_a^k(f_a(x))) = (f_a^{k+1}(x), 2a - f_a^{k+1}(x)), \end{aligned}$$

that is to say, when $n = k + 1$, (4) is still established, the certificate is finished. \square

To further study the theorem 3, we obtain the following

Corollary 1 Let $p, q \in \mathbb{R}$, $p \neq q$, $\lambda, \mu \in (0, 1)$, if

$$\left| \frac{(2\lambda - 1)p - (2\mu - 1)q}{p - q} \right| < 1,$$

then there is an open set $U \subset \mathbb{R}^2$ containing the diagonal $\Delta = (x, x) | x \in \mathbb{R}$ such that

$$\lim_{n \rightarrow \infty} \left(E_{\lambda, \mu}^{[p, q]}, E_{1-\lambda, 1-\mu}^{[-p, -q]} \right)^n (x, y) = \left(\frac{x + y}{2}, \frac{x + y}{2} \right), \quad (x, y) \in U.$$

Proof. According to theorem 3, we can obtain that

$$f_{[\lambda, \mu]; a}^{[p, q]}(a) = E(a, a) = a,$$

as well as

$$\begin{aligned} \left(f_{[\lambda, \mu]; a}^{[p, q]} \right)(x) &= \left[\ln \left(\frac{\lambda e^{px} + (1 - \lambda) e^{2a-x}}{\mu e^{qx} + (1 - \mu) e^{2a-x}} \right)^{\frac{1}{p-q}} \right] \\ &= \frac{1}{p-q} \left[p - \frac{2p(1-\lambda)e^{p(2a-x)}}{\lambda e^{px} + (1-\lambda)e^{p(2a-x)}} - q + \frac{2q(1-\mu)e^{q(2a-x)}}{\mu e^{qx} + (1-\mu)e^{q(2a-x)}} \right]. \end{aligned}$$

So,

$$\left(f_{[\lambda, \mu]; a}^{[p, q]} \right)(a) = \frac{(2\lambda - 1)p - (2\mu - 1)q}{p - q}.$$

It can be seen that any real number a , a are the fixed point of $f_{[\lambda, \mu]; a}^{[p, q]}$ and $\left(f_{[\lambda, \mu]; a}^{[p, q]} \right)(a)$ is independent of the value of a .

Since $\left(f_{[\lambda, \mu]; a}^{[p, q]} \right)(a) < 1$, according to the compression mapping principle,

$$\lim_{n \rightarrow \infty} \left(f_{[\lambda, \mu]; a}^{[p, q]} \right)^n (x) = f_{[\lambda, \mu]; a}^{[p, q]}(a) = a = \frac{x + y}{2},$$

which complete the proof. \square

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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