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Pseudo-Semi-Overlap Functions-Based Fuzzy Rough Sets Applied to Image Edge Extraction

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Abstract

As an extension of overlap functions, pseudo-semi-overlap functions are a crucial class of aggregation functions. Therefore, (I, PSO)-fuzzy rough sets are introduced, utilizing pseudo-semi-overlap functions, and further extended for applications in image edge extraction. Firstly, a new clustering function, the pseudo-semi-overlap function, is introduced by eliminating the symmetry and right continuity present in the overlap function. The relaxed nature of this function enhances its applicability in image edge extraction. Secondly, the definitions of (I, PSO)-fuzzy rough sets are provided, using (I, PSO)-fuzzy rough sets, a pair of new fuzzy mathematical morphological operators (IPSOFMM operators) is proposed. Finally, by combining the fuzzy C-means algorithm and IPSOFMM operators, a novel image edge extraction algorithm (FCM-IPSO algorithm) is proposed and implemented. Compared to existing algorithms, the FCM-IPSO algorithm exhibits more image edges and a 73.81% decrease in the noise introduction rate. The outstanding performance of (I, PSO)-fuzzy rough sets in image edge extraction demonstrates their practical application value.

Keywords

Pseudo-Semi-Overlap Functions, Fuzzy Rough Set, Fuzzy Mathematical Morphology, Image Edge Extraction

1. Introduction

Zadeh introduced fuzzy sets in 1965 [1], and Pawlak explored rough sets in 1982 [2]. In 1990, Dubois and Prade combined fuzzy sets and rough sets using the fuzzy operators min and max to create fuzzy rough sets [3]. Since then, numerous scholars have explored the theory of fuzzy rough sets and their practical applications in depth. In 2002, Radzikowska *et al.* employed a broader method for fuzzily rough sets and introduced a fuzzy rough set that relies on T-norm and fuzzy implication [4]. Subsequently, Qiao [5] and Wen *et al.* [6] formulated the (IO, O)-fuzzy rough sets. Zhang *et al.* [7] introduced (I, O)-fuzzy rough sets by substituting the IO with a broader I in the (IO, O)-fuzzy rough sets. Wu *et al.* [8] proposed a novel form of (I, T)-fuzzy rough sets, relying on the general fuzzy binary relation. Mieszkowicz Rolka *et al.* [9] and Zhan *et al.* [10] presented the theories of variable precision fuzzy rough sets and covering-based multi-granulation fuzzy rough sets, respectively. These theories have been widely used in digital image processing [11] [12], attribution reduction [13] [14], webpage classification [15], tumor detection [16], big data analysis [17], and other applications [18] [19].

With the advancement of fuzzy rough sets based on various operators, fuzzy rough sets based on clustering functions with overlap function as an important representative have performed well in image edge extraction [20]-[23] and decision-making application [24]-[26]. Along with the rapid development of overlap functions as a class of clustering functions, scholars have proposed more extensive clustering functions. For example, Zhang *et al.* [26] removed the symmetry in the overlap function, proposed the pseudo-overlap function, and discussed its applications in decision-making and image processing. In 2022, Zhang [27] updated the concept of overlap functions by removing the right continuity. Therefore, semi-overlap functions were proposed as new aggregation functions. Subsequently, a novel classification algorithm based on semi-overlap functions was discovered and successfully applied. In addition to clustering functions, other proposed functions include quasi-overlap functions [28], interval-valued pseudo-overlap functions [29], and general overlap functions [30]. Simultaneously, many scholars have combined the clustering function with fuzzy rough sets and proposed new fuzzy rough sets. Zhang *et al.* [31] proposed a fuzzy rough set comprising overlap functions and fuzzy implication and applied it to image edge extraction and attribute reduction. A link between a group of approximate operators in (I, O)-fuzzy rough sets and a group of fuzzy dilation and erosion operators is present in image edge extraction applications [7]. Thus, the IO-FCM image edge extraction algorithm was introduced and effectively implemented. However, for practical applications, due to the strict requirement of continuity aspects of the overlap functions, both left and right continuity must be satisfied. Hence, the flexibility of the algorithm is low, and its practical applications are limited.

Therefore, this paper conducts research by considering the broad range of applications of fuzzy rough sets, as well as the successful utilization of fuzzy rough sets based on clustering functions in image edge extraction. First, the symmetry of semi-overlap functions was removed, and the pseudo-semi-overlap functions with their two construction methods were proposed. Second, the (I, O)-fuzzy rough set was extended to the (I, PSO)-fuzzy rough set, and the overlap function was replaced by a pseudo-semi-overlap function. Further, the theory and properties related to the (*I*, *PSO*)-fuzzy rough set, along with the looser constraints of the *PSO* operator, were explored for wider application in image edge extraction. Compared to existing applications of fuzzy rough sets in image edge extraction, (*I*, *PSO*)-fuzzy rough sets are superior in the following aspects: 1) The pseudo-semi-overlap function is an important aggregation function that can effectively distinguish the foreground and background of an image. Compared to existing clustering functions [32], the pseudo-semi-overlap function has more relaxed requirements for continuity and does not need symmetry. Therefore, (*I*, *PSO*)-fuzzy rough sets have broader applications, better practical adaptability, and a higher theoretical conversion rate. 2) The upper and lower approximation operators in the (*I*, *PSO*)-fuzzy rough set correlate with the fuzzy dilation and fuzzy erosion operators, respectively, in fuzzy mathematical morphology. Therefore, a new set of morphological operators with higher flexibility, IPSOFMM operators, is proposed, and the relevant properties in fuzzy rough sets and fuzzy mathematical morphology are studied. 3) The FCM-IPSO image edge extraction algorithm obtained via a combination of the fuzzy C-means algorithm and the IPSOFMM operators exhibits superior image edge extraction results compared to those obtained using the Canny operator, Laplacian operator, Prewitt operator, Roberts operator, and Sobel operator. In other words, the FCM-IPSO algorithm provides improved image edge information with a minimum noise introduction rate.

The rest of this paper is organized as follows: Section 2 presents the fundamental concepts. Section 3 begins with the definition of a pseudo-overlap function and elaborates on two methods for constructing this function. Subsequently, (*I*, *PSO*)-fuzzy rough sets are defined, and the related theories and properties are systematically described. Section 4 introduces a new set of fuzzy mathematical morphological operators, IPSOFMM operators, and delves into their properties. In Section 5, the FCM-IPSO image edge extraction algorithm is proposed, and its performance is assessed using five gray images. The experimental results demonstrate the exceptional performance of the FCM-IPSO algorithm over existing classical algorithms. An overview of the study is presented in **Figure 1**. The concluding remarks, along with subsequent future studies, are summarized in Section 6.

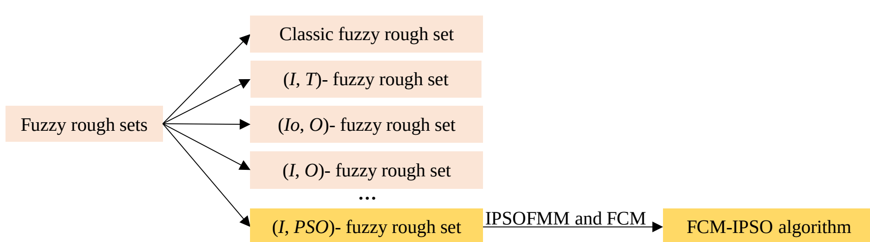


Figure 1. Outline of the study.

2. Fundamental Definitions

Definition 1 ([20]). A bivariate function $f:[0,1]^2 \to [0,1]$, for any $m,n \in [0,1]$,

(1) $f(m, n) = 0$ iff $mn = 0$;

(2) $f(m, n) = 1$ iff $mn = 1$;

(3) $f(m, n) = f(n, m)$;

(4) f is increasing;

(5) f is continuous.

A binary function is termed an overlap function (denoted as O) if it conforms to conditions (1)-(5).

Definition 2 ([24]). A binary function $f:[0,1]^2 \to [0,1]$, for any $m,n \in [0,1]$,

(1) $f(m, n) = 0$ iff $mn = 0$;

(2) $f(m, n) = 1$ iff $mn = 1$;

(3) $f(m, n) = f(n, m)$;

(4) f is increasing;

(5) f is left-continuous.

A binary function is termed a semi-overlap function (denoted as SO) if it conforms to conditions (1)-(5).

Definition 3 ([33]) A binary function $I:[0,1]^2 \to [0,1]$, for any $l,m,n \in [0,1]$,

($I1$) $I(1, 1) = I(0, 0) = 1$;

($I2$) $I(1, 0) = 0$;

($I3$) If $m \le n$, then $I(m,l) \ge I(n,l)$;

($I4$) If $n \le l$, then $I(m,n) \le I(m,l)$.

If the above conditions are satisfied, the binary function is called a fuzzy implication (denoted as I).

Definition 4 ([28]) Assume O is an overlap function, and I is a fuzzy implication. Consider the fuzzy approximation space (U, R), where U is the domain and R is a fuzzy binary relation on U. To define a pair of fuzzy sets on U, a fuzzy set B in U (*i.e.*, $B \in F(U)$) can be considered: for any $m \in U$,

$$\overline{R}(B)(m) = \sup_{n \in U} O(R(m,n), B(n)), \tag{1}$$

$$\underline{R}(B)(m) = \inf_{n \in U} I(R(m,n), B(n)). \tag{2}$$

where $\overline{R}(B)$ presents the fuzzy upper approximation and $\underline{R}(B)$ represents the fuzzy lower approximation in (I, O)-fuzzy rough sets of B.

Definition 5 ([7]) Let B and C be fuzzy subsets of $\mathcal{R}^2$. Assume O is an overlap function, and I is a fuzzy implication. The expressions for the fuzzy dilation D_O (B, C) and fuzzy erosion E_I (B, C) of a gray image B by a gray structuring element C are as follows ($d(C) = \{m \mid C(m) \ne 0\} \subseteq \mathcal{R}^2$) for $\forall m \in \mathcal{R}^2$:

$$D_O(B,C)(m) = \sup_{n \in d(C)} O(C(n), B(m+n)), \tag{3}$$

$$E_I(B,C)(m) = \inf_{n \in d(C)} I(C(n), B(m+n)). \tag{4}$$

3. Pseudo-Semi-Overlap Functions and (*I*, *PSO*)-Fuzzy Rough Sets

This section proposes pseudo-semi-overlap functions and defines essential characteristics of (I, PSO)-fuzzy rough sets.

Definition 6. A bivariate function $PSO:[0,1]^2 \to [0,1]$ is named a pseudo-semi-overlap function (denoted as *PSO*) when it fulfills the following conditions:

(PSO_1) For any $m,n \in [0,1]$, if $mn = 0$, then $PSO(m, n) = 0$;

(PSO_2) If $m = n = 1$, then $PSO(m, n) = 1$;

(PSO_3) *PSO* is increasing;

(PSO_4) *PSO* is left-continuous.

Example 1. A mapping $PSO:[0,1]^2 \to [0,1]$ defined for any $m,n \in [0,1]$, as (As shown in Figure 2).

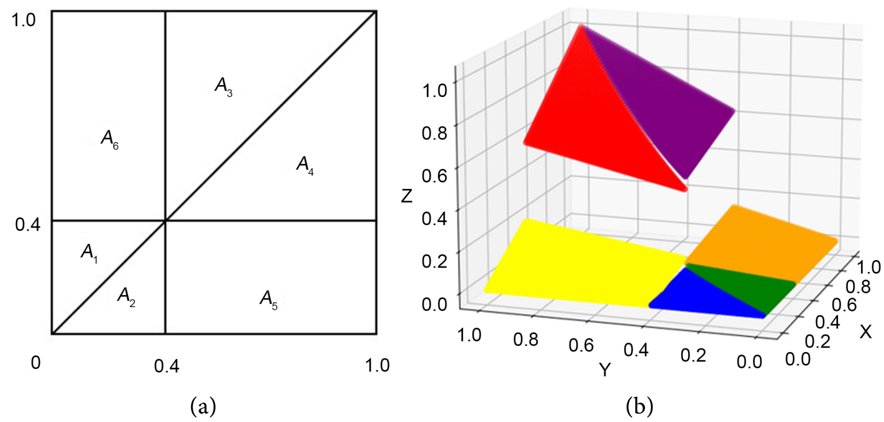


Figure 2. (a) Distribution of intervals of (2) in Example 1; (b) Visualization of the proposed function in Example 1.

$$PSO(m,n) = \begin{cases} \dfrac{mn}{7}, & \text{if } (m,n) \in A_1 \\ \dfrac{2mn}{7}, & \text{if } (m,n) \in A_2 \\ \dfrac{21mn+9}{30}, & \text{if } (m,n) \in A_3 \\ \dfrac{19mn+11}{30}, & \text{if } (m,n) \in A_4 \\ \dfrac{mn}{3}, & \text{if } (m,n) \in A_5 \\ \dfrac{mn}{2}, & \text{if } (m,n) \in A_6 \end{cases} \tag{5}$$

is a pseudo-semi-overlap function, where $A_1 = \{(m,n) \mid 0 \le m \le 0.4, m \le n \le 0.4\}$, $A_2 = \{(m,n) \mid 0 < m \le 0.4, 0 \le n < m\}$, $A_3 = \{(m,n) \mid 0.4 < m \le 1, m \le n \le 1\}$, $A_4 = \{(m,n) \mid 0.4 < m \le 1, 0.4 \le n < m\}$, $A_5 = \{(m,n) \mid 0.4 < m \le 1, 0 \le n < 0.4\}$, $A_6 = \{(m,n) \mid 0 \le m \le 0.4, 0.4 < n \le 1\}$.

The specific intervals are distributed as follows:

Theorem 1. Assume that $PSO:[0,1]^2 \to [0,1]$ is a pseudo-semi-overlap function. If *PSO* is commutative, then it is a semi-overlap function.

Proof. The proof follows from Definitions 1 and 5.

Theorem 2. A bivariate function $PSO:[0,1]^2 \to [0,1]$ is a pseudo-semi-overlap function if and only if two operators *f* and *g* exist on [0, 1] with

$$PSO(m,n)=\frac{f(m,n)}{f(m,n)+g(m,n)}. \tag{6}$$

Note: $f(m,n)+g(m,n)\neq 0$.

Where

(1) f is increasing and g is decreasing;

(2) If $mn = 0$, then $f(m, n) = 0$;

(3) If $m = n = 1$, then $g(m, n) = 0$;

(4) Both f and g satisfy continuity.

Proof. ($\Leftarrow$) By (2), for $mn = 0$, $f(m, n) = 0$. Then $PSO(m, n) = 0$, *i.e.*, the binary function PSO satisfies (PSO_1).

By (3), for $m = n = 1$, $g(m, n) = 0$. Then $PSO(m, n) = 1$, *i.e.*, the binary function PSO satisfies (PSO_2).

By (1), if $m_1 \le m_2$, for any $n\in[0,1]$, then $f(m_1,n)g(m_2,n)\le f(m_2,n)g(m_1,n)$. Next, by adding a non-negative number $f(m_1, n)\, f(m_2, n)$ to both sides of the equation simultaneously, $f(m_1,n)\left(f(m_2,n)+g(m_2,n)\right)\le f(m_2,n)\left(f(m_1,n)+g(m_1,n)\right)$ can be obtained, *i.e.*, $PSO(m_1,n)\le PSO(m_2,n)$. Following the same logic, if $n_1\le n_2$, then $PSO(m,n_1)\le PSO(m,n_1)$ can be obtained. Therefore, the binary function PSO satisfies (PSO_3).

By (4), it is straightforward to note that the binary function PSO is continuous, *i.e.*, the binary function PSO satisfies (PSO_4).

($\Rightarrow$) It is known that PSO satisfies (PSO_1)-(PSO_4), and suppose that $f(m, n) = PSO(m, n)$ and $g(m, n) = 1 - PSO(m, n)$. Then, $PSO(m, n)$ can be defined by $f(m, n)$, $g(m, n)$. Furthermore, note that conditions (1)-(4) are satisfied.

Theorem 3. Assume $PSO_1, PSO_2, \cdots, PSO_m$ be a pseudo-semi-overlap function and $r_1, r_2, \cdots, r_m$ be nonnegative weights with $\sum_{j=1}^{m} r_j = 1$. Then $PSO(u,v)=\sum_{j=1}^{m} r_j PSO_j(u,v)$ is also a pseudo-semi-overlap function.

Proof. (PSO_1)-(PSO_3) are easy proved. So, we prove PSO satisfies (PSO_4). If PSO is left-continuous, then for any $u\in[0,1]$ and for any $\{v_i \mid i\in I\}\subseteq[0,1]$, it follows that $PSO(u,\sup\{v_i \mid i\in I\})=\sup\{PSO(u,v_i)\mid i\in I\}$. Hence, we can get

$$\begin{aligned} PSO\left(u,\sup_{i\in I} v_i\right) &= \sum_{j=1}^{m} r_j PSO_j\left(u,\sup_{i\in I} v_i\right)=\sum_{j=1}^{m} r_j\left(\sup_{i\in I} PSO_j(u,v_i)\right) \\ &= \sup_{i\in I}\sum_{j=1}^{m} r_j PSO_j(u,v_i)=\sup_{i\in I} PSO_j(u,v_i) \end{aligned}$$

Proposition 1. Assume $\beta_1,\beta_2,\beta_3:[0,1]\to[0,1]$ are continuous and increasing operators. For any $i\in[1,2,3]$, iff $m = 0$, and $\beta_i(m)=1$ iff $m = 1$. Assuming that PSO is a binary pseudo-semi-overlap function, $PSO^{\beta_1,\beta_2,\beta_3}$ is defined as follows:

$$PSO^{\beta_1,\beta_2,\beta_3}(m,n)=\beta_1(PSO(\beta_2(m),\beta_3(n)). \tag{7}$$

Proof. It is easy to show that $PSO^{\beta_1,\beta_2,\beta_3}$ satisfies (PSO_3) and (PSO_4). If $m = 0$, $\beta_2(m)=0$, and consequently, $PSO(\beta_2(m),\beta_3(n))=0$. According to the known conditions, $PSO^{\beta_1,\beta_2,\beta_3}(m,n)=0$ can be easily obtained. Moreover, when $n = 0$, $PSO^{\beta_1,\beta_2,\beta_3}(m,n)=0$ can be obtained. Thus, $PSO^{\beta_1,\beta_2,\beta_3}$ satisfies condition (PSO_1). If $m = 1$, $\beta_1(m)=\beta_2(m)=0$ can be determined. Then, based on the known conditions, $PSO(\beta_2(m),\beta_3(n))=1$, and $PSO^{\beta_1,\beta_2,\beta_3}$ satisfies condition (PSO_2).

Definition 7. Assume that (U, R) is a fuzzy approximation space, where R represents the fuzzy binary relation on U. PSO represents a pseudo-semi-overlap function, while I represent a fuzzy implication. Given the fuzzy set B defined on the domain set U (*i.e.*, $B \in F(U)$), the following equation shows a couple of fuzzy sets in U for any $m \in U$.

$$\overline{R}_{PSO}(B)(m)=\sup_{n\in U} PSO(R(m,n),B(n)), \tag{8}$$

$$\underline{R}_I(B)(m)=\inf_{n\in U} I(R(m,n),B(n)). \tag{9}$$

Here, $\overline{R}_{PSO}(B)$ and $\underline{R}_I(B)$ are known as the (I, PSO)-fuzzy upper approximation and lower approximation of B, respectively.

Example 2. Assuming $U = \{m_1, m_2, m_3, m_4, m_5\}$, fuzzy set $B = \{0.4/m_1, 0.5/m_2, 0.7/m_3, 0.8/m_4, 0.6/m_5\}$. Table 1 lists the fuzzy relations R in the domain U.

Table 1. Fuzzy relation R in the domain U.

R	m_1	m_2	m_3	m_4	m_5
m_1	1	0.6	0.7	0.7	0.6
m_2	0.6	1	0.4	0.6	0.8
m_3	0.7	0.4	1	0.6	0.7
m_4	0.7	0.6	0.6	1	0.6
m_5	0.6	0.8	0.7	0.6	1

By Definition 7, the upper and lower approximations of the fuzzy set B in the approximation space (U, R) are deduced as follows (these relevant functions are used, including PSO_1 and I_1):

Note:

$$I_1(m,n)=\min(1,1-m+n), \tag{10}$$

$$PSO_1(m,n)=\begin{cases} m^2, & m^2 \le n^3, \\ n, & m^2 > n^3. \end{cases} \tag{11}$$

$$\overline{R}_{PSO}(B)(m_1)=\sup\{0.40, 0.50, 0.70, 0.49, 0.60\}=0.7;$$

$$\overline{R}_{PSO}(B)(m_2)=\sup\{0.40, 0.50, 0.16, 0.36, 0.60\}=0.6;$$

$$\overline{R}_{PSO}(B)(m_3)=\sup\{0.40, 0.50, 0.70, 0.36, 0.60\}=0.7;$$

$$\overline{R}_{PSO}(B)(m_4)=\sup\{0.40, 0.50, 0.70, 0.80, 0.60\}=0.8;$$

$$\overline{R}_{PSO}(B)(m_5)=\sup\{0.40, 0.50, 0.70, 0.36, 0.60\}=0.7;$$

$$\underline{R}_I(B)(m_1)=\inf\{0.40, 0.90, 1.00, 1.00, 1.00\}=0.4;$$

$$\underline{R}_I(B)(m_2)=\inf\{0.80, 0.50, 1.00, 1.00, 0.80\}=0.5;$$

$$\underline{R}_I(B)(m_3)=\inf\{0.70, 1.00, 0.70, 1.00, 0.90\}=0.7;$$

$$\underline{R}_I(B)(m_4)=\inf\{0.70, 0.90, 1.00, 0.80, 1.00\}=0.7;$$

$$\underline{R}_I(B)(m_5)=\inf\{0.80,0.70,1.00,1.00,0.60\}=0.6.$$

Subsequently, the upper and lower approximation sets of B in the approximate space are as follows:

$$\overline{R}_{PSO}(B)=\{0.7/m_1,0.6/m_2,0.7/m_3,0.8/m_4,0.7/m_5\};$$

$$\underline{R}_I(B)=\{0.4/m_1,0.5/m_2,0.7/m_3,0.7/m_4,0.6/m_5\}.$$

The example illustrates the calculation process of (I, PSO)-fuzzy rough sets, and then the properties of (I, PSO)-fuzzy rough sets are demonstrated.

Theorem 4. Assuming PSO as a pseudo-semi-overlap function, R as a fuzzy reflexive relation, and I as a fuzzy implication. For (I, PSO)-fuzzy rough sets, the following conditions apply for $\overline{R}_{PSO}(B)$ and $\underline{R}_I(B)$:

(1) $\overline{R}_{PSO}(\varnothing)=\varnothing$;

(2) $\underline{R}_I(U)=U$;

(3) For any $m,n\in[0,1]$, if $PSO(1,m)\geq m$ and $I(1,n)\leq n$, then $\underline{R}_I(B)\subseteq B\subseteq\overline{R}_{PSO}(B)$;

(4) If $B\subseteq C$, then $\overline{R}_{PSO}(B)\subseteq\overline{R}_{PSO}(C)$, $\underline{R}_I(B)\subseteq\underline{R}_I(C)$.

Proof. (1) $\overline{R}_{PSO}(\varnothing)=\varnothing$ can be proven by Definition 7. (2) $\underline{R}_I(U)=U$ can be proven according to Definition 7.

(3) For the fuzzy set B, according to Definition 7, $\forall m\in U$,

$$\overline{R}_{PSO}(B)(m)=\sup_{n\in U}PSO(R(x,n),B(n))=\begin{cases}I_M\left(\bigcup_{l\neq n}[l]_R,\alpha_M\right)(m), & R(m,n)=0,\\ 0, & R(m,n)\neq 0.\end{cases}$$

Thus, $\overline{R}_{PSO}(B)\supseteq B$.

Moreover, according to Definition 7,

$$\begin{aligned}\underline{R}_I(B)(m)&=\inf_{n\in U}I(R(m,n),B(n))\\ &\leq I(R(m,m),B(m))\\ &=I(1,B(m))\\ &\leq B(m)\end{aligned}$$

Therefore, $\underline{R}_I(B)\subseteq B$. Finally, it is proven that $\underline{R}_I(B)\subseteq B\subseteq\overline{R}_{SO}(B)$.

(4) If $B\subseteq C$, by (PSO_3) of Definition 6, $\forall m\in U$, PSO (R (m, n) can be obtained. $B(n)\leq PSO(R(m,n),C(n))$. Then

$$\sup_{n\in U}PSO(R(m,n),B(n))\leq\sup_{n\in U}PSO(R(m,n),C(m))$$

Therefore, $\overline{R}_{PSO}(B)\subseteq\overline{R}_{PSO}(C)$. By ($I2$), similarly, $\underline{R}_I(B)\subseteq\underline{R}_I(C)$.

Theorem 5. Suppose PSO is a pseudo-semi-overlap function, I is a fuzzy implication, and R_1 and R_2 represent a couple of fuzzy binary relations on U. If $R_1\subseteq R_2$, in this case,

(1) $\overline{R_1}_{PSO}(A)\subseteq\overline{R_2}_{PSO}(A)$;

(2) $\underline{R_2}_I(A)\subseteq\underline{R_1}_I(A)$.

Note: $\overline{R_1}_{PSO}(A)$ and $\overline{R_2}_{PSO}(A)$ represent the fuzzy sets A based on the (I, PSO)-fuzzy rough set upper-approximation operators of R_1 and R_2, respectively; $\underline{R_2}_I(A)$ and $\underline{R_1}_I(A)$ represent the fuzzy sets A based on (I, PSO)-fuzzy rough set lower-approximation operators of R_1 and R_2, respectively.

Proof. (1) If $R_1 \subseteq R_2$, then for any $x, y \in U$; according to (PSO_3) in Definition 6, the following expression can be written:

$$PSO\left(R_1(x,y), A(y)\right) \leq PSO\left(R_2(x,y), B(y)\right).$$

Then,

$$\sup_{y\in U} PSO\left(R_1(x,y), A(y)\right) \leq \sup_{y\in U} PSO\left(R_2(x,y), A(y)\right).$$

Therefore, $\overline{R_1}_{PSO}(A) \subseteq \overline{R_2}_{PSO}(A)$.

(2) By combining the proof strategies of (1) and (I_2) of Definition 3, it can be proven that $\underline{R_2}_I(A) \subseteq \underline{R_1}_I(A)$.

Theorem 6. Suppose PSO is a pseudo-semi-overlap function, and I is a fuzzy implication, where C and D are fuzzy sets in the domain U. Consequently, the following can be inferred:

(1) $\overline{R}_{PSO}(C \cup D) = \overline{R}_{PSO}(C) \cup \overline{R}_{PSO}(D)$;

(2) $\underline{R}_I(C \cup D) \supseteq \underline{R}_I(C) \cup \underline{R}_I(D)$;

(3) $\overline{R}_{PSO}(C \cap D) \subseteq \overline{R}_{PSO}(C) \cap \overline{R}_{PSO}(D)$;

(4) $\underline{R}_I(C \cap D) = \underline{R}_I(C) \cap \underline{R}_I(D)$.

Proof. The definition of (I, PSO)-fuzzy rough set is obtained directly from conditions (1) and (4). Proofs for (2) and (3) are given below.

(2) From Definition 7, $\forall m \in U$,

$$\begin{aligned}
\underline{R}_I(C \cup D)(m) &= \inf_{n\in U} I\left(R(m,n),(C \cup D)(n)\right) = \inf_{n\in U} I\left(R(m,n), C(n) \vee D(n)\right) \\
&= \inf_{n\in U}\left(I\left(R(m,n),C(n)\right) \vee I\left(R(m,n),D(n)\right)\right) \\
&\geq \inf_{n\in U} I\left(R(m,n),C(n)\right) \vee \inf_{n\in U} I\left(R(m,n),D(n)\right) \\
&= \underline{R}_I(C) \cup \underline{R}_I(D)(m)
\end{aligned}$$

Hence, $\underline{R}_I(C \cup D) \supseteq \underline{R}_I(C) \cup \underline{R}_I(D)$.

(3) By Definition 7, $\forall m \in U$,

$$\begin{aligned}
\overline{R}_{PSO}(C \cap D)(m) &= \sup_{n\in U} PSO\left(R(m,n),(C \cap D)(n)\right) \\
&= \sup_{n\in U} PSO\left(R(m,n), C(n) \wedge D(n)\right) \\
&= \sup_{n\in U}\left(PSO\left(R(m,n),C(n)\right) \wedge PSO\left(R(m,n),D(n)\right)\right) \\
&\leq \sup_{n\in U} PSO\left(R(m,n),C(n)\right) \wedge \sup_{n\in U} PSO\left(R(m,n),D(n)\right) \\
&= \overline{R}_{PSO}(C) \cap \overline{R}_{PSO}(D)(m)
\end{aligned}$$

Hence, $\overline{R}_{PSO}(C \cap D) \subseteq \overline{R}_{PSO}(C) \cap \overline{R}_{PSO}(D)$.

Proposition 2. Let (M, N, R) be a fuzzy approximation space, PSO be a pseudo-semi-overlap function, I be a fuzzy implication, and R be a fuzzy relation from M to N. For any $\alpha \in [0,1]$, the following statement holds:

(1) $\overline{R}_{PSO}(\alpha_M) = PSO_N\left(\bigcup_{m\in M} [m]_R, \alpha_N\right)$;

(2) $\underline{R}_I(\alpha_N) = I_M\left(\bigcup_{n\in N} [n]_R, \alpha_M\right)$;

(3) $\overline{R}_{PSO}(\alpha_N) = PSO_M([n]_R, \alpha_M)(\forall n \in N)$;

(4) If $\forall a \in [0,1]$, then $I(a,0) = 0$. The following statements hold: $\forall n \in N$,

Note: For $\forall m \in M$, $n \in N$, $[n]_R(m) = R(m,n)$ exists; the value of the fuzzy set N in the context of α is a set of constant α_N; the value of the fuzzy set M in the context of α_M is a set of constant α.

Proof. (1) By Definition 7, $\forall m \in M$,

$$\begin{aligned}\overline{R}_{PSO}(\alpha_M)(n) &= \operatorname*{Sup}_{m \in M} PSO(R(n,m), \alpha_X(m)) = \operatorname*{Sup}_{m \in M} PSO(R(n,m), \alpha) \\ &= PSO\left(\operatorname*{Sup}_{m \in M} R(n,m), \alpha\right) = PSO\left(\bigcup_{m \in M} [m]_R, \alpha_N\right)(n)\end{aligned}$$

Thus, $\overline{R}_{PSO}(\alpha_M) = PSO_N\left(\bigcup_{m \in M} [m]_R, \alpha_N\right)$.

(2) By Definition 7, $\forall m \in M$, $n \in N$,

$$\begin{aligned}\underline{R}_I(\alpha_N)(m) &= \inf_{n \in N} I(R(m,n), \alpha_N(n)) = \inf_{n \in N} I(R(m,n), \alpha) \\ &= I\left(\operatorname*{Sup}_{n \in N} R(m,n), \alpha\right) = I\left(\bigcup_{n \in N} [n]_R, \alpha_M\right)(m)\end{aligned}$$

Hence, $\underline{R}_I(\alpha_N) = I_M\left(\bigcup_{n \in N} [n]_R, \alpha_M\right)$.

(3) $\overline{R}_{PSO}(\alpha_N) = PSO_M([n]_R, \alpha_M)$ can be directly inferred from (1).

(4) By Definition 7, $\forall m \in M$, $n, l \in N$,

$$\begin{aligned}\underline{R}_I(\alpha_N - \{n\})(m) &= \inf_{l \in N} I(R(m,l), (\alpha_N - \{n\})(l)) \\ &= \inf_{l \in N} I(R(m,l), \alpha) \wedge I(R(m,l), 0)\end{aligned}$$

by $I(a, 0) = 0(\forall a \in [0,1])$,

$$\underline{R}_I(\alpha_N - \{n\})(m) = \begin{cases} \bigwedge_{l \neq n} I(R(m,l), \alpha), & R(m,n) = 0, \\ 0, & R(m,n) \neq 0. \end{cases}$$

$$\begin{aligned}\underline{R}_I(\alpha_N - \{n\})(m) &= \begin{cases} I\left(\bigvee_{l \neq n} [l]_R, \alpha\right), & R(m,n) = 0, \\ 0, & R(m,n) \neq 0, \end{cases} \\ &= \begin{cases} I_M\left(\bigcup_{l \neq n} [l]_R, \alpha_M\right)(m), & R(m,n) = 0, \\ 0, & R(m,n) \neq 0. \end{cases}\end{aligned}$$

Hence, statement (4) is true.

4. IPSOFMM Operators

This section presents the IPSOFMM operators, an innovative set of morphological operators based on pseudo-semi-overlap functions and fuzzy implications. Furthermore, an innovative algorithm for image edge extraction, called FCM-IPSO, was developed by integrating IPSOFMM operators and the fuzzy C-means algorithm.

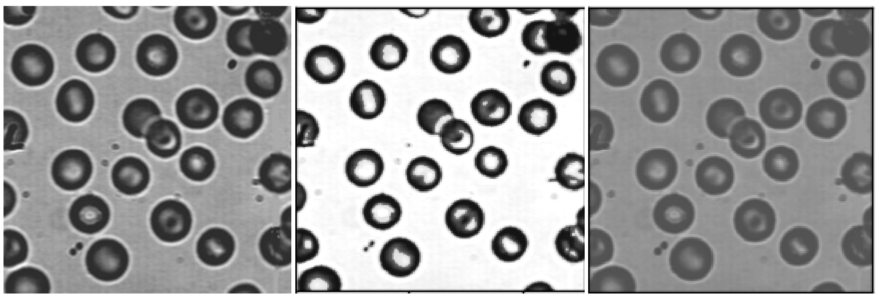
Definition 8. Consider R as a fuzzy binary relation on $\mathcal{R}^2$ (*i.e.*, $R: \mathcal{R}^2 \times \mathcal{R}^2 \to [0,1]$). The pair ($\mathcal{R}^2$, R) forms a fuzzy approximate space. Let

PSO represent a pseudo-semi-overlap function and I represent a fuzzy implication. B is a fuzzy subset of $\mathcal{R}^2$ (*i.e.*, $B:\mathcal{R}^2 \to [0,1]$). The fuzzy dilation operator, denoted as $D_{PSO}(B, R)$, and the fuzzy erosion operator, denoted as $E_I(B, R)$, are defined below: $\forall x, y \in \mathcal{R}^2$,

$$D_{PSO}(B,R)(x) = \sup_{y\in \mathrm{R}^2} PSO\big(R(x,y), B(y)\big), \tag{12}$$

$$E_I(B,R)(x) = \inf_{y\in \mathrm{R}^2} I\big(R(x,y), B(y)\big). \tag{13}$$

Example 3. Dilation and erosion examples per-formed by IPSOFMM operators are presented Figure 3.



(a) (b) (c)

Figure 3. Sample of dilation and erosion image. (a) Original image of cell. (b) Fuzzy dilation of cells. (c) Fuzzy erosion of cells.

Theorem 7. Consider B as a gray image, R as a fuzzy relation, PSO as a pseudo-semi-overlap function, I as a fuzzy implication, $D_{PSO}(B, R)$ as a fuzzy dilation operator, and $E_I(B, R)$ as a fuzzy erosion operator in the IPSOFMM operator. Then, for any $x, y \in R^2$, (the symbol $d(R)$ denotes the set of all points in R).

(1) $D_{PSO}(B,R)(x) = 0$ iff ($\forall y \in d(R), B(x+y) = 0$);

(2) $\exists y \in d(R)$, $R(y)=1$ and $B(x+y)=0$ iff $D_{PSO}(B,R)(x)=1$;

(3) $\exists y \in d(R)$, $R(y)=1$ and $B(x+y)=0$ iff $E_I(B,R)(x)=0$.

Proof. (1) Assume $\forall y \in d(R)$ satisfies $B(x+y)=0$. Moreover, by condition (2) in Definition 2.1, for $\exists m \in [0,1]$, then $PSO(0,m) = PSO(m,0) = 0$; hence,

$$\mathrm{Sup}_{y\in d(R)} PSO\big(R(x), B(x+y)\big) = 0$$

Assume $D_{PSO}(B,R)(y) = 0$; then,

$$D_{PSO}(B,R)(x) = \mathrm{Sup}_{y\in d(R)} PSO\big(R(y), B(x+y)\big) = 0.$$

Therefore, for $\forall y \in d(R)$, $PSO\big(R(y), B(x+y)\big) = 0$ can be obtained. By condition (2) in Definition 2.1, $R(y)\cdot B(x+y) = 0$, $\forall y \in d(R)$, $R(y) \neq 0$; hence, $B(x+y) = 0$.

(2) Assume $\exists y \in d(R)$, $R(y)=1$, and $B(x+y)=1$. By (3) in Definition 2.1, $PSO\big(R(y), B(x+y)\big) = 1$. Hence,

$$D_{PSO}(B,\ R)(x) = \mathrm{Sup}_{y\in d(R)} PSO\big(R(y), B(x+y)\big) = 1.$$

(3) This property is straightforward from the fuzzy implication and fuzzy erosion definitions.

Based on the integrated content in Sections 3 and 4, the (*I*, *PSO*)-fuzzy rough sets are more extensive than the (*I*, *O*)-fuzzy rough sets and have improved practical applicability while also retaining most of the characteristics of (*I*, *O*)-fuzzy rough sets. Moreover, the IPSOFMM operators exhibit greater scope than the IOFMM operators while retaining the properties of fuzzy rough sets.

5. FCM-IPSO Algorithm and Edge Extraction Experiment

In this section, the importance and advantages of the pseudo-semi-overlap functions in mathematical morphology and the field of image processing are demonstrated experimentally using the FCM-IPSO algorithm.

5.1. FCM-IPSO Algorithm

The core concept of the FCM-IPSO algorithm can be summarized as follows. First, the fuzzy C-means algorithm is applied for image clustering. This step aims to separate the background of the grayscale image from its foreground. Second, the fuzzy relation R is calculated based on the prior clustering outcomes, and $\overline{R}, \underline{R}$ are calculated. Third, the value of $\overline{R} - \underline{R}$ is calculated to obtain the fuzzy edge image. Finally, the image is deblurred and then binarization is applied to acquire a binary edge. The detailed procedures of the FCM-IPSO algorithm are outlined as follows.

Algorithm 5.1. An image edge extraction algorithm with (*I*, *PSO*)-fuzzy rough sets.

Input: gray image *GI*;

Output: edge image;

Step 1: $GI \leftarrow GI/255$;

Step 2: *GI* is subjected to clustering using the fuzzy C-means algorithm. *BG* represents the collection of all background points; Object represents the collection of all foreground points;

Step 3: for n in *GI*:

for m in *GI*:

Step 4: for n in *GI*:

Calculate $D_{PSO}(GI)(n)$, $E_I(GI)(n)$;

Step 5: fuzzyI_edge←$D_{PSO}(GI)$-$E_I(GI)$;

Step 6: grayI_edge←fuzzy_*edge*×255;

```
for i in edge:
    if gray_edge (GI, B_1)(i)>a:
        edge (i)←1
    else:
        edge (i)←0
return edge;
```

5.2. Experimental Step

Step 1. Choose the datasets.

Figures 4(a)-(f) displays the six standard images selected for the experiments. The Lena image was used to evaluate the FCM-ISO algorithm.

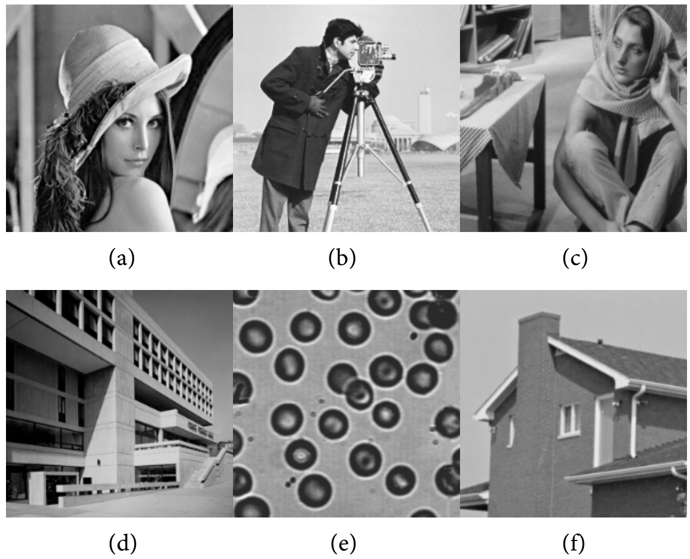


Figure 4. Datasets. (a) Lena, (b) Cameraman, (c) Barbara, (d) Bank, (e) Cell, and (f) House.

Step 2. Clustering analysis was performed on the Lena image using the fuzzy C-means algorithm. (Note: The approach is similar to the image clustering method outlined in [7]).

Step 3. Image edges were detected using the Canny, Prewitt, Roberts, Laplacian, and Sobel operators.

Step 4. The FCM-IPSO algorithm was employed to compute the image edges. The fuzzy relation R was calculated using B_1 and B_2 as follows.

$$B_1 = \begin{bmatrix} 0.7 & 0.7 & 0.7 \\ 0.7 & 0.8 & 0.7 \\ 0.7 & 0.7 & 0.7 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.6 & 0.6 & 0.6 \\ 0.6 & 0.7 & 0.6 \\ 0.6 & 0.6 & 0.6 \end{bmatrix} \tag{14}$$

5.3. Experimental Results

First, the grayscale Lena image was clustered using the FCM algorithm. The deblurred outcomes are depicted in Figure 5. (Note: Figures 5(a)-(c) belong to the *Object* set, whereas Figures 5(d)-(f) belong to the *BG* set).

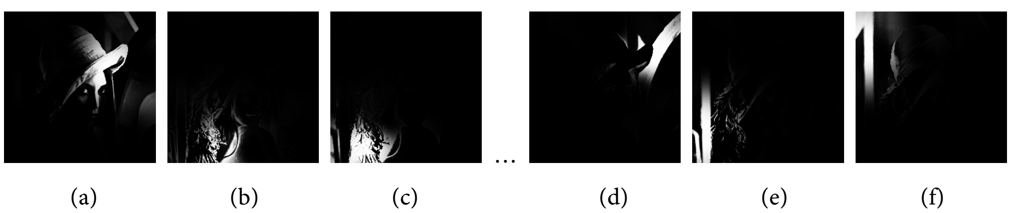
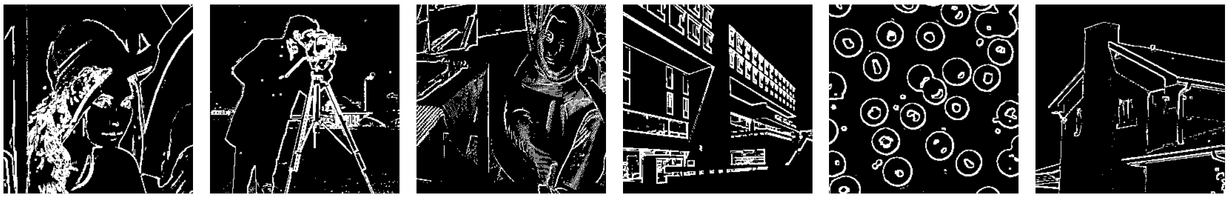


Figure 5. Results of applying FCM algorithm on Lena.

Second, the grayscale images in the dataset were processed using different edge detection algorithms. The output of the FCM-IPSO algorithm is shown in **Figure 6**. The application of classical operators to process five grayscale images is illustrated in **Figures 7-11**. The operators used include Canny, Laplace, Prewitt, Roberts, and Sobel.

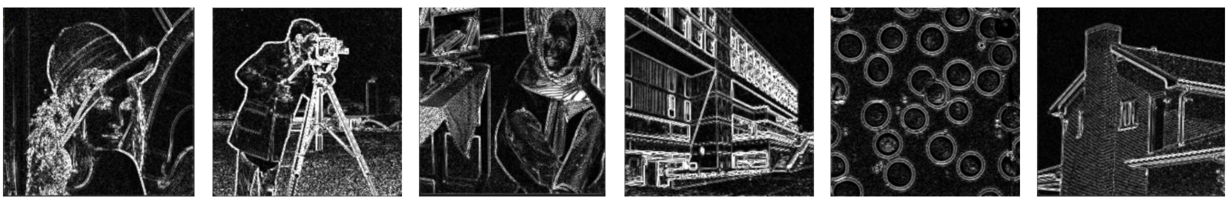


(a) (b) (c) (d) (e) (f)

Figure 6. Results of FCM-IPSO algorithm.

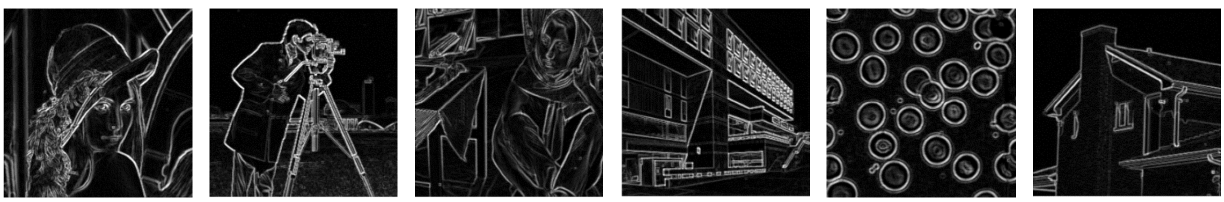
(a) (b) (c) (d) (e) (f)

Figure 7. Results from Canny operator.



(a) (b) (c) (d) (e) (f)

Figure 8. Results from Laplacian operator.

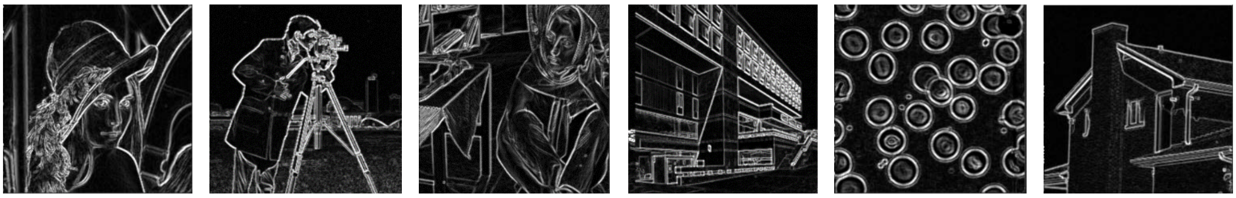


(a) (b) (c) (d) (e) (f)

Figure 9. Results from Prewitt operator.

(a) (b) (c) (d) (e) (f)

Figure 10. Results from Roberts operator.



(a) (b) (c) (d) (e) (f)

Figure 11. Results from Sobel operator.

5.4. Analysis of Experimental Results

Two central problems are frequently studied when implementing image edge algorithms: first, the feasibility of extracting the edges of a foreground object from an image, and second, whether the noise level in the image is excessively high [21]-[23].

Regarding the first problem, **Figure 6** shows the experimental results of the FCM-IPSO algorithm, indicating that it can extract edge information from each gray image. For example, the edge of the build-ing and button in image 5(b); the edge of the kerchief, tablecloths, and books in image 5(c); the edge of the small window in image 5(d); the edge of the bubbles in image 5(e); and the edge of the beams and columns in image 5(f). Furthermore, a minimal increase in noise due to ineffective background extraction was observed. Some of the classical algorithms underperformed when extracting the edges of fore-ground objects in images [7].

Subsequently, different (*I*, *PSO*) pairs as in the Equations (15)-(21), were used in the FCM-IPSO algorithm to test the noise introduction rate at the edges of the Lena, Cameraman, Barbara, Bank, Cell, and House images. The results are shown in **Figure 12** and **Table 2** and **Table 3**.

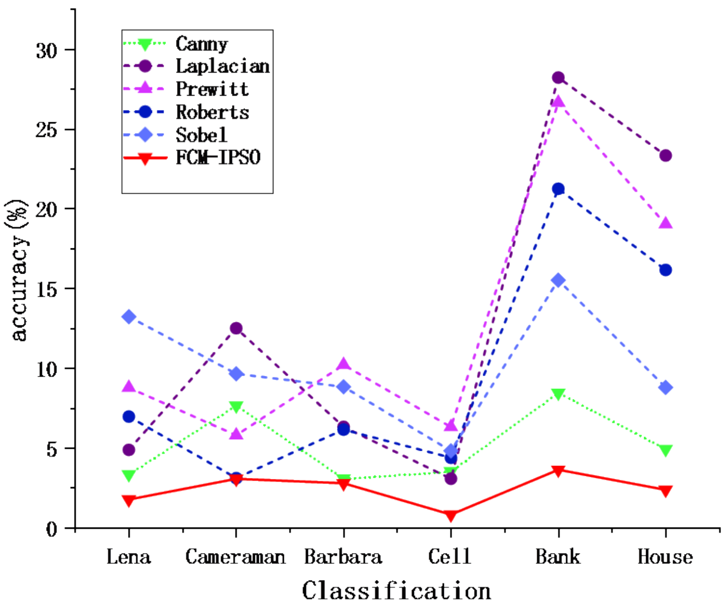


Figure 12. Noise introduction rates for each algorithm.

Table 2. Noise introduction rates of the FCM-IPSO algorithm (%).

	Lena	Cameraman	Barbara	Cell	Bank	House
(PSO_3, I_2)	1.51	3.07	2.78	0.72	4.61	3.06
(PSO_3, I_1)	2.38	3.06	3.05	1.24	5.17	2.82
(PSO_3, I_3)	2.17	2.92	3.02	0.89	6.21	2.88
(PSO_2, I_4)	1.77	3.05	2.21	0.85	3.97	2.28
(PSO_2, I_2)	1.94	3.08	3.1	0.57	2.2	1.48
(PSO_2, I_4)	1.24	3.09	2.85	0.84	3.78	2.67
(PSO_2, I_1)	1.72	3.05	2.77	1.02	4.07	2.11
(PSO_2, I_3)	1.25	3.03	2.94	0.82	1.46	2.29
(PSO_1, I_2)	2.16	3.06	2.64	0.58	2.06	2.63
(PSO_1, I_1)	1.48	3.11	3.22	0.7	2.82	1.56
Average noise rate	1.72	3.05	2.86	0.82	3.64	2.38

Table 3. Noise introduction rates of each algorithm (%).

	Lena	Cameraman	Barbara	Cell	Bank	House
Canny operator	3.34	7.65	3.07	3.52	8.45	4.92
Laplacian operator	4.87	12.51	6.32	3.06	28.24	23.35
Prewitt operator	8.77	5.78	10.21	6.31	26.67	19.03
Roberts operator	6.96	3.1	6.14	4.38	21.26	16.17
Sobel operator	13.23	9.65	8.83	4.81	15.52	8.79
FCM-IPSO algorithm	**1.72**	**3.05**	**2.86**	**0.82**	**3.64**	**2.38**

$$I_1 = \begin{cases} 1, & \sqrt{x} \le y, \\ \left(\dfrac{y}{\sqrt{x}}\right)^2, & \text{else}. \end{cases} \tag{15}$$

$$I_2(x,y) = \begin{cases} 1, & x \le y, \\ \dfrac{y}{x}, & \text{else}. \end{cases} \tag{16}$$

$$I_3(x,y) = \begin{cases} 0, & 0 \le y \le -x+1, \\ x+y-1, & -x+1 \le y \le 1. \end{cases} \tag{17}$$

$$I_4(x,y) = \frac{x+y}{2}. \tag{18}$$

$$PSO_1(x,y) = \begin{cases} \dfrac{xy}{6}, & 0 \le x \le 0.2, x \le y \le -x+0.4, \\ \dfrac{xy}{3}, & y < x \le -y+0.4, 0 \le y < 0.2, \\ xy^2, & 0 \le x \le 0.2, -x+0.4 < y \le 1, \\ xy^2, & 0.2 < x \le 0.4, -x+0.4 < y \le x, \\ xy, & 0.2 < x \le 0.4, -x+0.4 < y < x, \\ xy, & 0.4 < x \le 1, 0 \le y < x. \end{cases} \tag{19}$$

$$PSO_2(x,y)=\begin{cases}\frac{xy}{7}, & 0\le x\le 0.4,\ x\le y\le 0.4,\\ \frac{2xy}{7}, & 0\le x\le 0.4,\ 0\le y<x,\\ \frac{19xy+11}{30}, & 0.4<x\le 1,\ x\le y\le 1,\\ \frac{21xy+9}{30}, & 0.4<x\le 1,\ 0.4<y<x,\\ \frac{xy}{3}, & 0.4<x\le 1,\ 0\le y\le 0.4,\\ \frac{xy}{2}, & 0\le x\le 0.4,\ 0.4\le y\le 1.\end{cases} \tag{20}$$

$$PSO_3(x,y)=\begin{cases}x^2, & x^2\le y^3,\\ y, & x^2>y^3.\end{cases} \tag{21}$$

Figure 12 and Table 3 indicate that the average noise introduction rate of the FCM-IPSO algorithm was generally smaller than those of the other five algorithms. This is because the image is clustered using the fuzzy C-means algorithm before extracting the image edge, which effectively distinguishes the image background from the foreground and thus effectively reduces noise generation.

In summary, the proposed FCM-IPSO algorithm minimizes noise introduction compared to other conventional algorithms while simultaneously extracting as many complete foreground edges from the image as possible.

6. Conclusion

In this paper, the pseudo-semi-overlap function is defined, and two construction methods for it are presented. Subsequently, the (I, PSO)-fuzzy rough set is introduced, and its theoretical properties are explored. Following that, the integration of the upper and lower approximation operators within the (I, PSO)-fuzzy rough set with the fuzzy mathematical morphology operators leads to the proposal of the IPSOFMM operators, with a focus on investigating its properties. Finally, the fuzzy C-means algorithm is combined with the IPSOFMM operator to formulate the FCM-IPSO image edge extraction algorithm, subsequently applied to six grayscale images. The pseudo-semi-overlap function proposed in this paper requires only the properties of asymmetry and left continuity. The PSO function enhances the FCM-IPSO algorithm's ability to handle digital image data with ambiguity, non-completeness, and irregularity, making it flexible to be used in different application environments. However, constructing the pseudo-semi-overlap function becomes more intricate across diverse application contexts. Therefore, future research efforts will focus on devising pseudo-semi-overlap functions tailored to specific application backgrounds. Follow-up research work could further investigate the application of the FCM-IPSO algorithm in video image edge extraction in addition to the construction method of the PSO function.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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