Journal of Applied Mathematics and Physics, 2024, 12, 4163-4169
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Application of Laplacian Operator on Lightlike Warped Product Hypersurfaces

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How to cite this paper: Domitien, N., Gilbert, N. and Ménédore, K. (2024) Application of Laplacian Operator on Lightlike Warped Product Hypersurfaces. *Journal of Applied Mathematics and Physics*, **12**, 4163-4169.
https://doi.org/10.4236/jamp.2024.1212255

Received: October 24, 2024
Accepted: December 16, 2024
Published: December 19, 2024

Abstract

The Laplacian of a function measures the difference between the value of the function at a point and its average around that point. It is a differential operator appears in many differential equations describing many physical phenomena. We establish some inequalities involving the Laplacian of warping function for screen conformal lightlike warped product hypersurface in Lorentzian manifold of constant sectional curvature. The existence of such hypersurface is deduced in minimal case.

Keywords

Laplacian, Conformal Screen Manifold, Hamonic Function

1. Introduction

The warping function plays an important role in the study of the lightlike warped product geometry. One can make a study on the warping function to get desired properties on a lightlike warped product manifold.

In case of Riemannian warped product submanifold, many authors proved some inequalities relating the Laplacian of the warping function to various curvature terms with important applications [1]-[4].

Especially in analysis, it is well known that the Laplacian of a function f allows to discover important physical properties. For instance, solutions of Laplace's equation $\Delta f = 0$ occur in problems of gravitational potentials, magnetic, electrical, and of hydrodynamics. In this paper, the Laplacian of warping function is a central tool of our analysis.

The study of null submanifolds is complicated since the inclusion of a part of the normal bundle in the tangent bundle doesn't allow us to find projector to define induced geometric objects. Particularly, in case of lightlike hypersurface, the normal bundle is totally contained in the tangent bundle. In consequence, the Riemannian curvature tensor doesn't have symmetry properties. To deal with the above problems, some techniques are proposed in [5]-[7]. In [8], the authors explored the problem concerning isometric immersion of lightlike warped product manifold in semi-Riemannian space form.

In our approach, we consider isometric immersion of a lightlike warped product of a degenerate manifold with signature $(1,0,p)$ and a Riemannian manifold in a simply connected complete Lorentazian space form. In [9], it is proved that the Riemannian curvature tensor of coisotropic warped product manifold that is conformal screen is an algebraic curvature tensor. We consider screen conformal hypersurface case and establish some fundamental inequalities involving the Laplacian of the warping function.

2. Preliminaries

Let $(M_1^{m_1}, g_1),(M_2^{m_2}, g_2)$ be two semi-Riemannian manifolds, and f a positive differential function on M_1. The warped product manifold $M^m = M_1 \times_f M_2$ is the product $M_1 \times M_2$ equiped by the warped metric $g = \pi_1^*(g_1) + (f \circ \pi_1)^2 \pi_2^*(g_2)$ where π_1 and π_2 are the projection morphisms of $M_1 \times M_2$ on M_1 and M_2 respecively and f is called a warping function.

Let $\{E_1, \cdots, E_{m_1}, E_{m_1+1}, \cdots, E_m\}$ be an orthonormal frame of (M, g) where $\{E_1, \cdots, E_{m_1}\}$ is the basis on the horizontal bundle $\Gamma(\mathcal{H})$ and $\{E_{m_1+1}, \cdots, E_m\}$ is with respect to the vertical bundle $\Gamma(\mathcal{V})$. The Laplacian of the warping function is given by

$$\Delta f = \sum_{i=1}^{m_1} \left(\left(\nabla_{E_i} E_i \right) f - E_i^2 f \right). \tag{1}$$

If (M_1, g) is a r-degenerate manifold and (M_2, g_2) a semi-Riemannian manifold, then $(M_1 \times_f M_2, g_1 + f^2 g_2)$ is a r-lightlike warped product manifold and any screen structure has dimension $(m_1 - r) + m_2$.

The Riemmannian curvature tensor in $\Gamma(\mathcal{V})$ is given by

$$R(X,V)W = -\frac{\langle V, W \rangle}{f} \nabla_X (grad f). \tag{2}$$

It is well known from [10] that a normalized lightlike hypersurface of a semi-Riemannian manifold is called screen conformal if there exists a non vanishing differential function φ in a neighborhood $\mathcal{U}$ such that

$$A_N = \varphi A_\xi^*. \tag{3}$$

If φ is a non-zero constant then M is said to be screen homothetic.

Consequently the local second fundamental form B and the screen second fundamental form C of a normalized hyupersurface are related by

$$C^N(X, PY) = \varphi B^N(X, Y). \tag{4}$$

For any lightlike hypersurface M of a semi-Riemannian manifold $\bar{M}$ the

Gauss-Codazzi equation is given by

$$\langle R(X,Y)Z,PW\rangle = \langle \bar{R}(X,Y)Z,PW\rangle + B^N(Y,Z)C^N(X,PW) - B^N(X,Z)C^N(Y,PW) \tag{5}$$

for all $X,Y,Z,W \in \Gamma(TM)$.

Let $\{E_1,\cdots,E_m\}$ be an orthonormal basis of the screen bundle $\mathscr{S}(N)$ of a lightlike hypersurface M. The mean curvature μ of M is defined in [11] as follow

$$\mu = \frac{1}{m}\sum_{i=1}^{m} \epsilon_i B_{ii} \tag{6}$$

with $\epsilon_i = g(E_i,E_i)$, $B_{ii} = B(E_i,E_i)$. M is said to be minimal if μ vanishes identically.

Let $\pi = span\{X,Y\}$ be a 2-dimensional non-degenerate plane of T_pM. The number

$$K(\pi) = \frac{\langle R(X,Y)Y,X\rangle}{\langle X,X\rangle\langle Y,Y\rangle - \langle X,Y\rangle^2}, \tag{7}$$

is called the sectional curvature of π in M.

Let π_k be a non-null k-plane section of T_pM and $\{E_1,\cdots,E_k\}$ be any orthonormal basis of π_k.

The scalar curvature $\tau(\pi_k)$ of π_k is given by [12]

$$\tau(\pi_k) = \sum_{ij=1}^{k} K(E_i \wedge E_j) \tag{8}$$

where $K(E_i \wedge E_j) = \langle R(E_i,E_j)E_j,E_i\rangle$.

If the Riemannian curvature tensor R has the symetry properties we have

$$\tau(\pi_k) = 2\sum_{1\le i<j\le k} K(E_i \wedge E_j). \tag{9}$$

3. Main Results

In the following we consider a lightlike warped product of a 1-degenerate manifold M_1 with signature $(1,0,p)$ and a connected Riemannian manifold M_2. For a Lorentazian manifold space form $\bar{M}$ that is simply connected and complete, we consider an isometric immersion h of $M_1 \times_f M_2$ in $\bar{M}$ where $M_1 \times_f M_2$ is a screen conformal normalized lightlike warped product hypersurface.

We explore the following fundamental inequality given in [13] to establish some inequalities involving some geometrical materials on lightlike warped product hypersurface.

Lemma 3.1. *If* $m \ge 2$ *and* $a_1,\cdots,a_m,b$ *are real numbers such that*

$$\left(\sum_{i=1}^{m} a_i\right)^2 = (m-1)\left(\sum_{i=1}^{m} a_i^2 + b\right),$$

then

$$2a_1a_2 \ge b$$

with equality if and only if $a_1 + a_2 = a_3 = \cdots = a_m$.

Theorem 3.1. *Let* $(M = M_1 \times_f M_2, g)$ *be a lightlike warped product of* $(m_1 + 1)$ *-dimensional lightlike manifold with signature* $(1, 0, m_1)$ *and a* m_2 *-dimensional connected Riemannian manifold. Let* $h : (M, g) \to (\bar{M}_{(c)}, \bar{g})$ *be a screen conformal noemalized hypersurface of a* $(m + 2)$ *-dimensional Lorentzian manifold with constant sectional curvature c. Then we have*

$$\frac{\Delta f}{f} \geq \frac{m_1 m^2}{2(m-1)} \varphi \mu^2 - \frac{\varphi m_1}{2} \sum_{ij=1}^{m} B_{ij}^2 + m_1 c \tag{10}$$

where $m = m_1 + m_2$ and φ is a positive conformal function.

Proof. From (5), and (8) we have

$$\tau_{\mathscr{S}(N)} = \bar{\tau}_{\mathscr{S}(N)} + \varphi m^2 \mu^2 - \varphi \sum_{ij=1}^{m} B_{ij}^2. \tag{11}$$

Consider

$$\sigma = \tau_{\mathscr{S}(N)} - \frac{m^2(m-2)}{m-1} \varphi \mu^2 - \bar{\tau}_{\mathscr{S}(N)} \tag{12}$$

we have

$$\left(\sum_{i=1}^{m} B_{ii} \right)^2 = (m-1) \left(\frac{\sigma}{\varphi} + \sum_{i=1}^{m} B_{ii}^2 + \sum_{i \neq j=1}^{m} B_{ij}^2 \right) \tag{13}$$

and by lemma 3.1, take $a_1 = B_{11}$ and $a_2 = B_{m_1+1 m_1+1}$, we get

$$2 B_{11} B_{m_1+1 m_1+1} \geq \frac{\sigma}{\varphi} + \sum_{i \neq j=1}^{m} B_{ij}^2. \tag{14}$$

From 5 and 2 we have

$$\begin{aligned} \frac{1}{f}\left(\left(\nabla_{E_1} E_1 \right) f - E_1^2 f \right) &= c + \varphi B_{m_1+1 m_1+1} B_{11} \\ &\overset{(14)}{\geq} c + \frac{\sigma}{2} + \frac{\varphi}{2} \sum_{i \neq j=1}^{m} B_{ij}^2 \\ &\geq c + \frac{\sigma}{2}. \end{aligned} \tag{15}$$

By (1), 11) and (12) we get

$$\begin{aligned} \frac{\Delta f}{f} &\geq \frac{m_1}{2} \sigma + m_1 c \\ &= \frac{m_1 m^2}{2(m-1)} \varphi \mu^2 - \frac{\varphi m_1}{2} \sum_{ij=1}^{m} B_{ij}^2 + m_1 c \end{aligned} \tag{16}$$

and (10) is proved.■

Theorem 3.2. *Let* $(M = M_1 \times_f M_2, g)$ *be a lightlike warped product of* $(m_1 + 1)$ *-dimensional lightlike manifold with signature* $(1, 0, m_1)$ *and a* m_2 *-dimensional connected Riemannian manifold. Let* $h : (M, g) \to (\bar{M}_{(c)}, \bar{g})$ *be an isometric immersion such that M is a screen conformal normalized hypersurface of a* $(m + 2)$ *-dimensional Lorentzian space form. If* $\pi = span\{E_1, E_2\}$ *is a non-degenerate plane section of* $T_p M$ *then we have*

$$\frac{\Delta f}{f} \le \frac{(m-2)(m+1)}{2m_2}c + \frac{m^2(m-2)}{2m_2(m-1)}\varphi\mu^2 - \frac{\tau_{\mathscr{S}_1(N)} + \tau_{\mathscr{S}_2(N)}}{2m_2} + \frac{\tau(\pi)}{2m_2} + \frac{\varphi}{2m_2}\sum_{i=3}^{m} B_{ii}^2 \quad (17)$$

where $m = m_1 + m_2$ and φ is a positive conformal function and $\tau_{\mathscr{S}_1(N)}$, $\tau_{\mathscr{S}_2(N)}$ the scalar curvature on screen horizontal bundal and scren vertical bundle respecivelly.

Proof. By Lemma (3.1), take $a_1 = B_{11}$ and $a_2 = B_{22}$ in (13) we have

$$2B_{11}B_{22} \ge \frac{\sigma}{\varphi} + \sum_{i \ne j=1}^{m} B_{ij}^2. \quad (18)$$

Let $\pi = span\{E_1, E_2\}$, from (11) and (18) we have

$$\begin{aligned}
\tau(\pi) &= \overline{\tau}(\pi) + \varphi\sum_{ij=1}^{2} B_{ii}B_{jj} - \sum_{ij=1}^{2} \varphi B_{ij}^2 \\
&= \overline{\tau}(\pi) + 2\varphi B_{11}B_{22} - \sum_{i \ne j=1}^{2} \varphi B_{ij}^2 \\
&\ge \overline{\tau}(\pi) + \sigma + \sum_{i \ne j=1}^{m} \varphi B_{ij}^2 - \sum_{i \ne j=1}^{2} \varphi B_{ij}^2 \\
&= \overline{\tau}(\pi) + \sigma - \sum_{i=3}^{m} \varphi B_{ii}^2 + \sum_{ij=3}^{m} B_{ij}^2 \\
&\ge \overline{\tau}(\pi) + \sigma - \sum_{i=3}^{m} \varphi B_{ii}^2
\end{aligned}$$

and by (12) we have

$$2m_2\frac{\Delta f}{f} \le m(m-1)c - 2c + \frac{m^2(m-2)}{m-1}\varphi\mu^2 - \left(\tau_{\mathscr{S}_1(N)} + \tau_{\mathscr{S}_2(N)}\right) + \tau(\pi) + \sum_{i=3}^{m} \varphi B_{ii}^2$$

which leads to (17).■

Remark 3.1. *In case of a screen conformal warped product hypersurface with negative conformal function* φ*, we get the previous expressions with inversed sens of inequalities.*

The inequalities given in the following theorem hold with a positive or a negative conformal function.

Theorem 3.3. *Let* $(M = M_1 \times_f M_2, g)$ *be a lightlike warped product of* (m_1+1)*-dimensional lightlike manifold with signature* $(1,0,m_1)$ *and a* m_2*-dimensional connected Riemannian manifold. Let* $(\overline{M}_{(c)}, \overline{g})$ *be a* $(m+2)$*-dimensional simply connected complete Lorentzian manifold of constant sectional curvature . For any screen conformal normalized hypersurface isometric immersion* $h:(M,g) \to (\overline{M}_{(c)}, \overline{g})$ *we have*

$$\text{(a)} \quad \frac{\Delta f}{f} \le \frac{m(m-1)}{2m_2}c - \frac{\tau_{\mathscr{S}_1(N)} + \tau_{\mathscr{S}_2(N)}}{2m_2} + \frac{\varphi m^2\mu^2}{2m_2} + \frac{1+\varphi^2}{4m_2}\sum_{ij=1}^{m} B_{ij}^2 \quad (19)$$

with equality if and only if M is a screen homothetic lightlike hypersurface with $\varphi = -1$.

$$\text{(b)} \quad \frac{\Delta f}{f} \ge \frac{m(m-1)}{2m_2}c - \frac{\tau_{\mathscr{S}_1(N)} + \tau_{\mathscr{S}_2(N)}}{2m_2} + \frac{\varphi m^2\mu^2}{2m_2} - \frac{(1+\varphi)^2}{4m_2}\sum_{ij=1}^{m} B_{ij}^2 \quad (20)$$

Proof. From (8) and (5) we have

$$\frac{2m_2\Delta f}{f}+\tau_{\mathscr{S}_1(N)}+\tau_{\mathscr{S}_2(N)}$$
$$=\overline{\tau}_{\mathscr{S}(N)}+\sum_{ij=1}^{m}B_{jj}C_{ii}-\frac{1}{2}\sum_{ij=1}^{m}\left(B_{ij}+C_{ji}\right)^2+\frac{1}{2}\sum_{ij=1}^{m}\left[\left(B_{ij}\right)^2+\left(C_{ji}\right)^2\right]$$
$$\overset{(4)}{=}\overline{\tau}_{\mathscr{S}(N)}+\varphi m^2\mu^2-\frac{1}{2}(\varphi+1)^2\sum_{ij=1}^{m}B_{ij}^2+\frac{1}{2}\left(\varphi^2+1\right)\sum_{ij=1}^{m}B_{ij}^2$$

and the relations (19) and (20) hold.■

Application

If the warping function *f* is harmonic, with respect to the sign of the conformal function φ, from Theorem (3.1) and Remark (3.1) we have the following application.

Corollary 3.1. *Let* $\left(M=M_1\times_\rho M_2,g\right)$ *be a lightlike warped product of* $\left(m_1+1\right)$ *-dimensional lightlike manifold with signature* $\left(1,0,m_1\right)$ *and a* m_2 *-dimensional connected Riemannian manifold. If f is a harmonic function, then*

(a) there does not exist a minimal screen conformal hypersurface isometric immersion of *M* in a Lorentzian manifold of positive constant curvature such that

$$\frac{\varphi m_1}{2}\sum_{ij=1}^{m}B_{ij}^2<m_1c$$

with φ is a positive conformal function;

(b) there does not exist a minimal screen conformal hypersurface isometric immersion of *M* in a Lorentzian manifold of negative constant curvature such that

$$\frac{\varphi m_1}{2}\sum_{ij=1}^{m}B_{ij}^2>m_1c$$

with φ is a negative conformal function.

4. Conclusion

We explored a differential operator named Laplacian to compute some geometrical objects of screen conformal warped product hypersurface. Especially, we made an estimation of the Laplacian of the warping function and got some obstructions on existence of minimal screen conformal hypersurface of a Lorentzian space form.

Acknowledgements

The authors are grateful to the referees for their valuable comments and suggestions for the improvement of the paper.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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