

# The Coxeter Trisection and the Hadwiger Conjecture in Multidimensional Spaces

István Lénárt

Eötvös Loránd University, Budapest, Hungary

Email: h12572len@ella.hu

**How to cite this paper:** Lénárt, I. (2024) The Coxeter Trisection and the Hadwiger Conjecture in Multidimensional Spaces. *Journal of Applied Mathematics and Physics*, 12, 4301-4321.

<https://doi.org/10.4236/jamp.2024.1212264>

**Received:** October 17, 2024

**Accepted:** December 28, 2024

**Published:** December 31, 2024

Copyright © 2024 by author(s) and Scientific Research Publishing Inc. This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).

<http://creativecommons.org/licenses/by/4.0/>



Open Access

## Abstract

In this article, I consider the right triangle as the simplex in the Euclidean plane, and extend this definition to higher dimensions. The  $n$ -dimensional simplex has one hypotenuse and  $(n-1)$  legs (catheti). The  $(n-1)$  legs define an orthogonal path of edges in the solid with perpendicular adjacent edges along the path. The length of the hypotenuse and the volume of the solid can be calculated without the Cayley-Menger determinant, by direct extension of the corresponding right triangle formulas. I give a proof of the existence of these shapes, describe the distribution of right angles in them, give an algebraic proof of the Coxeter trisection of a right tetrahedron into three smaller right tetrahedra, and generalize this construction to  $n$ -dimensional spaces. Finally, I investigate the connection between the Coxeter partition and the Hadwiger conjecture on the partition of the simplex into orthoschemes, which I call Pythagorean simplexes.

## Keywords

Generalized Pythagoras Theorem, Description of a Pythagorean Simplex, Pythagorean Unit Simplex, Coxeter Partition of a Simplex in  $n$ -Dimensional Space, Relation to the Hadwiger Conjecture

## 1. Pythagorean Simplexes in $n$ -Dimensional Euclidean Spaces

### 1.1. Definition of a Pythagorean Simplex

The literature on higher-dimensional geometry usually treats the general triangle as a 2D simplex and the general tetrahedron as a 3D simplex. Since the mid-19<sup>th</sup> century, however, researchers have recognized the importance of the 2D right triangle and the 3D quadrirectangular tetrahedron as possible alternatives to the definition of the simplex [1].

I came to this idea through spherical geometry. The coincidence of a great circle with a point, and the perpendicularity of the great circle and the polar of the point are equivalent and interchangeable statements on the sphere. The sequence of elements “hypotenuse-vertex-leg-leg-vertex” in a right triangle represents a closed 5-cycle of incidence, because the two legs are perpendicular, that is, incident, and the vertex at the meeting point of the two legs can be omitted from the cycle. The spherical Napier pentagram is also a 5-cycle with five perpendicular adjacent sides, in contrast to the 6-cycle of “vertex-side-vertex-side-vertex-side” cycle in the general spherical triangle. Similarly, a hyperbolic Napier pentagon can be constructed from five hyperbolic segments [2].

It is, therefore, reasonable to consider the 5-cycle as the simplex in any 2D geometry, and the 6-cycle as a composite element derived from 5-cycles. Specifically, the right triangle can replace the general triangle in the definition of a simplex in the Euclidean plane.

This concept is extended to any dimension in the following way.

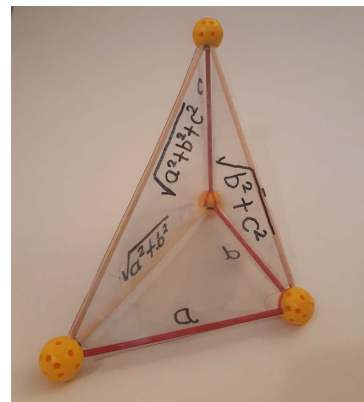
Consider an ordered chain of straight line segments  $a_1, a_2, \dots, a_{n-1}$  in an  $n$ -dimensional Euclidean space where the starting point of segment  $a_i$  ( $i > 1$ ) coincides with the endpoint of segment  $a_{i-1}$ , and segment  $a_i$  is perpendicular to the  $(i-1)$ -dimensional space formed by the preceding segments  $a_1, a_2, \dots, a_{i-1}$ .

Define segments  $a_1, a_2, \dots, a_{n-1}$  the **legs** (catheti) of the solid, and segment  $a_n$  the **hypotenuse** connecting the endpoint of segment  $a_{n-1}$  with the starting point of segment  $a_1$ .

The extension of the Pythagorean Theorem applies to this  $n$ -dimensional shape, namely,  $a_n^2 = a_1^2 + a_2^2 + \dots + a_{n-1}^2 = \sum_1^{n-1} a_i^2$  for the hypotenuse, and

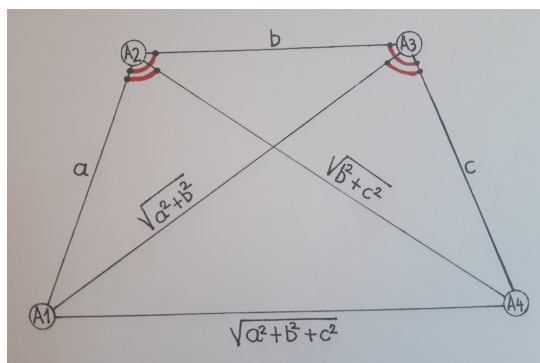
$$V = \frac{1}{n!} a_1 a_2 a_3 \dots a_{n-1} = \frac{\prod_1^{n-1} a_i}{n!}$$
 for the volume [3].

In 3D space, this shape is a tetrahedron, each face of which is a right triangle (**Figure 1**).



**Figure 1.** A Pythagorean tetrahedron, every face of which is a right triangle.

In this article, the Pythagorean tetrahedron is displayed on the plane, as seen in **Figure 2** (See explanation of marking in 1.2.).



**Figure 2.** A Pythagorean tetrahedron as displayed in this article.

The  $n$ -dimensional shape has been called by various names, such as a pyramid, an orthoscheme, Orthogonalsimplex, or in the 3D case, a quadrirectangular tetrahedron.

I propose the name a Pythagorean  $n$ -simplex for the  $n$ -dimensional shape with  $(n-1)$  legs and one hypotenuse. **Figure 1** and **Figure 2** show a Pythagorean tetrahedron, a Pythagorean 3D simplex.

In this paper, I shorten the name to a Pyth simplex, as for example, a Pyth triangle (right triangle), Pyth tetrahedraon (quadrirectangular tetrahedron), Pyth pentachoron, etc.

## 1.2. Representing Multidimensional Objects

Several methods have been developed to visualize multidimensional objects on the flat sheet. (See for example [4] or [5].)

I found the traditional right angle symbols confusing, so I chose a different marking method. A right angle is denoted here as an arc in the angle region terminating with two dots on the corresponding sides, regardless of the apparent length of the arc or the apparent (mostly non-perpendicular) position of the lines in the drawing.

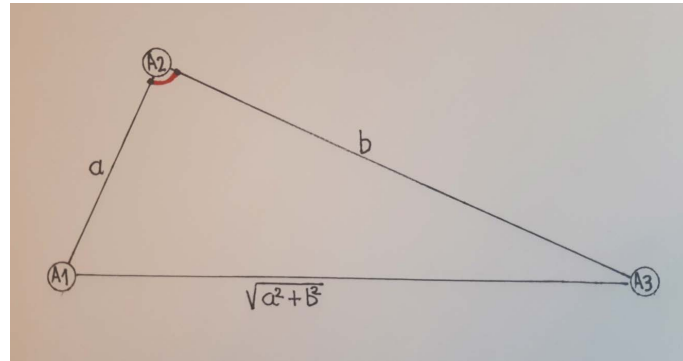
The points of intersection that really exist in the  $n$ -dimensional solid are only those which are marked with letters/numbers in the figure (as in graph theory). Lines that intersect in the flat figure without marking the point of intersection are skew lines, as  $A_1A_3$  and  $A_2A_4$  in **Figure 2**.

To better understand the essence of the geometric structure, the actual length of the segment and the measure of the angle are distorted in most of the drawings. Therefore, I ask the Reader to rely on the notation and legend of the lines and angles, rather than the apparent measure and position of the visual image.

## 2. Existence and Description of Pythagorean Simplexes

### 2.1. The Pyth Simplex in the 2D Euclidean Plane

The Euclidean right triangle has two freely given data as the length of the two legs. The hypotenuse is calculated by the Pythagorean Theorem. The measure of the angles and the sides are correct on the picture (**Figure 3**).



**Figure 3.** The Euclidean right triangle in 2D.

## 2.2. The Pyth Simplex in the 3D Euclidean Space

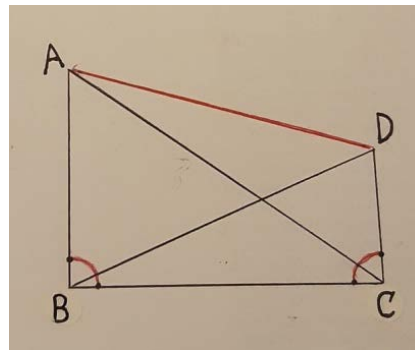
Construct a tetrahedron whose four faces are right triangles. The possible distribution of four right angles at the vertices are: 1) One right angle at each vertex; 2) Two right angles at two vertices, and two other vertices with no right angles; 3) Two right angles at one vertex, one right angle at two other vertices respectively, and the fourth vertex without a right angle; 4) Three right angles at one vertex, a fourth right angle at another vertex, and two vertices without right angles.

Simple geometric reasoning rules out all these options except for the second one with two right angles at two vertices each. How can we prove that this option does exist?

Consider two Pyth triangles (right triangles)  $ABC$  and  $BCD$ , with one common leg  $BC$ , located in two different planes in 3D space. Connect vertices  $A$  and  $D$ , and get a tetrahedron. Let  $AB = a$ ,  $BC = b$ ,  $CD = c$ .

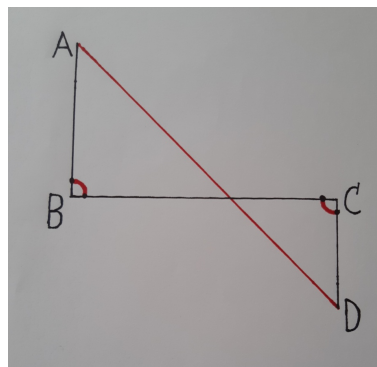
Rotate the two planes around the common axis  $BC$ .

If the dihedral angle between the two planes is equal to  $0^\circ$ , the two planes degenerate into one, and the distance of vertices  $A$  and  $D$ ,  $\sqrt{(a-c)^2 + b^2}$  is minimal (**Figure 4**).



**Figure 4.** Dihedral angle  $0^\circ$ ,  $AD = \sqrt{(a-c)^2 + b^2}$  in 2D.

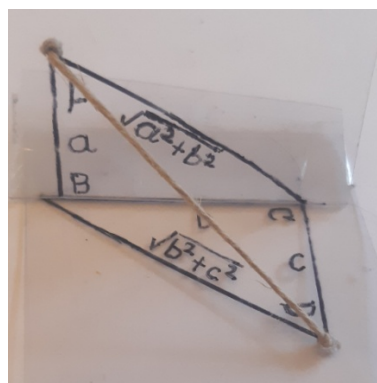
If the dihedral angle between the two planes is equal to  $180^\circ$ , then again, the two planes degenerate into one, and the distance of vertices  $A$  and  $D$ ,  $\sqrt{(a+c)^2 + b^2}$  is maximal (**Figure 5**).



**Figure 5.** Dihedral angle  $180^\circ$ ,  $AD = \sqrt{(a+c)^2 + b^2}$  in 2D.

The change between the two extremes is strictly monotonous during the rotation. There must be a position in the 3D space where the distance between  $A$  and  $D$  is equal to  $\sqrt{a^2 + b^2 + c^2}$  (**Figure 6**).

$$\sqrt{(a-c)^2 + b^2} < \sqrt{a^2 + b^2 + c^2} < \sqrt{(a+c)^2 + b^2}$$



**Figure 6.** Dihedral angle  $90^\circ$ ,  $AD = \sqrt{a^2 + b^2 + c^2}$  in 3D.

It follows from the Pythagorean Theorem that angles  $\angle ABD$  and  $\angle ACD$  are right angles in the position of **Figure 5**, and planes  $(ABC)$  and  $(BCD)$  are perpendicular to each other. This is a Pyth tetrahedron all of whose faces are Pyth triangles. The right angles are at vertices  $B$  and  $C$ . The legs are  $AB = a$ ,  $BC = b$ ,  $CD = c$ ; the hypotenuse is  $AD = \sqrt{a^2 + b^2 + c^2}$ . The volume of this shape is  $V = \frac{abc}{3!}$ .

### 2.3. Generalize the 3D Result to 2D Pyth Simplexes (Right Triangles)

Consider two straight line segments  $AB$  and  $BC$ , with one common point  $B$ , located in a 2D plane. Connect vertices  $A$  and  $C$ , and get a general planar triangle. Let  $AB = a$ ,  $BC = b$ .

Rotate the two segments in the 2D plane around the common point  $B$ .

If the angle of the two segments is equal to  $0^\circ$ , the two segments fall into a

straight line of one dimension, and the distance of vertices  $A$  and  $C$ ,  $\sqrt{(a-b)^2} = \sqrt{a^2+b^2-2ab}$  is minimal.

If the angle of the two segments is equal to  $180^\circ$ , then again, the two segments fall into a 1D straight line, and the distance of vertices  $A$  and  $C$ ,  $\sqrt{(a+b)^2} = \sqrt{a^2+b^2+2ab}$  is maximal.

The change between the two extremes is strictly monotonous. There must be a position in 2D where the distance between  $A$  and  $C$  is equal to  $\sqrt{a^2+b^2}$ .

$$\sqrt{(a-b)^2} < \sqrt{a^2+b^2} < \sqrt{(a+b)^2}$$

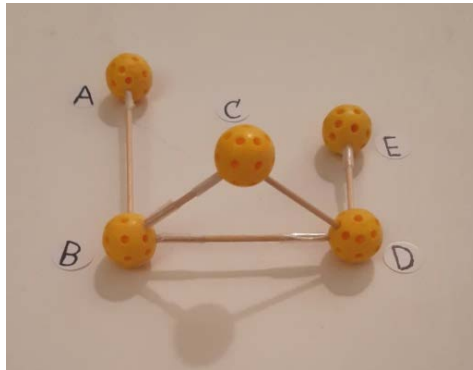
It follows from the Pythagorean Theorem that angle  $\angle ABC$  is a right angle in this case. This is a Pyth triangle with the right angle at vertex  $B$ . The legs are  $AB = a$  and  $BC = b$ , the hypotenuse is  $AC = \sqrt{a^2+b^2}$ . The volume (that is, the area) of this shape is  $V = \frac{ab}{2!}$ .

### 2.4. Generalize the 3D Result to 4D Pyth Simplexes (Pyth Pentachorons)

Consider two Pyth tetrahedra  $(ABCD)$  and  $(BCDE)$ , with one common Pyth triangle  $(BCD)$ , located in 4D Euclidean space. Connect vertices  $A$  and  $E$ , and get a pentachoron. Let  $AB = a$ ,  $BC = b$ ,  $CD = c$ ,  $DE = d$ .

Rotate the two tetrahedra around the common Pyth triangle  $(BCD)$  in 4D space.

If the dihedral 4D angle is equal to  $0^\circ$ , the two tetrahedra belong to the same 3D space, and the distance of vertices  $A$  and  $E$ ,  $\sqrt{(a-d)^2+b^2+c^2}$  is minimal (Figure 7).



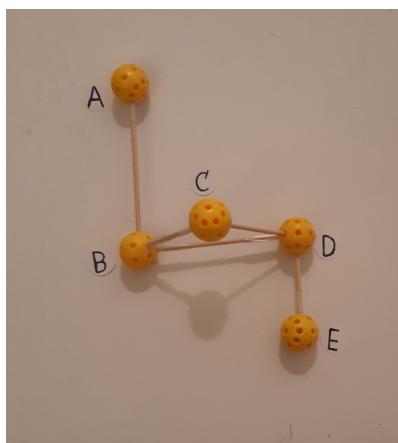
**Figure 7.** Dihedral angle  $0^\circ$ ,  $AE = \sqrt{(a-d)^2+b^2+c^2+d^2}$  in 3D; Right angles  $\angle ABC$ ,  $\angle BCD$ ,  $\angle CDE$ ,  $\angle ABD$ ,  $\angle BDE$ .

If the dihedral 4D angle is equal to  $180^\circ$ , then again, the two tetrahedra belong to the same 3D space, and the distance of vertices  $A$  and  $E$ ,  $\sqrt{(a+d)^2+b^2+c^2}$  is maximal (Figure 8).

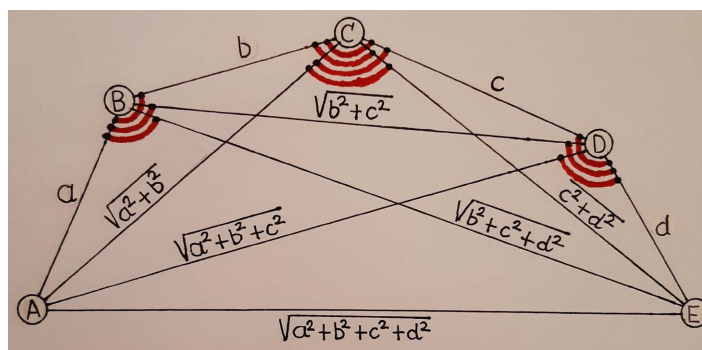
The change between the two extremes is strictly monotonous. There must be a position where the distance between  $A$  and  $E$  is equal to  $\sqrt{a^2+b^2+c^2+d^2}$

(Figure 9).

$$\sqrt{(a-d)^2 + b^2 + c^2} < \sqrt{a^2 + b^2 + c^2 + d^2} < \sqrt{(a+d)^2 + b^2 + c^2}$$



**Figure 8.** Dihedral angle  $180^\circ$ ,  $AE = \sqrt{(a+d)^2 + b^2 + c^2 + d^2}$  in 3D. Right angles  $\angle ABC$ ,  $\angle BCD$ ,  $\angle CDE$ ,  $\angle ABD$ ,  $\angle BDE$ .



**Figure 9.** Dihedral angle  $90^\circ$ ,  $AE = \sqrt{a^2 + b^2 + c^2 + d^2}$  in 4D. Right angles are marked by arcs between dots.

Only the two 3D cases can directly be displayed by 3D models in **Figure 7** and **Figure 8**. The 4D Pyth pentachoron is displayed here as a non-regular pentagon in **Figure 9**.

It follows from the generalized Pythagorean Theorem that the solid in **Figure 9** is a Pyth pentachoron with ten faces, each of which is a Pyth triangle. The ten right angles are at vertices  $B, C, D$ . The legs are  $AB = a$ ,  $BC = b$ ,  $CD = c$ ,  $DE = d$ . The hypotenuse is  $AE = \sqrt{a^2 + b^2 + c^2 + d^2}$ . The volume of this shape is  $V = \frac{abcd}{4!}$ .

Further generalizations can be achieved in the same manner, by induction from  $(n-1)$  to  $n$  dimensions.

$$\begin{aligned} \sqrt{(a_1 - a_{n-1})^2 + a_2^2 + \dots + a_{n-2}^2} &< \sqrt{a_1^2 + a_2^2 + \dots + a_{n-2}^2 + a_{n-1}^2} \\ &< \sqrt{(a_1 + a_{n-1})^2 + a_2^2 + \dots + a_{n-2}^2} \end{aligned}$$

### 2.5. Distribution of Right Angles in a Pyth $n$ -Simplex

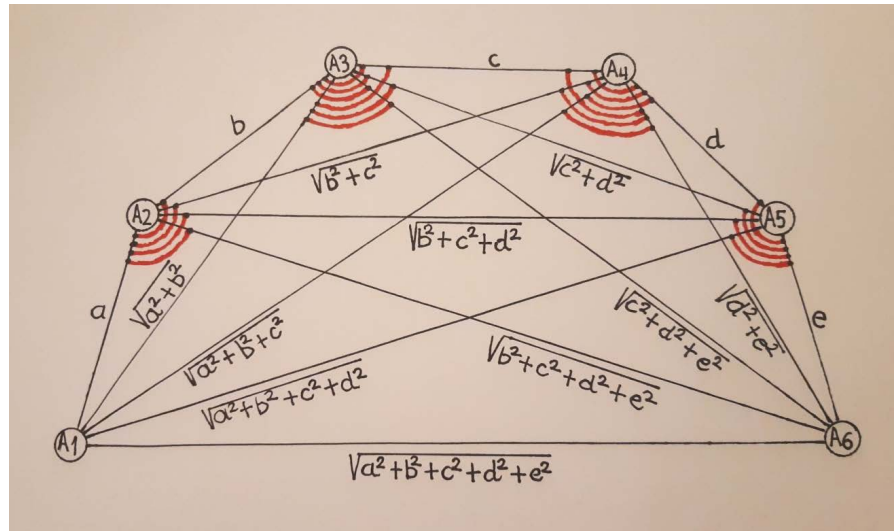


Figure 10. Pyth simplex in 5D.

Figure 10 shows the distribution of right angles in 5D (See also Figure 2, Figure 3, Figure 8 for the distribution in lower dimensions).

The distribution of the right angles in the respective simplexes is displayed in Table 1:

Table 1. Distribution of right angles in a Pyth  $n$ -simplex.

|    |     |     |     |     |     |     |     |         |                                  |
|----|-----|-----|-----|-----|-----|-----|-----|---------|----------------------------------|
| 2D |     |     | 1   |     |     |     |     | $s_2 =$ | 1                                |
| 3D |     |     | 2   | 2   |     |     |     | $s_3 =$ | $4 = 1 + (1 + 2)$                |
| 4D |     | 3   | 4   | 3   |     |     |     | $s_4 =$ | $10 = 1 + (1 + 2) + (1 + 2 + 3)$ |
| 5D |     | 4   | 6   | 6   | 4   |     |     | $s_5 =$ | 20                               |
| 6D | 5   | 8   | 9   | 8   | 5   |     |     | $s_6 =$ | 35                               |
| 7D | 6   | 10  | 12  | 12  | 10  | 6   |     | $s_7 =$ | 56                               |
| 8D | 7   | 12  | 15  | 16  | 15  | 12  | 7   | $s_8 =$ | 84                               |
|    | ... | ... | ... | ... | ... | ... | ... | ...     | ...                              |

$$(1) + (1+2) + (1+2+3) + \dots + (1+2+3+\dots+n)$$

$$= (n-0) \cdot 1 + (n-1) \cdot 2 + (n-2) \cdot 3 + \dots + (n-(n-1)) \cdot n$$

$$s_n = s_{n-1} + \frac{(n-1)n}{2}$$

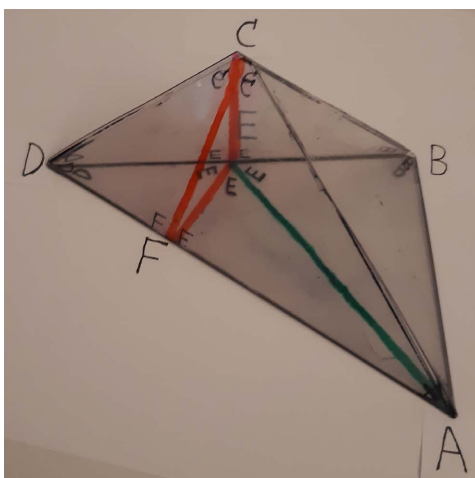
### 3. The Coxeter Partition in 3D Space

#### 3.1. The Coxeter Trisection of a 3D Pyth Simplex (Pyth Tetrahedron)

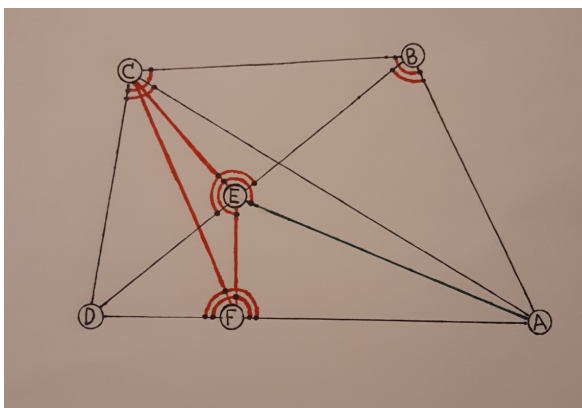
Consider a Pyth tetrahedron  $(ABCD)$  [6]. Drop a perpendicular from vertex  $C$  to edge  $(BD)$ , mark the foot as point  $E$ . Erect a perpendicular from  $E$  to

edge  $(AD)$ , mark the foot as point  $F$ . Connect  $F$  and  $C$ , and get a tetrahedron  $(CDEF)$ . Connect  $E$  and  $A$ , and get two further tetrahedra  $(ABCE)$  and  $(ACEF)$  (Figure 11).

In this paper, the Pyth tetrahedron is displayed on a flat surface, as shown in Figure 12.



**Figure 11.** The Coxeter trisection of a Pyth tetrahedron  $(ABCD)$ , divided into three smaller Pyth tetrahedra  $(ABCE)$ ,  $(ACEF)$ ,  $(CDEF)$ .



**Figure 12.** The Coxeter trisection of a Pyth tetrahedron  $(ABCD)$  divided into  $(ABCE)$ ,  $(ACEF)$ ,  $(CDEF)$ , displayed on a flat surface.

The triangle  $(CEF)$  is a right triangle with the right angle at vertex  $(E)$ , because the plane  $(ABD)$  is perpendicular to the plane  $(BCD)$  in the original Pyth tetrahedron  $(ABCD)$ . Therefore, the line  $(CE)$  in the plane  $(BCD)$  is perpendicular to the line  $(EF)$  in the plane  $(ABD)$ . This gives that the three smaller tetrahedra  $(ABCE)$ ,  $(ACEF)$ ,  $(CDEF)$  are also Pyth 3D simplexes, because each of their faces is a right triangle.

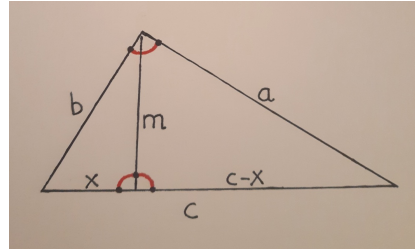
### 3.2. Theorem 1

Prove that the sum of the volumes of the three smaller tetrahedra  $(ABCE)$ ,

$(ACEF)$ ,  $(CDEF)$  is equal to the volume of tetrahedron  $(ABCD)$ .

**Lemma 1:**

The length of the non-trivial altitude  $m$  of a 2D right triangle with legs  $a, b$  is equal to  $\frac{ab}{\sqrt{a^2+b^2}}$  (Figure 13).



**Figure 13.** Right triangle with a non-trivial altitude.

Proof:

$$m^2 = a^2 - (c-x)^2 = b^2 - x^2 = a^2 - c^2 - x^2 + 2cx$$

$$x^2 = \frac{b^4}{c^2} = \frac{b^4}{a^2+b^2}$$

$$m^2 = b^2 - \frac{b^4}{a^2+b^2} = \frac{a^2b^2}{a^2+b^2}$$

$$m = \frac{ab}{\sqrt{a^2+b^2}}$$

**Proof of Theorem 1:**

Let  $(AB) = a$ ,  $(BC) = b$ ,  $(CD) = c$ ,  
then  $(AC) = \sqrt{a^2+b^2}$ ;  $(BD) = \sqrt{b^2+c^2}$ ;  $(AD) = \sqrt{a^2+b^2+c^2}$ .

$(CE) = \frac{bc}{\sqrt{b^2+c^2}}$ , because  $(CE)$  is the non-trivial altitude of right triangle

$(BCD)$ , with legs  $b$  and  $c$ .

$(CF) = \frac{c\sqrt{a^2+b^2}}{\sqrt{a^2+b^2+c^2}}$ , because  $(CF)$  is the non-trivial altitude of right triangle

$(ACD)$ , with legs  $\sqrt{a^2+b^2}$  and  $c$ .

Consequently,

$$(EF)^2 = (CF)^2 - (CE)^2 = \frac{(a^2+b^2)c^2}{a^2+b^2+c^2} - \frac{b^2c^2}{b^2+c^2} = \frac{a^2c^4}{(a^2+b^2+c^2)(b^2+c^2)}; \text{ so:}$$

$$(EF) = \frac{ac^2}{\sqrt{b^2+c^2}\sqrt{a^2+b^2+c^2}}$$

$$(FD)^2 = (CD)^2 - (FC)^2 = c^2 - \frac{(a^2+b^2)c^2}{a^2+b^2+c^2} = \frac{c^4}{a^2+b^2+c^2}; \text{ so:}$$

$$(FD) = \frac{c^2}{\sqrt{a^2+b^2+c^2}}$$

$$(AF) = (AD) - (FD) = \sqrt{a^2 + b^2 + c^2} - \frac{c^2}{\sqrt{a^2 + b^2 + c^2}} = \frac{a^2 + b^2}{\sqrt{a^2 + b^2 + c^2}}$$

The volumes of the three smaller tetrahedra:

$$\frac{1}{6}(AB)(BE)(EC) = \frac{1}{6}a \frac{b^2}{\sqrt{b^2 + c^2}} \frac{bc}{\sqrt{b^2 + c^2}} = \frac{1}{6} \frac{ab^3c}{b^2 + c^2}$$

$$\begin{aligned} \frac{1}{6}(CE)(EF)(FD) &= \frac{1}{6} \frac{bc}{\sqrt{b^2 + c^2}} \frac{ac^2}{\sqrt{b^2 + c^2} \sqrt{a^2 + b^2 + c^2}} \frac{c^2}{\sqrt{a^2 + b^2 + c^2}} \\ &= \frac{1}{6} \frac{abc^5}{(b^2 + c^2)(a^2 + b^2 + c^2)} \end{aligned}$$

$$\begin{aligned} \frac{1}{6}(AF)(FE)(EC) &= \frac{1}{6} \frac{a^2 + b^2}{\sqrt{a^2 + b^2 + c^2}} \frac{ac^2}{\sqrt{b^2 + c^2} \sqrt{a^2 + b^2 + c^2}} \frac{bc}{\sqrt{b^2 + c^2}} \\ &= \frac{1}{6} \frac{(a^2 + b^2)abc^3}{(b^2 + c^2)(a^2 + b^2 + c^2)} \end{aligned}$$

The sum of the three volumes adds up to the volume of the original tetrahedron:

$$\begin{aligned} &\frac{1}{6} \left( \frac{ab^3c}{b^2 + c^2} + \frac{abc^5}{(b^2 + c^2)(a^2 + b^2 + c^2)} + \frac{(a^2 + b^2)abc^3}{(b^2 + c^2)(a^2 + b^2 + c^2)} \right) \\ &= \frac{1}{6} abc \frac{b^2(a^2 + b^2 + c^2) + c^4 + (a^2 + b^2)c^2}{(b^2 + c^2)(a^2 + b^2 + c^2)} \\ &= \frac{1}{6} abc \frac{a^2b^2 + b^4 + b^2c^2 + c^4 + a^2c^2 + b^2c^2}{(b^2 + c^2)(a^2 + b^2 + c^2)} \\ &= \frac{1}{6} abc \frac{(b^2 + c^2)(a^2 + b^2 + c^2)}{(b^2 + c^2)(a^2 + b^2 + c^2)} = \frac{1}{6} abc \end{aligned}$$

Q.E.D.

## 4. Generalizing the Coxeter Division to $n$ -Dimensional Spaces

### 4.1. The 2D Case

This has already been settled, because Lemma 1 about the non-trivial altitude of a right triangle represents the Coxeter partition of a 2D Pyth simplex. The Pyth triangle with one right angle is divided into two smaller Pyth triangles by the non-trivial altitude (Figure 18). The volumes (areas) of the two smaller triangles add up to the volume of the original triangle.

$$\frac{1}{2!} [xm + (c - x)m] = \frac{1}{2!} \left( \frac{ab}{\sqrt{a^2 + b^2}} \frac{a^2}{\sqrt{a^2 + b^2}} + \frac{ab}{\sqrt{a^2 + b^2}} \frac{b^2}{\sqrt{a^2 + b^2}} \right) = \frac{1}{2!} ab$$

### 4.2. Into How Many Parts Does the Coxeter Partition Divide a Pyth $n$ -Simplex?

As shown above, the Coxeter partition divides a 2D Pyth simplex into 2 smaller

2D Pyth simplexes, and the 3D Pyth simplex into 3 smaller 3D Pyth simplexes. These results suggest that the extension of the Coxeter division to 4D yields 4 smaller 4D Pyth simplexes (Pyth pentachorons), it gives 5 smaller 5D Pyth simplexes in 5D, etc., and we get  $n$  pieces of simplexes in  $n$  dimensions.

The  $n = 1$  case can also be interpreted as an element of this series. Consider a degenerate 1D simplex, a segment viewed as a degenerate right triangle with zero right angle. The hypotenuse coincides with the leg. The Coxeter “division” yields one degenerate 1D triangle, that is, the segment itself.

### 4.3. Simplify the Proof by Introducing Unit Simplexes

The above computation about the 3D case with data  $(AB) = a$ ;  $(BC) = b$ ;  $(CD) = c$  is quite complex and difficult to extend. Even the 2D case is far from trivial. This is why I chose a simpler way to proceed.

Suppose that each leg in the 3D Pyth simplex is equal to 1, that is,  $(AB) = (BC) = (CD) = 1$ . This case is illustrated by the tetrahedron in **Figure 1**.

Define an  $n$ -dimensional unit Pyth simplex as an  $n$ -dimensional Pyth simplex in which the length of each leg is equal to 1 unit of distance, and the hypotenuse is equal to  $\sqrt{n}$  unit of distance.

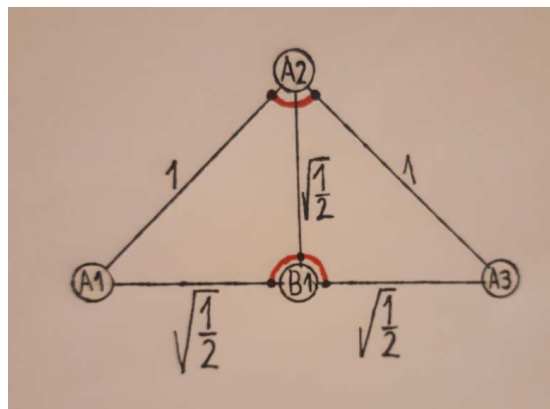
This convention makes calculations much easier. In Chapters 4 and 5 of this paper, I will discuss the partition of unit Pyth simplexes.

### 4.4. The 1D Case is the Unit Segment Interpreted as a Degenerate Right Triangle

$$V(A_2A_1) = \frac{1}{1!} \sqrt[1]{1} = \frac{1!}{(1!)^2} = \frac{1}{1!}.$$

### 4.5. The 2D Case: The 2D Unit Pyth Simplex Partitioned in Two Non-Unit Simplexes

As seen in **Figure 14**, drop a perpendicular from vertex  $A_2$  to line  $A_1A_3$  in the unit triangle. Mark the foot  $B_1$ .



**Figure 14.** The 2D unit Pyth simplex divided into two non-unit simplexes by the Coxeter division.

The respective volumes of the smaller simplexes are as follows:

$$V_1(A_2B_1A_1) = \frac{1}{2!} \sqrt{\frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{4}$$

$$V_2(A_2B_1A_3) = \frac{1}{2!} \sqrt{\frac{1}{2} \cdot \frac{1}{2}} = \frac{1}{4}$$

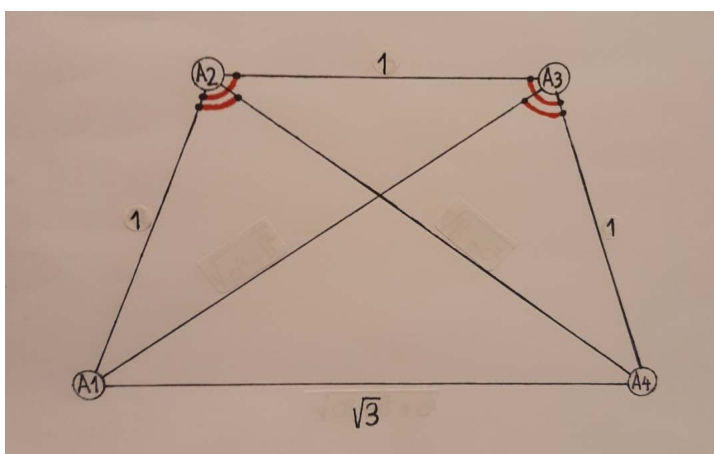
The volumes of the two smaller Pyth 2D simplexes (Pyth triangles) add up to the volume of the original unit Pyth triangle:

$$\begin{aligned} V(A_2A_3A_1) &= \frac{1}{2!} \sqrt{\frac{1}{1} \cdot \frac{1}{1}} = \frac{1}{2} \\ &= V_1(A_2B_1A_1) + V_2(A_2B_1A_3) \\ &= \frac{1}{4} + \frac{1}{4} = \frac{2}{4} = \frac{2!}{(2!)^2} = \frac{1}{2!}. \end{aligned}$$

This result completes the proof of the Coxeter partition in 2D space.

#### 4.6. The 3D Case: The 3D Unit Pyth Simplex Partitioned in Three Non-Unit Simplexes

The 3D Pyth unit simplex without partitioning is shown in **Figure 15**.



**Figure 15.** The original 3D Pyth unit simplex with three unit legs and a hypotenuse.

The trisection of a non-unit Pyth tetrahedron was shown in **Figures 11 and 12**. Following is the trisection of a unit Pyth simplex (tetrahedron) into three non-unit simplexes (**Figure 16**).

Note the distortion of measures: Right angles are only those marked by arcs in the angle region, whether they look perpendicular or not. Similarly, the length of segments corresponds to the marking at the line, not the actual length of the visual image.

There are 12 right angles on the figure, two at vertex  $A_2$ , two at vertex  $A_3$ , four at vertex  $B_1$ , and four at vertex  $B_2$ .

Drop a perpendicular from vertex  $A_2$  to line  $A_1A_3$  in the original tetrahedron. Mark the foot  $B_1$ . Drop a perpendicular from  $B_1$  to line  $A_1A_4$ . Mark the



Figure 22):

$$\begin{aligned}
 V(A_2A_3A_4A_1) &= \frac{1}{3!} \sqrt{\frac{1 \cdot 1 \cdot 1}{1 \cdot 1 \cdot 1}} = \frac{6}{36} \\
 &= V_1(A_2B_1B_2A_1) + V_2(A_2B_1B_2A_4) + V_3(A_2B_1A_3A_4) \\
 &= \frac{1}{36} + \frac{2}{36} + \frac{3}{36} = \frac{6}{36} = \frac{3!}{(3!)^2} = \frac{1}{3!}
 \end{aligned}$$

This result completes the proof of the Coxeter partitioning in 3D space.

In higher dimensions, I proceed in the same way, omitting steps that can be easily determined from the 3D pattern.

### 4.7. The 4D Case: The 4D Unit Pyth Simplex Partitioned in Four Non-Unit Simplexes

Figure 17 shows the original 4D unit Pyth simplex with four legs and a hypotenuse before the partition.

Figure 18 shows the 4D unit Pyth simplex partitioned into four non-unit simplexes.

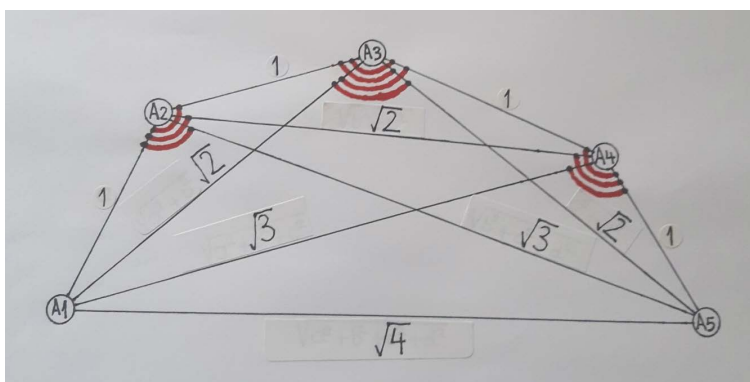


Figure 17. The original 4D unit Pyth simplex with four legs and a hypotenuse: a Pyth pentachoron depicted on a distorted planar figure.

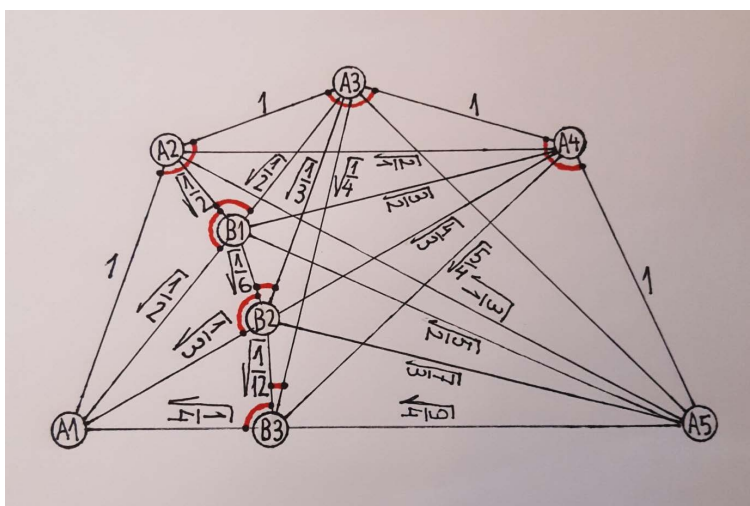


Figure 18. The Coxeter partition of a 4D unit Pyth simplex.

For better clarity, only certain right angles are shown. The missing right angles can be found by simple Pythagorean calculation. For example, triangle  $B_1B_2A_5$  is a right triangle with right angle at vertex  $B_2$ , because  $\frac{7}{3} + \frac{1}{6} = \frac{5}{2}$ .

Again, the length of the sides and the measure of the angles may appear in a strongly distorted form. For example, triangle  $A_3B_2B_3$  has its right angle at vertex  $B_3$ , although it appears much smaller than the angle at  $B_2$ , and the hypotenuse  $A_3B_2$  seems shorter than leg  $A_3B_3$ .

The original unit Pyth simplex is  $A_2A_3A_4A_5A_1$ , the four smaller non-unit Pyth 4D simplexes are

$$A_2B_1B_2B_3A_1, A_2B_1B_2B_3A_4, A_2B_1B_2A_4A_5, A_2B_1A_3A_4A_5.$$

The respective volumes of the smaller simplexes are as follows:

$$V_1(A_2B_1B_2B_3A_1) = \frac{1}{4!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{1}{4}} = \frac{1}{576}$$

$$V_2(A_2B_1B_2B_3A_5) = \frac{1}{4!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{9}{4}} = \frac{3}{576}$$

$$V_3(A_2B_1B_2A_4A_5) = \frac{1}{4!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{3} \cdot \frac{1}{1}} = \frac{8}{576}$$

$$V_4(A_2B_1A_3A_4A_5) = \frac{1}{4!} \sqrt{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{12}{576}$$

The volumes of the four smaller Pyth 4D simplexes (Pyth pentachorons) add up to the volume of the original unit Pyth pentachoron:

$$\begin{aligned} V(A_2A_3A_4A_5A_1) &= \frac{1}{4!} \sqrt{\frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{24}{576} \\ &= V_1(A_2B_1B_2B_3A_1) + V_2(A_2B_1B_2B_3A_5) \\ &\quad + V_3(A_2B_1B_2A_4A_5) + V_4(A_2B_1A_3A_4A_5) \\ &= \frac{1}{576} + \frac{3}{576} + \frac{8}{576} + \frac{12}{576} = \frac{24}{576} = \frac{4!}{(4!)^2} = \frac{1}{4!} \end{aligned}$$

This result completes the proof of the Coxeter partitioning in 4D space.

#### 4.8. The 5D Case

**Figure 19** shows the 5D unit Pyth simplex partitioned into five non-unit simplexes.

Again, for better clarity, only certain right angles are shown. The missing right angles can be found by simple Pythagorean calculation.

The original unit Pyth simplex is  $A_2A_3A_4A_5A_6A_1$ , the five smaller non-unit Pyth 5D simplexes are:

$$A_2B_1B_2B_3B_4A_1, A_2B_1B_2B_3B_4A_6, A_2B_1B_2B_3A_5A_6, A_2B_1B_2A_4A_5A_6, A_2B_1A_3A_4A_5A_6.$$

The respective volumes of the smaller simplexes are as follows:

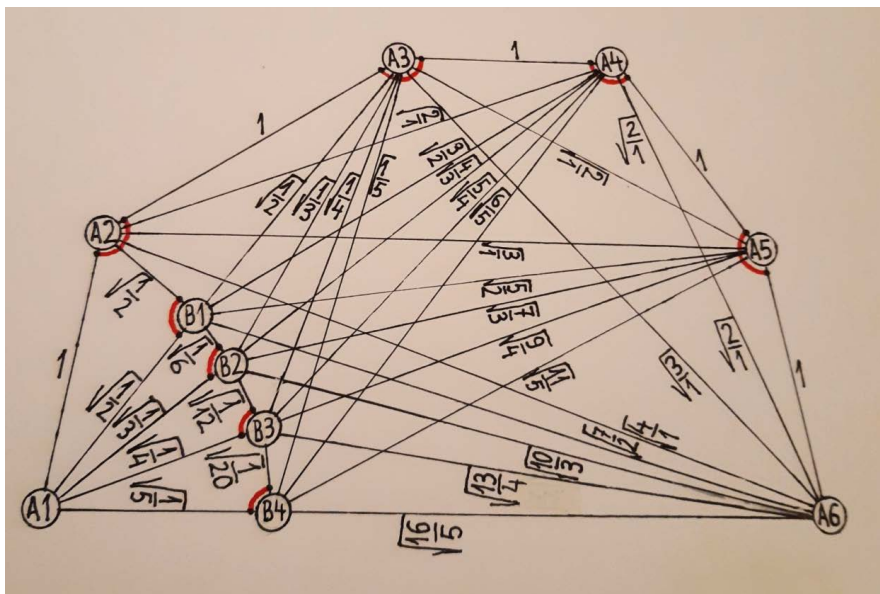


Figure 19. The Coxeter partition of a5D unit Pyth simplex.

$$\begin{aligned}
 V_1(A_2B_1B_2B_3B_4A_1) &= \frac{1}{5!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{1}{20} \cdot \frac{1}{5}} = \frac{1}{14400} \\
 V_2(A_2B_1B_2B_3B_4A_6) &= \frac{1}{5!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{1}{20} \cdot \frac{16}{5}} = \frac{4}{14400} \\
 V_3(A_2B_1B_2B_3A_5A_6) &= \frac{1}{5!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{9}{4} \cdot \frac{1}{1}} = \frac{15}{14400} \\
 V_4(A_2B_1B_2A_4A_5A_6) &= \frac{1}{5!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{4}{3} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{40}{14400} \\
 V_5(A_2B_1A_3A_4A_5A_6) &= \frac{1}{5!} \sqrt{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{60}{14400} \\
 \\
 V(A_2A_3A_4A_5A_6A_1) &= \frac{1}{5!} \sqrt{\frac{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}{1 \cdot 1 \cdot 1 \cdot 1 \cdot 1}} = \frac{120}{14400} \\
 &= V_1(A_2B_1B_2B_3B_4A_6) + V_2(A_2B_1B_2B_3B_4A_6) \\
 &\quad + V_3(A_2B_1B_2B_3A_5A_6) + V_4(A_2B_1B_2A_4A_5A_6) \\
 &\quad + V_5(A_2B_1A_3A_4A_5A_6) \\
 &= \frac{1}{14400} + \frac{4}{14400} + \frac{15}{14400} + \frac{40}{14400} + \frac{60}{14400} \\
 &= \frac{5!}{(5!)^2} = \frac{120}{14400} = \frac{1}{5!}
 \end{aligned}$$

This result completes the proof of the Coxeter partitioning in 5D space.

### 4.9. The 6D Case

$$V_1(A_2B_1B_2B_3B_4B_5A_1) = \frac{1}{6!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{1}{20} \cdot \frac{1}{30} \cdot \frac{1}{6}} = \frac{1}{720^2}$$

$$\begin{aligned}
 V_2(A_2B_1B_2B_3B_4B_5A_7) &= \frac{1}{6!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{1}{20} \cdot \frac{1}{30} \cdot \frac{25}{6}} = \frac{5}{720^2} \\
 V_3(A_2B_1B_2B_3B_4A_6A_7) &= \frac{1}{6!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{1}{20} \cdot \frac{16}{5} \cdot \frac{1}{1}} = \frac{24}{720^2} \\
 V_4(A_2B_1B_2B_3A_5A_6A_7) &= \frac{1}{6!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{1}{12} \cdot \frac{9}{4} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{90}{720^2} \\
 V_5(A_2B_1B_2A_4A_5A_6A_7) &= \frac{1}{6!} \sqrt{\frac{1}{2} \cdot \frac{1}{6} \cdot \frac{4}{3} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{240}{720^2} \\
 V_6(A_2B_1A_3A_4A_5A_6A_7) &= \frac{1}{6!} \sqrt{\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{360}{720^2} \\
 V(A_2A_3A_4A_5A_6A_7A_1) &= \frac{1}{6!} \sqrt{\frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{720}{720^2} \\
 &= \frac{1}{720^2} + \frac{5}{720^2} + \frac{24}{720^2} + \frac{90}{720^2} + \frac{240}{720^2} + \frac{360}{720^2} \\
 &= \frac{720}{720^2} = \frac{1}{6!}
 \end{aligned}$$

#### 4.10. The 7D Case

$$\begin{aligned}
 V(A_2A_3A_4A_5A_6A_7A_8A_1) &= \frac{1}{7!} \sqrt{\frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1} \cdot \frac{1}{1}} = \frac{5040}{5040^2} \\
 &= \frac{1}{5040^2} + \frac{6}{5040^2} + \frac{35}{5040^2} + \frac{168}{5040^2} + \frac{630}{5040^2} \\
 &\quad + \frac{1680}{5040^2} + \frac{2520}{5040^2} \\
 &= \frac{5040}{5040^2} = \frac{1}{7!}
 \end{aligned}$$

### 5. The Principle of Recursion for the $n$ -Dimensional Simplex

#### 5.1. The Points of Partition

In the  $n$ -dimensional space, the partition points are  $B_1, B_2, B_3, \dots, B_{n-1}$ .

The partition of the original unit simplex can be carried out by the same algorithm as in the previous cases (see **Table 2**).

#### 5.2. Simplify the Addends

For calculating the volumes of the smaller simplexes in the  $n$ -dimensional space, simplify the addends to Egyptian fractions with number 1 in the numerator as shown in **Table 3**.

The sum is 1 in each row.

Each row is expanded from the previous one by one element with the entry

$\frac{1}{(n-2)!n}$  in the penultimate position. For example,  $n = 4$  gives

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{8} + \frac{1}{24} = 1.$$

**Table 2.** Vertices of the  $n$  simplexes from the Coxeter division of the unit simplex.

|       |       |       |       |       |     |           |           |           |       |
|-------|-------|-------|-------|-------|-----|-----------|-----------|-----------|-------|
| $A_2$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | ... | $B_{n-3}$ | $B_{n-2}$ | $B_{n-1}$ | $A_1$ |
| $A_2$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | ... | $B_{n-3}$ | $B_{n-2}$ | $B_{n-1}$ | $A_n$ |
| $A_2$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | ... | $B_{n-3}$ | $B_{n-2}$ | $A_{n-1}$ | $A_1$ |
| $A_2$ | $B_1$ | $B_2$ | $B_3$ | $B_4$ | ... | $B_{n-3}$ | $A_{n-2}$ | $A_{n-1}$ | $A_1$ |
| ...   | ...   | ...   | ...   | ...   | ... | ...       | ...       | ...       | ...   |
| $A_2$ | $B_1$ | $B_2$ | $B_3$ | $A_5$ | ... | $A_{n-3}$ | $A_{n-2}$ | $A_{n-1}$ | $A_1$ |
| $A_2$ | $B_1$ | $B_2$ | $A_4$ | $A_5$ | ... | $A_{n-3}$ | $A_{n-2}$ | $A_{n-1}$ | $A_1$ |
| $A_2$ | $B_1$ | $A_3$ | $A_4$ | $A_5$ | ... | $A_{n-3}$ | $A_{n-2}$ | $A_{n-1}$ | $A_1$ |

**Table 3.** The volume of the smaller simplexes compared with the volume of the original unit simplex.

| nD  |                              |                              |                              |                               |                                |                                |                                 |                |  |  |
|-----|------------------------------|------------------------------|------------------------------|-------------------------------|--------------------------------|--------------------------------|---------------------------------|----------------|--|--|
| 1D  | $\frac{1}{1} = \frac{1}{1!}$ |                              |                              |                               |                                |                                |                                 |                |  |  |
| 2D  | $\frac{1}{2}$                | $\frac{1}{2} = \frac{1}{2!}$ |                              |                               |                                |                                |                                 |                |  |  |
| 3D  | $\frac{1}{2}$                | $\frac{1}{3}$                | $\frac{1}{6} = \frac{1}{3!}$ |                               |                                |                                |                                 |                |  |  |
| 4D  | $\frac{1}{2}$                | $\frac{1}{3}$                | $\frac{1}{8}$                | $\frac{1}{24} = \frac{1}{4!}$ |                                |                                |                                 |                |  |  |
| 5D  | $\frac{1}{2}$                | $\frac{1}{3}$                | $\frac{1}{8}$                | $\frac{1}{30}$                | $\frac{1}{120} = \frac{1}{5!}$ |                                |                                 |                |  |  |
| 6D  | $\frac{1}{2}$                | $\frac{1}{3}$                | $\frac{1}{8}$                | $\frac{1}{30}$                | $\frac{1}{144}$                | $\frac{1}{720} = \frac{1}{6!}$ |                                 |                |  |  |
| 7D  | $\frac{1}{2}$                | $\frac{1}{3}$                | $\frac{1}{8}$                | $\frac{1}{30}$                | $\frac{1}{144}$                | $\frac{1}{840}$                | $\frac{1}{5040} = \frac{1}{7!}$ |                |  |  |
| ... | ...                          |                              |                              |                               |                                |                                |                                 |                |  |  |
| nD  | $\frac{1}{2}$                | $\frac{1}{3}$                | $\frac{1}{8}$                | ...                           |                                |                                | $\frac{1}{(n-2)!n}$             | $\frac{1}{n!}$ |  |  |

**Table 3** refers to the volume of the simplexes. The structure of the solids can be displayed using the same pattern as in **Figure 14**, **Figure 16**, **Figure 18** and **Figure 19**.

The same algorithm applies for a simplex in any  $n$ -dimensional space.

## 6. The Hadwiger Conjecture

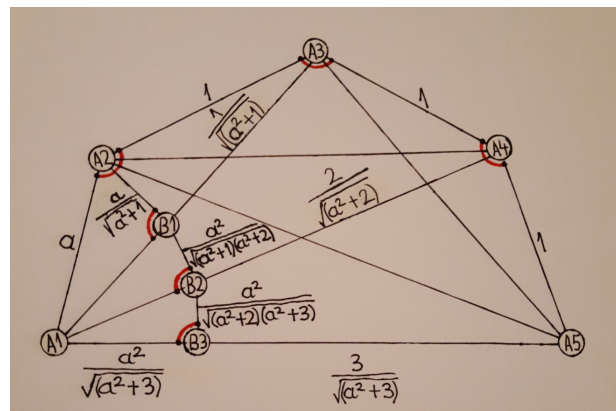
Hadwiger conjectured in 1956 that every orthoscheme can be dissected into finitely many orthoschemes [7]. The conjecture has been proven up to 5D by Tschirpke in 1994 [5], but remains unsolved in higher dimensions.

The above results in this article show that any  $n$ -dimensional unit Pythagorean simplex can be dissected into  $n$  pieces of Pythagorean  $n$ -dimensional simplexes.

I assume that this method of partitioning has relevance to the Hadwiger conjecture, but I am not sure that it proves the conjecture in the general case. My assumption is that the same method of partitioning applies, not only to the unit simplex, but to any Pythagorean simplex as well. This has already been proven in 2D and 3D spaces in 3.2. and 4.1, but I find the exact calculation too complicated to perform here in  $n$ -dimensional space with sides of arbitrary length of legs  $a_1, a_2, \dots, a_{n-1}$ .

However, I give a proof for a non-unit simplex in the 4D case when the length of legs is  $a, 1, 1, 1$ .

This non-unit simplex differs from the Pyth unit simplex in the length of only one leg. If the length of one leg is changed from the unit to an arbitrary length  $a$ , the remaining data will change accordingly, as seen in **Figure 20**:



**Figure 20.** The Coxeter partition of a 4D non-unit Pyth simplex.

**Figure 20** only shows the data that are relevant for calculating the volume of the simplexes.

The original non-unit Pyth simplex is  $A_2A_3A_4A_5A_1$ , the four smaller non-unit Pyth 4D simplexes are

$$A_2B_1B_2B_3A_1, A_2B_1B_2B_3A_4, A_2B_1B_2A_4A_5, A_2B_1A_3A_4A_5.$$

$$\begin{aligned} V_1(A_2B_1B_2B_3A_1) &= \frac{1}{4!} \frac{a}{\sqrt{a^2+1}} \cdot \frac{a^2}{\sqrt{(a^2+1)(a^2+2)}} \cdot \frac{a^2}{\sqrt{(a^2+2)(a^2+3)}} \cdot \frac{a^2}{\sqrt{a^2+3}} \\ &= \frac{1a^7}{(a^2+1)(a^2+2)(a^2+3)} \end{aligned}$$

$$\begin{aligned} V_2(A_2B_1B_2B_3A_5) &= \frac{1}{4!} \frac{a}{\sqrt{a^2+1}} \cdot \frac{a^2}{\sqrt{(a^2+1)(a^2+2)}} \cdot \frac{a^2}{\sqrt{(a^2+2)(a^2+3)}} \cdot \frac{3}{\sqrt{a^2+3}} \\ &= \frac{3a^5}{(a^2+1)(a^2+2)(a^2+3)} \end{aligned}$$

$$\begin{aligned}
 V_3(A_2B_1B_2A_4A_5) &= \frac{1}{4!} \frac{a}{\sqrt{a^2+1}} \cdot \frac{a^2}{\sqrt{(a^2+1)(a^2+2)}} \cdot \frac{2}{\sqrt{a^2+2}} \cdot 1 \\
 &= \frac{2a^3}{(a^2+1)(a^2+2)} = \frac{2a^3(a^2+3)}{(a^2+1)(a^2+2)(a^2+3)} \\
 V_4(A_2B_1A_3A_4A_5) &= \frac{1}{4!} \frac{a}{\sqrt{a^2+1}} \cdot \frac{1}{\sqrt{a^2+1}} \cdot 1 \cdot 1 = \frac{a}{a^2+1} \\
 &= \frac{a(a^2+2)(a^2+3)}{(a^2+1)(a^2+2)(a^2+3)}
 \end{aligned}$$

The sum of the three volumes adds up to the volume of the original tetrahedron:

$$\begin{aligned}
 V(A_2A_3A_4A_5A_1) &= V_1(A_2B_1B_2B_3A_1) + V_2(A_2B_1B_2B_3A_5) \\
 &\quad + V_3(A_2B_1B_2A_4A_5) + V_4(A_2B_1A_3A_4A_5) \\
 &= \frac{1}{4!} \frac{1a^7 + 3a^5 + 2a^3(a^2+3) + a(a^2+2)(a^2+3)}{(a^2+1)(a^2+2)(a^2+3)} \\
 &= \frac{a}{4!} \frac{a^6 + 3a^4 + 2a^2(a^2+3) + (a^2+2)(a^2+3)}{(a^2+1)(a^2+2)(a^2+3)} \\
 &= \frac{a}{4!} \frac{a^6 + 6a^4 + 11a^2 + 6}{(a^2+1)(a^2+2)(a^2+3)} = \frac{1}{4!} a
 \end{aligned}$$

This result completes the proof of the Coxeter partitioning of a special non-unit Pyth simplex in 4D space. I assume that we can proceed in the same way to prove the general case, changing the unit legs step by step to any value  $a_1, a_2, \dots, a_n$  in any dimension.

## Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

## References

- [1] Schläfli, L. (1901) Theorie der vielfachen Kontinuität. Birkhäuser Verlag Basel, 10-22.
- [2] Lénárt, I. (2020) The Algebra of Projective Spheres on Plane, Sphere and Hemisphere. *Journal of Applied Mathematics and Physics*, **8**, 2286-2333. <https://doi.org/10.4236/jamp.2020.810171>
- [3] Lénárt, I. (2022) The Right Triangle as the Simplex in 2D Euclidean Space, Generalized to N Dimensions. *Journal of Applied Mathematics and Physics*, **10**, 2837-2850. <https://doi.org/10.4236/jamp.2022.109189>
- [4] Petrie, J.F. (1938) Petrie Polygons. In: Coxeter, H.S.M., Val, P., Flather, H.T. and Petrie, J.F., eds., *The Fifty-Nine Icosahedra*, University of Toronto Studies, 1-26.
- [5] Tschirpke, Katrin (1994): The Dissection of Five-Dimensional Simplexes into Orthoschemes. *Beiträge zur Algebra und Geometrie*, **35**, 1-11.
- [6] Coxeter, H.S.M. (1989) Trisecting an Orthoscheme. *Computers & Mathematics with Applications*, **17**, 59-71. [https://doi.org/10.1016/0898-1221\(89\)90148-x](https://doi.org/10.1016/0898-1221(89)90148-x)
- [7] Hadwiger, H. (1956) Ungelöste Probleme. *Elemente der Mathematik*, **11**, 109-110.