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A Represent of Generators of the Cyclic Group of Higher Even, Odd and Prime Order for Composition Being Multiplication

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Abstract

This paper aims to treat a study of generators of the cyclic group of higher even, odd, and prime order for composition being multiplication. In fact we developed order of a group, order of element of a group and generators of the cyclic group in real numbers. Also we express cyclic and generators of the group for composition in real numbers. Here we discuss the higher order of groups in different types of order, and generators of the cyclic group which will give us practical knowledge to see the applications of the composition. In order to find out the order of an element $a^m \in G$ in which $a^n = e$ = identity element, then find Highest Common Factor *i.e.* (H.C.F) of m and n. When G is a finite group, every element must have finite order but the converse is false. There are infinite groups where each element has a finite order. There may be more than one generator of a cyclic group. Also every cyclic group is necessarily abelian. But show that every infinite cyclic group contains only two generators. Finally, we find out the generators of the cyclic group of higher even, odd and prime order in different types of the group for composition being multiplication.

Keywords

Order of Element of a Group, Multiplication Composition, Highest Common Factor, Generator

1. Introduction

We propose to study the groups of order of an element of a group, order of group,cyclic, generators and the integral powers of an element of a group etc. Then discuss all the order of every element in the higher order, and generators of the cyclic group for composition. The group notation is o or *. We will frequently omit the symbol for the group operation but we will also often write the operation as · or + when it represents multiplication or addition in a group, and write 1 or 0 for the corresponding identity elements respectively. It's addition +, multiplication × or (.) is used as binary operation. If the group operation is denoted as a multiplication, then an element $a \in G$ is said to be order n if n is the least positive integer such that $a^n = e$ or $O(a) \le n$ *i.e.*, if $a^n = e$ and $a^r \ne e$ $\forall r \in N$ s.t. $r < n$. The order of a is denoted by $O(a)$. If $a^n \ne e$ for any $n \in N$, then a is said to be of zero order or infinite order [1]. Let e is the identity element in $(G,+)$. An element $a \in G$ is said to be order n if $n \in Z^+$ such that $na = e$ or $O(a) \le n$. *i.e.*, if $na = e$ and $ar \ne e$ $\forall r \in N$ s.t. $0 < r < n$. The order of a is denoted by $O(a)$. If $na \ne e$ for any $n \in N$, then a is said to be of zero order or infinite order [2]. Also a group G is said to be cyclic if every element $x \in G$ is expressed as $x = a^n$ for some $n \in \mathbb{Z}$ (multiplication) the elements of G is $\cdots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \cdots$. Then the classification of finite simple groups (*i.e.* [3]-[5]) comes into play and one has to be able to handle the three different families of simple groups with appropriate techniques. Nonetheless the classification problem for finite groups into two problems: 1) identify the simple groups; 2) identify the ways these simple groups may be put together to form bigger groups. Next, we discuss the extension of the associative property to products with any number of factors. Then we find out the generators of the cyclic group of higher even, odd and prime order in different types of the group for composition being multiplication [6].

2. Integral Powers of an Element of a Group

Multiplication Composition [7]

Let $(G,.)$ be a group. Let $a \in G$ be arbitrary element.

By closure property, all the elements $a, aa, aaa, \cdots$ etc. belong to G.

Since the composition in G is associative. Hence $aaa\cdots$ to n factors is independent of manner in which the factors are grouped.

If n is a positive integer, then define $a^n = a \cdot a \cdot a \cdots$ to n factors.

$a^n \in G$, by closure property.

If e is identity in G, then we define $a^0 = e$.

If n is a negative integer, then by define $a^{-n} = \left(a^n\right)^{-1}$, where $\left(a^n\right)^{-1}$ is the inverse of a^n.

Consequently, $\left(a^n\right)^{-1} \in G$, since inverse of every element of G belong to G. $\therefore$ $a^{-n} \in G$.

According to the definition

$$\begin{aligned}\left(a^n\right)^{-1} &= \left(aaa\cdots \text{ to } n \text{ factors}\right)^{-1}\\ &= \left(a^{-1}\right)\left(a^{-1}\right)\left(a^{-1}\right)\cdots \text{ to } n \text{ factors} = \left(a^{-1}\right)^n\end{aligned}$$

$$\therefore\ a^{-n} = \left(a^n\right)^{-1} = \left(a^{-1}\right)^n .$$

The following law of indices can be easily proved

$$\left(a^m\right)^n = a^{mn} \quad \forall a \in G \text{ and } \forall m, n \in Z$$
$$\text{and } a^m a^n = a^{m+n} \quad \forall a \in G \text{ and } \forall m, n \in Z$$

Thus we defined a^n for all integral values of n, positive, negative or zero.

Thus we defined na for all integral values of n, positive, negative or zero.

3. General Properties of the Order, Cyclic and Generators of an Element of a Group

We begin this section of the following theorem related significance of the cyclic and generators of an element of a group.

1) Theorem [8]**:** The order of a cyclic group is equal to the order of any generator of the group

Proof: Let a be a generator of a group $G = \{a\}$ and let a be a finite order n so that

$$o(a) = n,\ a^n = e,\ a^r \neq e \text{ for } 0 < r < n .$$

To prove that $o(G) = 0$, i.e, to prove that $o(G) = n$.

This will be provided in two steps.

Step 1: To prove that G contains n elements.

The elements of the cyclic group G are given below:

$$a, a^2, a^3, \cdots, a^n = e = a^0 .$$

If possible, let G contains an element a^m besides these elements, where $m > n$.

By division algorithm,

$$m = nq + r \quad 0 \le r < n \text{ and } q, r \in N .$$

$$a^m = a^{nq+r} = a^{nq} \cdot a^r = \left(a^n\right)^q \cdot a^r = e^q \cdot a^r = e \cdot a^r = a^r$$

$$\therefore\ a^m = a^r,\ 0 \le r < n .$$

i.e. a^r is already contained in the set of n elements and so a^m is also contained.

Consequently, G contains n elements.

Step 2: To prove that any two elements of G are not equal.

For this we have to show that $a^r \neq a^s$ when $r \neq s$, where $0 < r < n$, $0 < s < n$.

To be particular at this point, let $r < s < n$.

Then $s - r > 0$.

$$a^r = a^s \Rightarrow ea^r = a^s \Rightarrow a^{s-r} = e$$
$$\Rightarrow o(a) \le s - r \text{ and } s - r < n$$
$$\Rightarrow o(a) < n.$$

This is a contradiction. For $o(a)=n$.

Hence $a^r \neq a^s$ when $r \neq s$.

Thus we have shown that G contains n distinct elements and hence $o(G)=n$.

Therefore, the order of a cyclic group is equal to the order of any generator of the group.

2) Theorem: Show that the order of every element of a finite group is finite.

Proof: Let G be a finite group with multiplication composition.

Let $a \in G$ be an arbitrary element.

Now we will prove that $O(a)$ is finite.

By closure property, all the elements $a^2 = a \cdot a$, $a^3 = a \cdot a \cdot a$, $\cdots$ etc. belong to G

i.e. $a, a^2, a^3, a^4, a^5, a^6, a^7, \cdots$ etc. belong to G.

But all these elements are not distinct. Since G is finite.

Let e be the identity in G, then $a^0 = e$.

Let us suppose that

$$a^m = a^n \text{ where } m > n$$
$$\Rightarrow a^m a^{-n} = a^n a^{-n} = a^0 = e$$
$$\Rightarrow a^{m-n} = e \Rightarrow a^p = e, \text{ where } p = m - n > 0, \text{ as } m > n$$

Also m and n are finite and hence p is a finite positive integer.

Now p is a positive integer s.t. $a^p = e$.

This proves that

$$o(a) \leq p = \text{finite number}$$
$$\text{i.e. } o(a) \leq \text{a finite number} \Rightarrow o(a) \text{ is finite}$$

Remark: The order of any element of a finite group can never exceed the order of the group.

3) Theorem: Show that the order of any integral power of an element of a group G is less than or equal of a. *i.e.* $o(a^m) \leq o(a)$ $\forall a \in G$ and $m \in N$.

Proof: Let $a \in G$ be an arbitrary element s. t. $o(a) = n$ where n is a natural number

$$\text{Such that } a^n = e = \text{identity of } G. \tag{1}$$

Let a^m be any power of a and let $o(a^m) = p$.

Now we will prove that $o(a^m) \leq o(a)$ *i.e.* $p \leq n$.

We have,

$$o(a) = n \Rightarrow a^n = e \Rightarrow a^{nm} = e^m = e \Rightarrow (a^m)^n = e \Rightarrow o(a^m) \leq n \Rightarrow p \leq n$$

Remark: This theorem can also be expressed in the following ways.

i. The order of any integral power of an element a of a group cannot exceed the order of a.

ii. If $a \in G$ *i.e.* G being a group, then

$$o(a^m) \leq o(a) \quad \forall a \in G \text{ and } m \in N$$

iii. If G is a group and $a \in G$, then order of any power of an element a is almost equal to the order of a.

4. Result and Discussion

We discuss the result of generators of the cyclic group of higher even, odd and prime order in different types of the group for composition being multiplication. But we can easily use composition related theorem to evaluate order and generators of group of different orders such as order 2, 3, 4, 5, ..., 20 etc., *i.e.* whose order is not so high (Not Higher Order Groups). As a result, we use multiplication related theorem to evaluate order of group, cyclic and generators of the group of a higher order of group for composition. For that reason, here we discuss the generators of the cyclic group of higher even, odd and prime order in different types of the group for composition being multiplication group as like 90, 95, and 59.

Find the generators of the cyclic group of order 59, 90 and 95 for composition being multiplication [9] [10].

Solution:

The Cyclic Group of Order 59:

Let a cyclic group G of order 59 be generated by an element a, then

$$o(a) = o(G) = 59$$

Now we determine the number of generators of *G*.

Evidently, Evidently, $G = \{a, a^2, a^3, a^4, \cdots, a^{59} = e\}$.

An element $a^m \in G$ is also a generator of G if H. C. F of m and 59 is 1.

H.C.F of 1 and 59 is 1, H.C.F of 2 and 59 is 1, H.C.F of 3 and 59 is 1, H.C.F of 4 and 59 is 1, H.C.F of 5 and 59 is 1, H.C.F of 6 and 59 is 1, H.C.F of 7 and 59 is 1, H.C.F of 8 and 59 is 1, H.C.F of 9 and 59 is 1, H.C.F of 10 and 59 is 1, H.C.F of 11 and59 is 1, H.C.F of 12 and 59 is 1, H.C.F of 13 and 59 is 1, H.C.F of 14 and 59 is 1, H.C.F of 15 and 59 is 1, H.C.F of 16 and 59 is 1, H.C.F of 17 and 59 is 1, H.C.F of 18 and 59 is 1, H.C.F of 19 and 59 is 1, H.C.F of 20 and 59 is 1, H.C.F of 21 and 59 is 1, H.C.F of 22 and 59 is 1, H.C.F of 23 and 59 is 1, H.C.F of 24 and 59 is 1, H.C.F of 25 and 59 is 1, H.C.F of 26 and 59 is 1, H.C.F of 27 and 59 is 1, H.C.F of 28 and 59 is 1, H.C.F of 29 and 59 is 1, H.C.F of 30 and 59 is 1, H.C.F of 31 and 59 is 1, H.C.F of 32 and 59 is 1, H.C.F of 33 and 59 is 1, H.C.F of 34 and 59 is 1, H.C.F of 35 and 59 is 1, H.C.F of 36 and 59 is 1, H.C.F of 37 and 59 is 1, H.C.F of 38 and 59 is 1, H.C.F of 39 and 59 is 1, H.C.F of 40 and 59 is 1, H.C.F of 41 and 59 is 1, H.C.F of 42 and 59 is 1, H.C.F of 43 and 59 is 1, H.C.F of 44 and 59 is 1, H.C.F of 45 and 59 is 1, H.C.F of 46 and 59 is 1, H.C.F of 47 and 59 is 1, H.C.F of 48 and 59 is 1, H.C.F of 49 and 59 is 1, H.C.F of 50 and 59 is 1, H.C.F of 51 and 59 is 1, H.C.F of 52 and 59 is 1, H.C.F of 53 and 59 is 1, H.C.F of 54 and 59 is 1, H.C.F of 55 and 59 is 1, H.C.F of 56 and 59 is 1, H.C.F of 57 and 59 is 1, H.C.F of 58 and 59 is 1.

There are fifty eight generators of G as $G = \{a, a^2, a^3, a^4, \cdots, a^{58}\}$.

The Cyclic Group of Order 90:

Let a cyclic group G of order 60 be generated by an element a, then

$$o(a) = o(G) = 60$$

Now we determine the number of generators of *G*.

Evidently, $G = \{a, a^2, a^3, a^4, \cdots, a^{90} = e\}$

An element $a^m \in G$ is also a generator of G if H. C. F of m and 60 is 1.

H.C.F of 1 and 90 is 1, H.C.F of 7 and 60 is 1, H.C.F of 11 and 60 is 1, H.C.F of 13 and 60 is 1, H.C.F of 17 and 60 is 1, H.C.F of 19 and 90 is 1, H.C.F of 23 and 90 is 1, H.C.F of 29 and 90 is 1, H.C.F of 31 and 90 is 1, H.C.F of 37 and 90 is 1, H.C.F of 39 and 90 is 1, H.C.F of 41 and 90 is 1, H.C.F of 43 and 90 is 1, H.C.F of 47 and 90 is 1,H.C.F of 49 and 90 is 1, H.C.F of 51 and 90 is 1, H.C.F of 53 and 90 is 1, H.C.F of 59 and 90 is 1,H.C.F of 61 and 90 is 1, H.C.F of 67 and 90 is 1, H.C.F of 71 and 90 is 1, H.C.F of 73 and 90 is 1, H.C.F of 77 and 90 is 1, H.C.F of 79 and 90 is 1, H.C.F of 83 and 90 is 1, H.C.F of 89 and 90 is 1.

There are twenty six generators of G as $G = \{a, a^7, a^{11}, a^{13}, \cdots, a^{89}\}$.

The Cyclic Group of Order 95:

Let a cyclic group G of order 95 be generated by an element a, then

$$o(a) = o(G) = 95$$

Now we determine the number of generators of *G*.

Evidently, $G = \{a, a^2, a^3, a^4, \cdots, a^{95} = e\}$.

An element $a^m \in G$ is also a generator of G if H. C. F of m and 95 is 1.

H.C.F of 1 and 95 is 1, H.C.F of 2 and 95 is 1, H.C.F of 3 and 95 is 1, H.C.F of 4 and 95 is 1, H.C.F of 6 and 95 is 1, H.C.F of 7 and 95 is 1, H.C.F of 8 and 95 is 1, H.C.F of 9 and 95 is 1, H.C.F of 11 and 95 is 1, H.C.F of 12 and 95 is 1, H.C.F of 13 and 95 is 1, H.C.F of 14 and 95 is 1, H.C.F of 16 and 95 is 1, H.C.F of 17 and 95 is 1, H.C.F of 18 and 95 is 1, H.C.F of 21 and 95 is 1, H.C.F of 22 and 95 is 1, H.C.F of 23 and 95 is 1, H.C.F of 24 and 95 is 1, H.C.F of 26 and 95 is 1, H.C.F of 27and 95 is 1, H.C.F of 28 and 95 is 1, H.C.F of 29and 95 is 1, H.C.F of 31 and 95 is 1, H.C.F of 32 and 95 is 1, H.C.F of 33 and 95 is 1, H.C.F of 34 and 95 is 1, H.C.F of 36 and 95 is 1, H.C.F of 37 and 95 is 1, H.C.F of 39 and 95 is 1, H.C.F of 41 and 95 is 1, H.C.F of 42 and 59 is 1, H.C.F of 43 and 95 is 1, H.C.F of 44 and 95 is 1, H.C.F of 46 and 95 is 1, H.C.F of 47 and 95 is 1, H.C.F of 48and 95 is 1, H.C.F of 49and 95 is 1, H.C.F of 51 and 95 is 1, H.C.F of 52 and 95 is 1, H.C.F of 53 and 95 is 1, H.C.F of 54 and 95 is 1, H.C.F of 56 and 95 is 1, H.C.F of 58and 95 is 1, H.C.F of 59 and 95 is 1, H.C.F of 61 and 95 is 1, H.C.F of 63 and 95 is 1, H.C.F of 64 and 95 is 1, H.C.F of 66 and 95 is 1, H.C.F of 67 and 95 is 1, H.C.F of 68 and 95 is 1, H.C.F of 69 and 95 is 1, H.C.F of 71 and 95 is 1, H.C.F of 72 and 95 is 1, H.C.F of 73 and 95 is 1, H.C.F of 74 and 95 is 1, H.C.F of 77 and 95 is 1, H.C.F of 78 and 95 is 1, H.C.F of 79 and 95 is 1, H.C.F of 81 and 95 is 1, H.C.F of 82 and 95 is 1, H.C.F of 83 and 95 is 1, H.C.F of 84 and 95 is 1, H.C.F of 86 and 95 is 1, H.C.F of 87and 95 is 1, H.C.F of 88 and 95 is 1, H.C.F of 89and 95 is 1, H.C.F of 91 and 95 is 1, H.C.F of 92 and 95 is 1, H.C.F of 93 and 95 is 1, H.C.F of 94 and 95 is 1.

There are seventy one generators of G as $G = \{a, a^2, a^3, a^4, \cdots, a^{71}\}$.

5. Conclusion

This work will be useful for group theory related to the higher order of element of a group, also cyclic and generators of the group. The result is generators of the cyclic group of a group in different types of the higher order of group. This result has found extensive use in statistics, physics, information theory and geometrics, etc. After that, all expected results in this paper will help us to understand better solution to complicate the higher order of the generators of the cyclic group.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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