

# Surreal Numbers, Dyadic Rational Numbers and Cell Proliferation

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## Abstract

The growth of multicellular organisms is directly related to cell proliferation through the cell cycle, in which a single cell grows and divides to produce two daughter cells. This process leads to an exponential increase in cell number and serves as a rapid mechanism for tissue growth. In this article, we aim to establish a connection between cell proliferation and surreal numbers, demonstrating that dyadic rational numbers emerging from surreal number theory may be related to growing cell numbers.

## Keywords

Surreal Numbers, Dyadic Rational, Cell Cycle, Cell Proliferation

## 1. Introduction

In this paper, we argue that a link can be established between cell proliferation (see section 2) and surreal numbers [1]-[3], which can lead to a better understanding of cell growth and proliferation. Specifically, we explain how the dyadic rational numbers (see Ref. [4] and references therein) that arise in surreal number theory can be related to growing cell numbers.

The above program can be understood as a mathematical bridge construction between surreal numbers and cell proliferation, serving as an intriguing motivation for uncovering hidden symmetries in the dynamics of cell proliferation. For instance, the parity associated with dyadic fractions can be used as wagering with Zeno which in turn may be considered as a statistic framework for the study of

evolution of a Zeno-wagering game corresponding to a special kind of random walk [5]. A player's gains and losses are represented by the movement of a walker along an interval. It is tempting to believe that this Zeno mechanism can shed some light on hidden symmetries in cell proliferation. In a sense, we can say that this program eventually provides us with a deeper understanding of cell proliferation. This idea is illustrated in Moshinsky's book "Simetrías en la Naturaleza" (Symmetries in Nature) [6], which offers a compelling example of what we are exploring. The author elucidates, using Siqueiros' mural "La Nueva Democracia" (The New Democracy), how the original sketched structure reveals a series of hidden symmetries—such as circles, triangles, and squares—that were crucial in the development of the final mural.

A natural question arises: what might be the specific motivation for attempting to build a mathematical bridge between surreal numbers and cell proliferation? Currently, we only have a partial answer to this question, so what follows can be seen as a progress report, in hopes of motivating others to expand upon it. The initial idea to establish a bridge between surreal numbers stems from the observation that dyadic rational fractions play a central role in both contexts. It is expected that such a bridge will facilitate the transfer of valuable information between these two domains.

Technically, this work is organized as follows: In Section 2, we provide several general comments about cell proliferation. Section 3 describes the standard mathematical tools used to study cell proliferation. In Section 4, we offer a brief review of surreal numbers, with a particular focus on dyadic fractional numbers. Furthermore, Section 5 establishes the connection between cell proliferation and dyadic rational numbers.

## 2. Short Comments above Cell Proliferation

The development of multicellular living systems is closely related to cell proliferation, a complex process that requires simultaneous cell growth and cell division through the cell cycle, which consists of four successive stages, namely G1, S (DNA synthesis), G2, and M. In somatic cells, the latter consists of mitosis and cytokinesis, resulting in genetically identical cells. Specifically, a single cell grows and divides to produce two daughter cells, resulting in an exponential increase in cell number and providing a mechanism for rapid tissue growth. The cell cycle is tightly regulated at multiple checkpoints, in particular G1/S progression, G2/M transition, and mitotic spindle attachment [7]. Cell cycle regulation and the extracellular environment limit the proliferative response in normal cells, but it is disrupted in most cancers [8]. The cell cycle can be arrested by three different mechanisms: the first one occurs when a cell has fully developed the specific characteristics of the cell line, this state is also known as G0; the second one occurs when the environment is not suitable for cell division, maintaining a state known as quiescence; and finally, when the DNA of the cell suffers damage leading to cell senescence or apoptosis [9] [10]. The cell lines have different needs to induce

division and divergent times to complete the whole cycle, being able to categorize the process as heterogeneous growth, these variations are affected by the type of organism, even in the same subject that possesses several cell lines present variance in the time of cell division [11].

During the early stages of development, embryonic cells rapidly proliferate and then divide into different specialized cell types. This means that in the development of multicellular organisms a single cell divides repeatedly to produce numerous differentiated cells, in a final pattern of spectacular complexity and precision [12] [13]. This process is accompanied by an adapted metabolic response in which proteins and factors that control cell cycle progression simultaneously regulate the required metabolic changes [14]. The link between mathematics and cell biology has led to a deeper understanding of cancerous tumour growth [15]. Cell division has long been a subject of interest, and through research and experimental analysis observing and interpreting various cell division processes, models have been developed to study the cell cycle and mitotic factors [16]. There is a high degree of variability in the times at which cells divide, so there are multiple generations in the population at any given time. This is the main reason why it has been difficult to develop a more accurate model to explain heterogeneous cell proliferation [17].

Mathematical models have made significant contributions [18] as effective tools for tackling the complexity of the cell cycle control system and various signaling networks. Some models focus on specific stages of the cycle, while others are constructed based on well-established biological processes and interactions [19]. These models enable representation of biochemical reaction rates through non-linear ordinary differential equations [20].

### 3. Growing Cells

It is known that the growing cellular mass  $M$  and the growing individual cells parameter  $m$  are related by (see Ref. [21], section 6.5.2, page 211, in Spanish Book)

$$dM = Mmdt. \quad (1)$$

where  $m$  is the proliferation rate. Integrating we obtain

$$M = M_0 e^{m(t-t_0)}. \quad (2)$$

On the other hand, if you start with a single cell, it divides to make two cells, and those two cells then divide to make four cells, and so on. This result can be expressed in terms of the geometric series

$$1, 2, 2^2, 2^3, 2^4, \dots, 2^n, \quad (3)$$

where  $n$  is the number of generations.

Therefore, for a  $N_0$  original cells we have that the total number of generation  $N$  is given by

$$N = N_0 2^n. \quad (4)$$

But the number of generation can be computed if we know the time of

generation  $t$ . In fact, we have

$$N = N_0 2^{\frac{t-t_0}{g}}. \quad (5)$$

Here, the quantity  $g$  denotes a constant called generation time.

If we now combine (2) and (5), we obtain the expression

$$m(t-t_0) = \frac{t-t_0}{g} \ln 2, \quad (6)$$

which can be simplified to

$$m = \frac{\ln 2}{g}. \quad (7)$$

This expression indicates a relationship between the two base generation numbers  $m$  and  $g$ . It turns out that (7) can be linked with so called “doubling time” of cells [22] [23] (official terminology used in experimental cell biology). In fact, the doubling time is the time it takes for a population to double in size/value. It is applied to population growth which tend to grow over time such as the cell growth and the volume of malignant tumours. When the relative growth rate (not the absolute growth rate) is constant, the quantity undergoes exponential growth and has a constant doubling time or period, which can be calculated directly from the growth rate. Moreover, this time can be calculated by dividing the natural logarithm of 2 by the exponent of growth. It is worth mentioning that the cancer cells have the fastest doubling time. But the doubling rate depends on the type and aggressiveness of the cancer.

#### 4. A Short View of Surreal Numbers

In 1973 Conway [1] (see also Refs. [24] and [25]) introduced a “set” of numbers called surreal numbers  $\mathcal{S}$ . It can be shown that  $\mathbb{R} \subset \mathcal{S}$ , where  $\mathbb{R}$  denotes the set of real numbers. Surprisingly, the surreal number “set”  $\mathcal{S}$  is constructed from two simple axioms applied to the structure

$$x = \{X_L \mid X_R\}, \quad (8)$$

in  $\mathcal{S}$ , where  $X_L$  and  $X_R$  denote the left and right sets of  $x$ .

**Axiom 1.** Every surreal number corresponds to two sets  $X_L$  and  $X_R$  of previously created numbers, such that no member of the left set  $x_L \in X_L$  is greater than or equal to any member  $x_R$  of the right set  $X_R$ .

Let us denote by the symbol  $\not\geq$  the notion of not greater than or equal to. So the axiom states that if  $x$  is a surreal number then for each  $x_L \in X_L$  and  $x_R \in X_R$  one has  $x_L \not\geq x_R$ . This is denoted by  $X_L \not\geq X_R$ .

**Axiom 2.** One number  $x = \{X_L \mid X_R\}$  is less than or equal to another number  $y = \{Y_L \mid Y_R\}$  if and only if the two conditions  $X_L \not\geq y$  and  $x \not\geq Y_R$  are satisfied.

This can be simplified by saying that  $x \leq y$  if and only if  $X_L \not\geq y$  and  $x \not\geq Y_R$ .

It is worth mentioning that the Conway definition of surreal numbers is based on a recursive method; before a surreal number  $x$  is introduced, we need to know the two sets  $X_L$  and  $X_R$  of previously created surreal numbers. Using the Conway algorithm one finds that at the  $l_2$ -day one obtains  $2^{l_2+1} - 1$  numbers, all of which are of the form  $x = \frac{m}{2^n}$ , where  $m$  is an integer  $Z$  and  $n$  is a natural number  $N$ ,  $n > 0$ . Of course, these numbers correspond to the so called dyadic rational numbers which are obtained from the set

$$D = \left\{ \frac{m}{2^n} \mid m \in Z, n \in N, n > 0 \right\}. \tag{9}$$

In 1986, Gonshor [2] introduced a different but equivalent definition of surreal numbers:

**Definition 1.** A surreal number is a function  $f$  of the initial segment of the ordinals in the set  $\{+, -\}$ .

An example of Gonshor approach is

$$(+ + - + - +) = 2 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} = \frac{27}{16}. \tag{10}$$

It is possible to show that the Conway and Gonshor definitions of surreal numbers are equivalent (see Ref. [2] for details). Thus, either the Conway or the Gonshor approach leads to the dyadic rationals  $D$  defined by the set (9). Indeed, the dyadic rational numbers play a fundamental role in the structure of surreal numbers.

### 4.1. Two Parameter Function

It turns out that (9) can be clarified by the two-parameter function [4]:

$$\mathcal{J}_{(+)}(l_1, l_2) = \begin{cases} \text{(I)} l_1, & \text{if } l_2 - l_1 = 0, \\ \text{(II)} l_1 - \frac{1}{2}, & \text{if } l_2 - l_1 = 1, \\ \text{(III)} l_1 - \frac{1}{2} + \sum_{k=1}^{l_2 - (l_1 + 1)} \frac{\varepsilon_{k+1}}{2^{k+1}}, & \text{if } l_2 - l_1 > 1 \end{cases} \tag{11}$$

Here,  $l_1$  and  $l_2$  take positive integer values, that is  $l_1, l_2 \in Z_{(+)}$ . Note that according to this expression one always has  $l_2 - l_1 \geq 0$ . We may prove that the negative sector can be obtained by  $\mathcal{J}_{(-)}(l_1, l_2) = -\mathcal{J}_{(+)}(l_1, l_2)$ .

Now let us assume  $l_1 = 1$ . In this case (11) becomes

$$\mathcal{J}_{(+)}(1, l_2) = \begin{cases} \text{(I)} 1, & \text{if } l_2 = 1, \\ \text{(II)} \frac{1}{2}, & \text{if } l_2 = 2, \\ \text{(III)} \frac{1}{2} + \sum_{k=1}^{l_2 - 2} \frac{\varepsilon_{k+1}}{2^{k+1}}, & \text{if } l_2 > 2 \end{cases} \tag{12}$$

This implies that from (I) and (II) one gets  $\mathcal{J}_{(+)}(1, 1) = 1$ ,  $\mathcal{J}_{(+)}(1, 2) = \frac{1}{2}$  and from (III) one obtains  $\mathcal{J}_{(+)}(1, 3) = \left\{ \frac{1}{4}, \frac{3}{4} \right\}$ ,  $\mathcal{J}_{(+)}(1, 4) = \left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}$  and so on.

Since  $\mathcal{J}_{(-)}(l_1, l_2) = -\mathcal{J}_{(+)}(l_1, l_2)$  one also has  $\mathcal{J}_{(-)}(1, 1) = -1$ ,  $\mathcal{J}_{(-)}(1, 2) = -\frac{1}{2}$  and  $\mathcal{J}_{(-)}(1, 3) = \left\{-\frac{1}{4}, -\frac{3}{4}\right\}$ ,  $\mathcal{J}_{(-)}(1, 4) = \left\{-\frac{1}{8}, -\frac{3}{8}, -\frac{5}{8}, -\frac{7}{8}\right\}$  and so on.

An interesting combination between (11) and (12) is the formula [4]

$$\mathcal{J}_{(+)}(l_1, l_2) = \mathcal{J}_{(+)}(1, l_2^{(0)}) + (l_1 - 1), \tag{13}$$

with  $l_2^{(0)} = l_2 - (l_1 - 1)$ . This means that the tree  $\mathcal{J}_{(+)}(1, l_2^{(0)})$  (and  $\mathcal{J}_{(-)}(1, l_2^{(0)})$ ) plays the role of a main building block. Observe that in the 0-day one starts with the number 0 and in the 1-day the numbers -1 and +1 are created, namely (-) and (+). While in the 2-day 4 numbers are created, namely  $(++) = 2$ ,  $(+-) = \frac{1}{2}$ ,  $(--)$  = -2,  $(-+) = -\frac{1}{2}$ , and so on. So, we have that the series

$\mathcal{T} = 1 + 2(1 + 2 + 4 + 8 + \dots + 2^{l_2-1})$  determines the total numbers of surreal numbers that at the  $l_2$ -day are created. But using the identity

$$2 + 2 + 4 + 8 + 16 + \dots + 2^{l_2} = 2^{l_2+1}, \tag{14}$$

we discover that  $\mathcal{T} = 2^{l_2+1} - 1$ .

Another interesting aspect of the structure  $\mathcal{J}(l_1, l_2) = \mathcal{J}_{(+)}(l_1, l_2) \cup \mathcal{J}_{(-)}(l_1, l_2) \cup \mathcal{J}_{(\pm)}(0, 0)$  is that the inequality  $-l_2 \leq -l_1 \leq \mathcal{J}(l_1, l_2) \leq l_1 \leq l_2$  holds.

It is worth mentioning the Refs [24] in which dyadic rational fractions are discussed in different contexts.

### 4.2. From Surreal to Real Numbers

Now, a surreal number [25]

$$t = \{T_L | T_R\} \tag{16}$$

can be introduced by considering that

- ★  $T_L$  is the set of all dyadic fractions  $\frac{m}{2^n}$ , where  $\frac{m}{2^n} < t$ ,
- ★  $T_R$  is the set of all dyadic fractions  $\frac{m}{2^n}$ , where  $\frac{m}{2^n} > t$ .

The quantity  $t = \frac{1}{3}$  provides the typical example of (16). In fact, the surreal number  $\frac{1}{3}$  is not a dyadic rational, but can be obtained from the dyadic rational using (16). One obtains the result

$$\frac{1}{3} = \left\{0, \frac{1}{4}, \frac{5}{16}, \frac{21}{64}, \frac{85}{256}, \dots | \dots, \frac{43}{128}, \frac{11}{32}, \frac{3}{8}, \frac{1}{2}\right\}. \tag{17}$$

Several observations can be made from this expression. First, we see that the elements of  $T_L$  and  $T_R$  are dyadic rational numbers, but the value of all

members of the set  $T_L$  is less than the value of any member of the set  $T_R$ . Furthermore, the elements of  $T_L$  get increasing values moving from the left to the right, while the elements of  $T_R$  get decreasing values moving from the right to the left. In general, this increasing/decreasing correspondence can be expressed in the dual form

$$t = \{\rightarrow, \dots | \dots, \leftarrow\}. \tag{18}$$

Of course, the question may emerge about what exactly one means by the dots  $\dots | \dots$ , in both (17) and (18). It turns out that in general there is not clear definition of what exactly such dots mean. But here, there is no such ambiguity because one has the parameter  $l_2$ -day which determines that in  $T_L$  moving from the right to the left, one needs to write younger dyadic fractions, while in  $T_R$  moving to the left one needs also write younger dyadic fractions. Note, however, that all members of the set  $T_L$  are less than the value of any member of the set  $T_R$ .

Let us consider another useful example:

$$\pi = \left\{ \frac{3}{1}, \frac{25}{8}, \frac{201}{64}, \dots | \dots, \frac{101}{32}, \frac{51}{16}, \frac{13}{4}, \frac{7}{2} \right\}. \tag{19}$$

This is, of course, the number  $\pi$  written as a surreal number.

From these examples and general arguments, one might conclude that the raw (basic) material to build any real number are the dyadic fractions. However, according to the previous result, even the most basic dyadic fractions are the family of subsets

$$\mathcal{D}^{(+)} = \left\{ \{0\}, \{1\} \right\} \cup \left\{ \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{4}, \frac{3}{4} \right\}, \left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}, \dots \right\} \tag{20}$$

and the corresponding negative family set  $\mathcal{D}^{(-)} = -\mathcal{D}^{(+)}$ . Notes that

$\mathcal{D}\mathcal{A}\mathcal{D}^{(+)} \cup \mathcal{D}^{(-)} \subset D$ . Moreover, as it was mentioned in (11), (I) and (II) one gets

$\mathcal{J}_{(+)}(1,1) = 1$ ,  $\mathcal{J}_{(+)}(1,2) = \frac{1}{2}$ , while from (III) one obtains  $\mathcal{J}_{(+)}(1,3) = \left\{ \frac{1}{4}, \frac{3}{4} \right\}$ ,

$\mathcal{J}_{(+)}(1,4) = \left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}$  and so on. Thus,  $\mathcal{D}^{(+)}$  can be obtained from

$\mathcal{J}_{(+)}(1, l_2^{(0)})$  as given in (12).

An interesting relation can be obtained from the different subsets of  $\mathcal{D}^{(+)}$ . So, if now one introduces the definition

$$m = 2^{l_2-2}, \tag{21}$$

then (21) becomes

$$\sum_{l=0}^{m-1} (2l+1) = m^2. \tag{22}$$

This result is well known and can even be proved by induction.

### 4.3. Parity and Dyadic Rational Numbers

Using the extended definition of dyadic rational numbers [26]

$$\mathbb{D} = \left\{ \frac{m}{2^n} \mid m \in \mathbb{Z}, n \in \mathbb{Z} \right\}, \tag{23}$$

it is shown that there exists a partition of rational groups into subgroups with a rich algebraic structure. This development can be understood by writing (23) in the equivalent form

$$\mathbb{D} = \left\{ \frac{2l-1}{2^k} \mid k \in \mathbb{Z}, l \in \mathbb{Z} \right\} \cup \{0\}, \tag{24}$$

which allows to consider the  $k$  level sets

$$\mathbb{D}_k = \left\{ \frac{2l-1}{2^k} \mid l \in \mathbb{Z} \right\} \text{ and } \mathbb{D}_\infty = \{0\}. \tag{25}$$

Of course, we have

$$\mathbb{D} = \bigcup_k \mathbb{D}_k \cup \mathbb{D}_\infty. \tag{26}$$

Thus, we see that

$$\begin{aligned} \mathbb{D}_0 &= Z_O, \\ \mathbb{D}_{k < 0} &= Z_E. \end{aligned} \tag{27}$$

Here,  $Z_O$  and  $Z_E$  denote the odd and even integers respectively. The relevant aspect of this construction is that the set  $\mathcal{D}^{(\pm)}$  in (20), emerging from the surreal numbers structure, is linked to  $\mathbb{D}_{k > 0}$  in the form

$$\mathcal{D}^{(+)} = \bigcup_k \mathbb{D}_{k > 0}^{*(+)} \cup \{0, 1\}, \tag{28}$$

where

$$\mathbb{D}_{k > 0}^{*(+)} = \left\{ \frac{2l-1}{2^k} \mid l = 1, 2, \dots, 2^{k-1} \right\}. \tag{29}$$

In fact, if one identifies  $k = l_2 - 1$  then (29) becomes

$$\mathbb{D}_{l_2}^{*(+)} = \left\{ \frac{2l-1}{2^{l_2-1}} \mid l = 1, 2, \dots, 2^{l_2-2}, l_2 \geq 2 \right\} \tag{30}$$

and therefore, we find that

$$\mathbb{D}_{l_2}^{*(+)} = \mathcal{D}^{(+)} = \left\{ \left\{ \frac{1}{2} \right\}, \left\{ \frac{1}{4}, \frac{3}{4} \right\}, \left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}, \dots \right\}. \tag{31}$$

The attractive feature of this result is that the parity concept of irrational numbers can be identified with the  $l_2$ -day notion of surreal numbers.

### 5. Link Between Growing Cell Numbers and Dyadic Rational Numbers

Below (12) we argue that from (II) we get  $\mathcal{J}_{(+)}(1, 2) = \frac{1}{2}$  and from (III) we obtain

$\mathcal{J}_{(+)}(1, 3) = \left\{ \frac{1}{4}, \frac{3}{4} \right\}$ ,  $\mathcal{J}_{(+)}(1, 4) = \left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}$  and so on. Therefore, the sequence

of created numbers corresponds to the sequence

$$1, 2, 2^2, 2^3, \dots, 2^{l_2-2} \tag{32}$$

The same sequence of growing cell numbers given in (3) and mentioned in section 2. Thus, the creation number parameter  $l_2$  and the  $n$  generation number must be related by

$$l_2 = n + 2 \tag{33}$$

This suggests to consider the sum identity (14)

$$2 + 2 + 4 + 8 + 16 + \dots + 2^{l_2} = 2^{l_2+1}, \tag{34}$$

to find that the total number of cells created in the  $n$  generation is given by

$$\mathcal{N} = 2^{n+1} - 1 \tag{35}$$

which corresponds to the total numbers created with  $\mathcal{J}_{(+)}(1, l_2)$ , namely

$$\mathcal{T} = 2^{l_2-1} - 1 \tag{36}$$

So, this thin connection between growing cell numbers and dyadic rational numbers opens many questions. First, observing formula (1) in section 2, what could be the corresponding analysis in the scenario of surreal numbers. In (1) differential calculus is used and therefore the quantities  $M$  and  $t$  in such expression take values in the real numbers  $R$ . But according to section 3.2 the real numbers can be obtained from the dyadic rational numbers. This means that it must be possible to consider the correspondence

$$M = M(\text{dyadic rational numbers}) \tag{37}$$

and

$$t = t(\text{dyadic rational numbers}) \tag{38}$$

as explained in section 3.2.

Second, in the tree of surreal numbers we start with the numbers  $\frac{1}{2}$  via  $\mathcal{J}_{(+)}(1, 2)$  and then we consider the sets  $\mathcal{J}_{(+)}(1, 3) = \left\{ \frac{1}{4}, \frac{3}{4} \right\}$ ,  $\mathcal{J}_{(+)}(1, 4) = \left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}$ . At the level of counting the amount of numbers created we have already found a connection with the number of growing cells, but we wonder what that may mean to associate these numbers with the cells themselves. Assume that we start with a single cell which somehow we printed the number  $\frac{1}{2}$ , of the two cells created in the division we printed one with  $\frac{1}{4}$  and the other with  $\frac{3}{4}$ . This method shall continue with four cells with the printer numbers  $\left\{ \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8} \right\}$  in each cell, and so on. From this point of view, it seems that at each level of division is keeping some kind of memory in the cell itself.

### 6. Conclusion

In this paper, we proposed the idea of relating cell proliferation to surreal number theory. Although this idea seems interesting, the geometric study of cell growth

presented here should be viewed as a preliminary approximation to a complex biological process. However, since any real number can be expressed in terms of dyadic rationals, as explained in section 4.2, it is expected that the whole structure of a mathematical cell growth model can be considered as a model in terms of the underlying surreal number structure.

Cell division is tightly regulated by multiple evolutionary conserved cell cycle control mechanisms encoded by the DNA sequences (see section 3). It turns out that the genetic code has been linked to a hidden up-down and left-right duality symmetry structure [27], allowing for a reduction, *via* duality, the  $64 = 4 \times 4 \times 4 = 2^6$  triality codons to only  $32 = 2^5$  possible independent codons. This is because in RNA, adenine pairs with uracil (*A-U*), and guanine pairs with cytosine (*G-C*). It's worth noting that both numbers  $2^6$  and  $2^5$  admit a link with the dyadic numbers  $\frac{1}{2^6}$  and  $\frac{1}{2^5}$ . It would be interesting for further studies to see if this duality or description of the genetic code is related to the cell cycle, and therefore to cell division and cell growth.

It may be helpful if we further clarify the sequence of created numbers (32) and the dyadic rational numbers (23). As it was mentioned in section 4, the surreal numbers are based on a recursive method; before a surreal number  $x$  is introduced, we need to know the two sets  $X_L$  and  $X_R$  of previously created surreal numbers. In this sense, we can start with the vacuum set  $\phi$  and then to construct the zero number

$$0 = \{\phi \mid \phi\}, \quad (39)$$

In the sequence, this is called 0-day. In the 1-day we may have the combinations

$$1 = \{0 \mid \phi\}, \quad (40)$$

and

$$-1 = \{\phi \mid 0\}, \quad (41)$$

While in the 2-day we may construct the new numbers  $-2 = \{\phi \mid -1\}$ ,  $-\frac{1}{2} = \{0 \mid -1\}$ ,  $\frac{1}{2} = \{0 \mid 1\}$  and  $2 = \{1 \mid \phi\}$  and so on. Thus, the result of this algorithm is that the set of dyadic rational numbers (23) are created. Thus, the question emerges in the  $l_2$ -day how many numbers  $\mathcal{T}$  of the form (23) are created? The answer explained in section (4) is  $\mathcal{T} = 2^{l_2+1} - 1$ . To prove this, we used (32). For better explanation of this development we recommend the Ref. [28].

Finally, it is known that uncontrolled cell growth proliferation causes cancer, and one of the solutions is to regulate cell growth back to normal. In particular, the importance of the role that cancer stem cells play in the initiation, progression, and recurrence of cancer is worth mentioning [29]. In addition, the behavior of cancer cell proliferation has motivated the introduction of mathematical models in the literature to model of cancer initiation, proliferation and metastasis. In particular, the growth rate of the cancer cell has been derived from both deterministic and stochastic considerations [30]. Moreover, some mathematical models

describing the interactions between a malignant tumor and the immune system have been considered (see Ref. [31] and references therein). In particular, simple ordinary differential equation models provide an attractive approach, as they start with explanations for tumor growth and then progressively build more complex differential equations that may establish a bridge between tumor growth and the immune system. The key observation is that the use of differential equations in these models indicates that the numerical theory underlying their developments is based on the real numbers  $R$ , which, in turn, are contained within the surreal numbers  $S$ . Therefore, understanding cell growth by establishing a link between the growing cell and the surreal number can be considered a very important and inspiring relationship that can motivate further work in bio-medicine.

It is worth noting that our research on uncovering hidden symmetries in the dynamics of cell proliferation is, to the best of our knowledge, entirely novel. Therefore, we cannot provide additional references beyond the one cited in [27]. It may also be helpful to mention that extensive research and experimental studies have provided valuable insights into various cell proliferation, which have led to the development of models aimed at understanding the cell cycle and mitotic factors. There are a variety of methods available for measuring cell proliferation, each differing in the specific phase of cellular growth and division they evaluate, as well as in the types of cells or tissues they can assess (see Ref. [16] and references therein).

For further and more detailed explanations of surreal number theory, readers are referred to Refs [24] and [25], which offer a comprehensive treatment of the topic. Finally, Refs [24] and [25] also provide insights into how dyadic rational fractions can be used to represent any real number.

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## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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