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Coupling Product on Permutations

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Abstract

In combinatorics, permutations are important objects with many operations. In this paper, we define a coupling product on permutations and prove that the space spanned by permutations is a graded algebra.

Keywords

Permutation, Coupling Product, Graded, Algebra

1. Introduction

Algebraic structures not only provide a solid theoretical foundation for the mathematical systems, but also offer precise mathematical methodologies for solving practical problems and they have facilitated the cross-integration of mathematics with other disciplines. Therefore, algebraic structures are extremely significant in mathematics. In combinatorics, many objects have algebraic structures. As time goes by, mathematicians have found many algebraic structures on different objects, such as algebraic structures on permutations [1] [2], planar trees [3], simple graphs [4], posets [5] and parking functions [6]-[8].

Permutations are important combinatorial objects. A permutation of degree n is an arrangement of n elements. The symmetric group of degree n, denoted by S_n, contains all permutations of $[n] := \{1, 2, \cdots, n\}$. Let $\mathbb{K}S_n$ be the vector space spanned by S_n over field $\mathbb{K}$. Define $\mathbb{K}\mathbb{S} := \oplus_{n\geq 0}\mathbb{K}S_n$, where $S_0 = \{\epsilon\}$ and ϵ is the empty permutation. Then $\mathbb{K}\mathbb{S}$ is graded and its n-th component is $\mathbb{K}S_n$. In 1995, Malvenuto and Reutenauer defined the classic operation on permutations [9], which is the shuffle product. In 2005, Aguiar and Sottile introduced global descents on permutations [10]. In 2018, Bergeron, Ceballos and Pilaud defined gaps on permutations [11], which play a vital role. In 2020, based on the global descents, Zhao and Li defined a new shuffle product on permutations and proved that $\mathbb{K}S_n$ with the new shuffle product is a graded algebra [12]. In 2014,

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Vargas defined super-shuffle product on permutations [13], and in 2021, Liu and Li proved that $\mathbb{K}S_n$ with the super-shuffle product is a graded algebra [14].

The organization of this paper is as follows. In Section 2, we review the basic definitions of graded algebras and basic notations on permutations. We introduce the definitions of gaps, absolute ascents and atoms of permutations. In Section 3, we define a coupling product $\ltimes$ on permutations and prove that ($\mathbb{K}\mathbb{S}, \ltimes, \mu$) is a graded algebra. In this paper, we also provide some examples to help readers understand the coupling product on permutations.

2. Preliminaries

2.1. Graded Algebra

Let R be a commutative ring and A be a R-module.

Define a *product* $m: A \otimes_R A \to A$ and a *unit* $\mu: R \to A$, respectively, satisfying the following diagrams, then (A, m, μ) is an R-*algebra*.

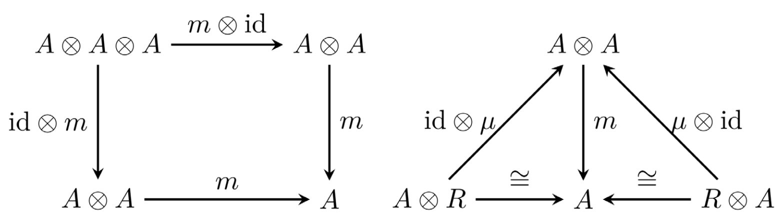

Figure 1. Associative law and unit.

The algebra A is *graded* if there is a direct sum decomposition $A = \oplus_{i \ge 0} A_i$ such that the product of homogeneous elements of degrees p and q is homogeneous of degree $p+q$, that is, $m(A_p \otimes A_q) \subseteq A_{p+q}$, and $\mu(R) \subseteq A_0$.

2.2. Basic Notations

A permutation a in S_n is a bijection of the set $[n] := \{1, 2, \cdots, n\}$, denoted by $a = a_1 a_2 \cdots a_n$, where $a_i = a(i) \in [n]$. In particular, $S_0 = \{\epsilon\}$, where ϵ is the empty permutation. Denote $\mathbb{S} := \uplus_{n \ge 0} S_n$ and $\mathbb{K}\mathbb{S} := \oplus_{n \ge 0} \mathbb{K}S_n$, where $\mathbb{K}S_n$ is the vector space spanned by S_n over field $\mathbb{K}$.

For any sequence of distinct positive integers $a = a_1 a_2 \cdots a_n$, the position between the i-th element a_i and the $(i+1)$-th element a_{i+1} is called the i-th *gap* of a, $1 \le i \le n-1$. The position in front of the first element a_1 is called the 0-th *gap* of a and the position behind the last element a_n is called the n-th *gap* of a. For example, the gaps of 38749 are the numbers in blue: ${}_0 3_1 8_2 7_3 4_4 9_5$.

For $1 \le i \le n-1$, we say that i is an *absolute ascent* of a if $a_i < a_j$ for any $i < j \le n$. Let $W_a = \{i_1, i_2, \cdots, i_r\}$ be the set of all absolute ascents of a, where $i_1 < i_2 < \cdots < i_r$. Denote

$$\alpha_1 = a_1 a_2 \cdots a_{i_1},$$

$$\alpha_2 = a_{i_1+1} a_{i_1+2} \cdots a_{i_2},$$

$$\vdots$$

$$\alpha_r = a_{i_{r-1}+1} \cdots a_n,$$

and call that α_i is an *atom* of a, for $1 \le i \le r$. For any permutation a in S_n, put the symbol $\circ$ at all absolute ascents of a, then we get the *decomposition* of a, denote by $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r$. If a has no absolute ascent, we call it *indecomposable*.

Example 2.1. For $a = 314659$, we have $W_a = \{2,3,5\}$ and

$$\alpha = \alpha_1 \circ \alpha_2 \circ \alpha_3 \circ \alpha_4 = 31 \circ 4 \circ 65 \circ 9.$$

The sequence 342 is indecomposable.

3. Coupling Product on Permutations

In this section, we give the definition of the coupling product on permutations.

Let $a = a_1 a_2 \cdots a_n$ be a permutation of degree n and $b = b_1 b_2 \cdots b_m$ be a permutation of degree m with $W_a = \{i_1, i_2, \cdots, i_r\}$ and $W_b = \{j_1, j_2, \cdots, j_s\}$, respectively. Then their atoms are

$$\alpha_1 = a_1 a_2 \cdots a_{i_1},$$

$$\alpha_2 = a_{i_1+1} a_{i_1+2} \cdots a_{i_2},$$

$$\vdots$$

$$\alpha_r = a_{i_{r-1}+1} a_{i_{r-1}+2} \cdots a_n,$$

$$\beta_1 = b_1 b_2 \cdots b_{j_1},$$

$$\beta_2 = b_{j_1+1} b_{j_1+2} \cdots b_{j_2},$$

$$\vdots$$

$$\beta_s = b_{j_{s-1}+1} b_{j_{s-1}+2} \cdots b_m,$$

and $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r$ and $\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_s$ are the decompositions of a and b, respectively. Denote $\hat{b} = \hat{b}_1 \hat{b}_2 \cdots \hat{b}_m$ as the sequence of positive integers obtained by adding n to each element in b, *i.e.*, $\hat{b}_j = b_j + n$, $1 \le j \le m$. Then the atoms of $\hat{b}$ are

$$\hat{\beta}_1 = (b_1 + n)(b_2 + n) \cdots (b_{j_1} + n),$$

$$\hat{\beta}_2 = (b_{j_1+1} + n)(b_{j_1+2} + n) \cdots (b_{j_2} + n),$$

$$\vdots$$

$$\hat{\beta}_s = (b_{j_{s-1}+1} + n)(b_{j_{s-1}+2} + n) \cdots (b_m + n),$$

i.e., its decomposition is $\hat{\beta} = \hat{\beta}_1 \circ \hat{\beta}_2 \circ \cdots \circ \hat{\beta}_s$.

Next we define the coupling product on permutations in a recursive way.

Definition 3.1. Define a *coupling product* $\ltimes$ on $\mathbb{KS}$ by

1) $a \ltimes \epsilon = \epsilon \ltimes a = a$,

2)
$$\begin{aligned} a \ltimes b &= \alpha \ltimes \hat{\beta} \\ &= (\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r) \ltimes (\hat{\beta}_1 \circ \hat{\beta}_2 \circ \cdots \circ \hat{\beta}_s) \\ &= \alpha_1 \circ (\alpha_2 \circ \cdots \circ \alpha_r \ltimes \hat{\beta}) + \sum_{k=1}^{s} \hat{\beta}_k \left(\alpha_1 \circ \left(\alpha_2 \circ \cdots \circ \alpha_r \ltimes (\hat{\beta} \setminus \hat{\beta}_k)\right)\right), \end{aligned}$$

where $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r$ and $\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_s$ are the decompositions of permutations a and b, respectively, and $\hat{\beta}\setminus\hat{\beta}_k$ is obtained by eliminating the k-th atom $\hat{\beta}_k$ from $\hat{\beta}$. Define the *unit* $\mu:\mathbb{K}\to\mathbb{KS}$ by $\mu(1)=\epsilon$.

From the definition, the coupling product $\ltimes$ is non-commutative.

Example 3.2. Let $a=12$ and $b=3124$. Then $\alpha=\alpha_1\circ\alpha_2=1\circ 2$, $\beta=\beta_1\circ\beta_2\circ\beta_3=31\circ 2\circ 4$ and $\hat{\beta}=\hat{\beta}_1\circ\hat{\beta}_2\circ\hat{\beta}_3=53\circ 4\circ 6$. Then

$$
\begin{aligned}
& a\ltimes b \\
&= \alpha_1\circ\left(\alpha_2\ltimes\hat{\beta}\right)+\hat{\beta}_1\alpha_1\circ\left(\alpha_2\ltimes\hat{\beta}_2\circ\hat{\beta}_3\right)+\hat{\beta}_2\alpha_1\circ\left(\alpha_2\ltimes\hat{\beta}_1\circ\hat{\beta}_3\right) \\
&\quad +\hat{\beta}_3\alpha_1\circ\left(\alpha_2\ltimes\hat{\beta}_1\circ\hat{\beta}_2\right) \\
&= \alpha_1\circ\alpha_2\circ\left(\epsilon\bowtie\hat{\beta}\right)+\alpha_1\circ\left(\hat{\beta}_1\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_2\circ\hat{\beta}_3\right)\right. \\
&\quad \left.+\hat{\beta}_2\alpha_2\circ\left(\epsilon\bowtie\hat{\beta}_1\circ\hat{\beta}_3\right)+\hat{\beta}_3\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_1\circ\hat{\beta}_2\right)\right) \\
&\quad +\hat{\beta}_1\alpha_1\circ\left(\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_2\circ\hat{\beta}_3\right)+\hat{\beta}_2\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_3\right)+\hat{\beta}_3\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_2\right)\right) \\
&\quad +\hat{\beta}_2\alpha_1\circ\left(\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_1\circ\hat{\beta}_3\right)+\hat{\beta}_1\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_3\right)+\hat{\beta}_3\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_1\right)\right) \\
&\quad +\hat{\beta}_3\alpha_1\circ\left(\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_1\circ\hat{\beta}_2\right)+\hat{\beta}_1\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_2\right)+\hat{\beta}_2\alpha_2\circ\left(\epsilon\ltimes\hat{\beta}_1\right)\right) \\
&= \alpha_1\circ\alpha_2\circ\hat{\beta}_1\circ\hat{\beta}_2\circ\hat{\beta}_3+\alpha_1\circ\hat{\beta}_1\alpha_2\circ\hat{\beta}_2\circ\hat{\beta}_3 \\
&\quad +\alpha_1\circ\hat{\beta}_2\alpha_2\circ\hat{\beta}_1\circ\hat{\beta}_3+\alpha_1\circ\hat{\beta}_3\alpha_2\circ\hat{\beta}_1\circ\hat{\beta}_2 \\
&\quad +\hat{\beta}_1\alpha_1\circ\alpha_2\circ\hat{\beta}_2\circ\hat{\beta}_3+\hat{\beta}_1\alpha_1\circ\hat{\beta}_2\alpha_2\circ\hat{\beta}_3+\hat{\beta}_1\alpha_1\circ\hat{\beta}_3\alpha_2\circ\hat{\beta}_2 \\
&\quad +\hat{\beta}_2\alpha_1\circ\alpha_2\circ\hat{\beta}_1\circ\hat{\beta}_3+\hat{\beta}_2\alpha_1\circ\hat{\beta}_1\alpha_2\circ\hat{\beta}_3+\hat{\beta}_2\alpha_1\circ\hat{\beta}_3\alpha_2\circ\hat{\beta}_1 \\
&\quad +\hat{\beta}_3\alpha_1\circ\alpha_2\circ\hat{\beta}_1\circ\hat{\beta}_2+\hat{\beta}_3\alpha_1\circ\hat{\beta}_1\alpha_2\circ\hat{\beta}_2+\hat{\beta}_3\alpha_1\circ\hat{\beta}_2\alpha_2\circ\hat{\beta}_1 \\
&= 1\circ 2\circ 53\circ 4\circ 6+1\circ 532\circ 4\circ 6 \\
&\quad +1\circ 42\circ 53\circ 6+1\circ 62\circ 53\circ 4 \\
&\quad +531\circ 2\circ 4\circ 6+531\circ 42\circ 6+531\circ 62\circ 4 \\
&\quad +41\circ 2\circ 53\circ 6+41\circ 532\circ 6+41\circ 62\circ 53 \\
&\quad +61\circ 2\circ 53\circ 4+61\circ 532\circ 4+61\circ 42\circ 53.
\end{aligned}
$$

From the above example, permutations in the coupling product of $\alpha=\alpha_1\circ\alpha_2$ and $\beta=\beta_1\circ\beta_2\circ\beta_3$ must consist of some atoms from the set

$$\left\{\alpha_1,\alpha_2,\hat{\beta}_1,\hat{\beta}_2,\hat{\beta}_3,\hat{\beta}_1\alpha_1,\hat{\beta}_2\alpha_1,\hat{\beta}_3\alpha_1,\hat{\beta}_1\alpha_2,\hat{\beta}_2\alpha_2,\hat{\beta}_3\alpha_2\right\},$$

and each atom in α and $\hat{\beta}$ appears and appears only once. When such atoms are given, the first atom must contain α_1, the 2nd atom must contain α_2, then the remaining atoms with $\hat{\beta}_j$ are arranged in ascending order of its index j. For example, in $a\ltimes b$ the permutation consists of the atoms $\hat{\beta}_3,\hat{\beta}_2\alpha_1,\hat{\beta}_1\alpha_2$ is $\hat{\beta}_2\alpha_1\circ\hat{\beta}_1\alpha_2\circ\hat{\beta}_3$, and the permutation consists of the atoms $\alpha_2,\hat{\beta}_1,\hat{\beta}_3,\hat{\beta}_2\alpha_1$ is $\hat{\beta}_2\alpha_1\circ\alpha_2\circ\hat{\beta}_1\circ\hat{\beta}_3$. Furthermore, the subsets that satisfy that each atom in α and $\hat{\beta}$ appears and appears only once give all permutations in the coupling product of a and b. And these permutations are distinct and they appear and appear only once in the coupling product.

In general, when $a=a_1a_2\cdots a_n$ is a permutation of degree n and $b=b_1b_2\cdots b_m$ is a permutation of degree m with decompositions $\alpha=\alpha_1\circ\alpha_2\circ\cdots\circ\alpha_r$ and $\beta=\beta_1\circ\beta_2\circ\cdots\circ\beta_s$, respectively. Any permutation in $a\ltimes b$ must be given by a subset of $\{\alpha_i, \hat{\beta}_j, \hat{\beta}_j\alpha_i \mid i=1,2,\cdots,r; j=1,2,\cdots,s\}$, in which each atom in α and $\hat{\beta}$ appears and appears only once. Once the atoms are given, all atoms containing α_i are arranged in ascending order of its index i, then the remaining atoms $\hat{\beta}_j$ are arranged in ascending order of its index j. Furthermore, the subsets that satisfy that each atom in α and $\hat{\beta}$ appears and appears only once give all permutations in the coupling product of a and b. These permutations are distinct and they appear only once.

Thus, we have an equivalent definition of coupling product.

Definition 3.3. Define the *coupling product* $\ltimes$ on $\mathbb{KS}$ by

$$\begin{aligned} a \ltimes b &= \alpha \ltimes \hat{\beta} \\ &= (\alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r) \ltimes (\hat{\beta}_1 \circ \hat{\beta}_2 \circ \cdots \circ \hat{\beta}_s) \\ &= \sum_{L \subseteq [r]} \sum_{f: L \to [s]} \ddot{\alpha}_1 \circ \ddot{\alpha}_2 \circ \cdots \circ \ddot{\alpha}_r \circ \hat{\beta}_{[s] \setminus f(l)}, \end{aligned}$$

where $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r$ and $\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_s$ are the decompositions of permutations a and b, respectively, f is an injective map from L to $[s]$, and

$$\ddot{\alpha}_i = \begin{cases} \hat{\beta}_{f(i)}\alpha_i, & i \in L, \\ \alpha_i, & i \notin L. \end{cases}$$

Remark 3.4. We can also define the coupling product on sequences of distinct positive integers. Let a and b be sequences of distinct positive integers. Denote $a_{\max}$ as the largest element of a and add $a_{\max}$ to each element of b and obtain $\hat{b} = \hat{b}_1\hat{b}_2 \cdots \hat{b}_m$. Then the atoms of $\hat{b}$ are

$$\hat{\beta}_1 = (b_1 + a_{\max})(b_2 + a_{\max}) \cdots (b_{j_1} + a_{\max}),$$

$$\hat{\beta}_2 = (b_{j_1+1} + a_{\max})(b_{j_1+2} + a_{\max}) \cdots (b_{j_2} + a_{\max}),$$

$$\vdots$$

$$\hat{\beta}_s = (b_{j_{s-1}+1} + a_{\max})(b_{j_{s-1}+2} + a_{\max}) \cdots (b_m + a_{\max}),$$

and its decomposition is $\hat{\beta} = \hat{\beta}_1 \circ \hat{\beta}_2 \circ \cdots \circ \hat{\beta}_s$.

Theorem 3.5. $(\mathbb{KS}, \ltimes, \mu)$ is a graded algebra.

Proof: Let $a = a_1a_2 \cdots a_n$ be a permutation of degree n and $b = b_1b_2 \cdots b_m$ be a permutation of degree m with decompositions $\alpha = \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_r$ and $\beta = \beta_1 \circ \beta_2 \circ \cdots \circ \beta_s$, respectively. We denote $\hat{b} = \hat{b}_1\hat{b}_2 \cdots \hat{b}_m$ as the sequence of positive integers obtained by adding n to each element in b, *i.e.*, $\hat{b}_j = b_j + n$, $1 \le j \le m$. Then denote $\hat{\beta} = \hat{\beta}_1 \circ \hat{\beta}_2 \circ \cdots \circ \hat{\beta}_s$ as the decomposition of $\hat{b}$. From above, any permutation in $a \ltimes b$ must be given by a subset of

$$\{\alpha_i, \hat{\beta}_j, \hat{\beta}_j\alpha_i \mid i=1,2,\cdots,r; j=1,2,\cdots,s\},$$

where each atom in α and $\hat{\beta}$ appears and appears only once. Suppose c is a permutation of degree l with $W_c = \{k_1, k_2, \cdots, k_t\}$. Then the atoms of c are

$$\gamma_1 = c_1c_2 \cdots c_{k_1},$$

$$\gamma_2 = c_{k_1+1}c_{k_1+2}\cdots c_{k_2},$$

$$\vdots$$

$$\gamma_t = c_{k_{t-1}+1}c_{k_{t-1}+2}\cdots c_l,$$

and the decomposition of c is $\gamma = \gamma_1 \circ \gamma_2 \circ \cdots \circ \gamma_t$. Denote $\tilde{c}$ as the sequence of positive integers obtained by adding $n+m$ to each element in c. Then the atoms of $\tilde{c}$ are

$$\tilde{\gamma}_1 = (c_1+n+m)(c_2+n+m)\cdots(c_{k_1}+n+m),$$

$$\tilde{\gamma}_2 = (c_{k_1+1}+n+m)(c_{k_1+2}+n+m)\cdots(c_{k_2}+n+m),$$

$$\vdots$$

$$\tilde{\gamma}_t = (c_{k_{t-1}+1}+n+m)(c_{k_{t-1}+2}+n+m)\cdots(c_l+n+m),$$

and the decomposition of $\tilde{c}$ is $\tilde{\gamma} = \tilde{\gamma}_1 \circ \tilde{\gamma}_2 \circ \cdots \circ \tilde{\gamma}_t$. Any permutation in $(a \ltimes b) \ltimes c$ must be given by a subset of

$$A = \left\{\alpha_i, \hat{\beta}_j, \tilde{\gamma}_k, \hat{\beta}_j\alpha_i, \tilde{\gamma}_k\alpha_i, \tilde{\gamma}_k\hat{\beta}_j, \tilde{\gamma}_k\hat{\beta}_j\alpha_i \mid i=1,2,\cdots,r;\ j=1,2,\cdots,s; k=1,2,\cdots,t\right\},$$

and each atom in α, $\hat{\beta}$ and $\tilde{\gamma}$ appears and appears only once.

Denote $\breve{c} = \breve{c}_1\breve{c}_2\cdots\breve{c}_l$ as the sequence of positive integers obtained by adding m to each element in c, *i.e.*, $\breve{c}_i = c_i + m$, $1 \le i \le l$. Then the atoms of $\breve{c}$ are

$$\breve{\gamma}_1 = (c_1+m)(c_2+m)\cdots(c_{k_1}+m),$$

$$\breve{\gamma}_2 = (c_{k_1+1}+m)(c_{k_1+2}+m)\cdots(c_{k_2}+m),$$

$$\vdots$$

$$\breve{\gamma}_t = (c_{k_{t-1}+1}+m)(c_{k_{t-1}+2}+m)\cdots(c_l+m),$$

and the decomposition of $\breve{c}$ is $\breve{\gamma} = \breve{\gamma}_1 \circ \breve{\gamma}_2 \circ \cdots \circ \breve{\gamma}_t$. Denote $\bar{\breve{c}}$ as the sequence of positive integers obtained by adding n to each element in $\breve{c}$. Then the atoms of $\bar{\breve{c}}$ are

$$\bar{\breve{\gamma}}_1 = (\breve{c}_1+n)(\breve{c}_2+n)\cdots(\breve{c}_{k_1}+n),$$

$$\bar{\breve{\gamma}}_2 = (\breve{c}_{k_1+1}+n)(\breve{c}_{k_1+2}+n)\cdots(\breve{c}_{k_2}+n),$$

$$\vdots$$

$$\bar{\breve{\gamma}}_t = (\breve{c}_{k_{t-1}+1}+n)(\breve{c}_{k_{t-1}+2}+n)\cdots(\breve{c}_l+n),$$

and the decomposition of $\bar{\breve{c}}$ is $\bar{\breve{\gamma}} = \bar{\breve{\gamma}}_1 \circ \bar{\breve{\gamma}}_2 \circ \cdots \circ \bar{\breve{\gamma}}_t$.

Any permutation in $b \ltimes c$ must be given by a subset of

$$\left\{\beta_j, \breve{\gamma}_k, \breve{\gamma}_k\beta_j \mid j=1,2,\cdots,s; k=1,2,\cdots,t\right\},$$

where each atom in β and $\breve{\gamma}$ appears and appears only once. Any permutation in $a \ltimes (b \ltimes c)$ must be given by a subset of

$$B = \left\{\alpha_i, \hat{\beta}_j, \bar{\breve{\gamma}}_k, \hat{\beta}_j\alpha_i, \bar{\breve{\gamma}}_k\alpha_i, \bar{\breve{\gamma}}_k\hat{\beta}_j, \bar{\breve{\gamma}}_k\hat{\beta}_j\alpha_i \mid i=1,2,\cdots,r;\ j=1,2,\cdots,s; k=1,2,\cdots,t\right\},$$

where each atom in α, $\hat{\beta}$ and $\bar{\breve{\gamma}}$ appears and appears only once.

From above, $\tilde{\gamma} = \overline{\overline{\gamma}}$ since $A = B$. Hence, $(a \ltimes b) \ltimes c = a \ltimes (b \ltimes c)$, for any permutations a, b and c. It is easy to verify that μ is a unit. So $(\mathbb{KS}, \ltimes, \mu)$ is an algebra. Obviously, $\mathbb{K}S_n \ltimes \mathbb{K}S_m \subseteq \mathbb{K}S_{n+m}$. Thus, $(\mathbb{KS}, \ltimes, \mu)$ is a graded algebra. □

4. Conclusion

In this paper, we define a new product operation on permutations, which is called the coupling product $\ltimes$. Then, we prove that the vector space spanned by permutations with the coupling product is a graded algebra.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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