Scientific Research Publishing

Journal of Applied Mathematics and Physics, 2024, 12, 3690-3696
https://www.scirp.org/journal/jamp
ISSN Online: 2327-4379
ISSN Print: 2327-4352

Magnetic Monopoles, the Poynting Vector and Lenz's Law

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How to cite this paper: Shiekh, A.Y. (2024) Magnetic Monopoles, the Poynting Vector and Lenz's Law. *Journal of Applied Mathematics and Physics*, **12**, 3690-3696.
https://doi.org/10.4236/jamp.2024.1211221

Received: October 10, 2024
Accepted: November 3, 2024
Published: November 6, 2024

Abstract

The Poynting vector includes time-independent contributions which, since they represent photons of zero frequency, need to be absent; this removal resolves several paradoxes that arise when they are left in place; furthermore, magnetic monopoles, if they existed, would be accompanied by a Lenz's law that defies conservation of energy suggesting that magnetic charges might not be found in Nature.

Keywords

Magnetic Monopoles, Poynting Vector, Lenz's Law

1. Introduction

Various paradoxes can be resolved by recognizing that photons of zero frequency don't exist, and their removal is consistent with electromagnetism as shown in [1] and Section 2.1, and this is implicitly done in the texts when deriving the Abraham-Lorentz equation of motion as shown in Section 2.2. Further, the idea that a stationary charge/monopole system has angular momentum [2] is also looked at in this context in Section 2.3.

Another related paradox arises in general relativity because, by the equivalence principle, a charge at rest in a gravitational field is equivalent to one accelerating and so would radiate, but in the gravitational case, there is no energy source to power this radiation. This paradox is also solved by the exclusion of zero-frequency photons, as shown in [1] and the Appendix.

A separate Section 3 looks at the inclusion of magnetic monopoles in electromagnetism, concluding that they may violate energy conservation and so possibly explaining why they are not found in Nature.

2. Larmor Versus Feynman

We begin by looking at ambiguities in the formula for electromagnetic radiated power. The formula for the power resulting from an accelerating charge is normally given (in SI units) by the Larmor formula. [3] [4]

$$P_L = \frac{q^2}{6\pi\varepsilon_0 c^3} a^2 \qquad \text{(Larmor power)}$$

q being the charge, $\boldsymbol{a}$ the acceleration, c the speed of light and ε_0 the electric permittivity; however, this leads to various contradictions (see Section 2.2) that are resolved by the Feynman form for the power. [1] [5]

$$\boxed{P_F = \begin{cases} -\dfrac{q^2}{6\pi\varepsilon_0 c^3}\dot{\boldsymbol{a}}\cdot\boldsymbol{v} & \text{if } \dot{\boldsymbol{a}}\cdot\boldsymbol{v} < 0 \\ 0 & \text{if } \dot{\boldsymbol{a}}\cdot\boldsymbol{v} \geq 0 \end{cases}} \qquad \text{(Feynman power)}$$

where $\boldsymbol{v}$ is the velocity (relative to whatever is causing the acceleration). An ambiguity in the derivation of the Larmor equation permits the Feynman form, which is detailed in Section 2.1.

These two versions of the radiated power seem so different that one might wonder why it was not immediately obvious which was correct. This becomes clearer using integration by parts

$$\int_a^b fg\mathrm{d}x \equiv \left[f\int g\mathrm{d}x \right]_a^b - \int_a^b \left(f'\int g\mathrm{d}x \right)\mathrm{d}x \qquad \text{(Integration by parts)}$$

that tells us

$$\int_i^f \left(P_L - P_F\right)\mathrm{d}t = \frac{q^2}{6\pi\varepsilon_0 c^3}\left[\boldsymbol{a}\cdot\boldsymbol{v}\right]_i^f \qquad (1)$$

and so, there is no difference in predicted radiated energy if the motion is periodic or if the initial and final accelerations were zero; explaining why the difference is not obvious for radio transmissions. A notable difference, however, would be for constant acceleration, where the Feynman form would predict zero radiated power, thus resolving the gravitational paradox [1] [5] as described in the Appendix.

2.1. Ambiguities in the Poynting Vector

The Poynting vector $\boldsymbol{S}$ gives the energy flux density of an electromagnetic field

$$\boxed{\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}} \qquad \text{(Poynting vector)}$$

along with the linear momentum density $\boldsymbol{g}$

$$\boldsymbol{g} = \frac{\boldsymbol{S}}{c^2} = \varepsilon_0 \boldsymbol{E}\times\boldsymbol{B} \qquad (2)$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively, while ε_0 and μ_0 are the electric permittivity and magnetic permeability and c is the speed of light (we use SI units throughout).

Because it is only the divergence of the Poynting vector that appears in the conservation law from which it is derived [4] [6], one can limit oneself to looking at

$$\nabla \cdot \boldsymbol{S} \tag{3}$$

Now, using the vector identity

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) \equiv (\nabla \times \boldsymbol{E}) \cdot \boldsymbol{B} - \boldsymbol{E} \cdot (\nabla \times \boldsymbol{B}) \tag{4}$$

and then applying Faraday's and Ampère's law in free space

$$\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t} \qquad \text{(Faraday's Law)}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} \qquad \text{(Ampère's Law)}$$

we arrive at

$$\nabla \cdot (\boldsymbol{E} \times \boldsymbol{B}) = -\boldsymbol{B} \cdot \frac{\partial \boldsymbol{B}}{\partial t} - \frac{1}{c^2}\boldsymbol{E} \cdot \frac{\partial \boldsymbol{E}}{\partial t} \tag{5}$$

So, we find from Equation (5) that there is no contribution from the Poynting vector in the conservation law from non-time-varying fields, and therefore, such components can be dropped. That they should be dropped can be seen from the wave equations:

$$\nabla^2 \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2} \tag{6a}$$

$$\nabla^2 \boldsymbol{B} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{B}}{\partial t^2} \tag{6b}$$

The second time derivative of $\boldsymbol{E}$ or $\boldsymbol{B}$ cannot be zero for a wave, so constant fields will not give rise to radiation and therefore should not be included in the Poynting vector. This is the classical version of the simpler quantum perspective that zero-frequency photons don't exist. Thus, we see that non-parallel electric and magnetic fields alone are not sufficient to produce an energy flux: they must also change with time. We are not the only ones to be concerned about the contribution of static fields to the Poynting vector; some are aware of paradoxes arising from these contributions [7] and others have tried to cancel them out with 'hidden momentum' [8].

This exclusion of non-time-varying fields allows us to get the Feynman form from the Larmor form as we showed previously for Equation (1).

2.2. Equation of Motion for a Constant Pushing Force

The Larmor version of power gives rise to non-physical equations of motion that are resolved by the Feynman form

2.2.1. Larmor Version

Using $P = \boldsymbol{F} \cdot \boldsymbol{v}$ we can include the radiation reaction force to get

$$ma = F - \frac{P}{v} \tag{7}$$

and so, for the Larmor power (using a constant pushing force F_c), we get

$$\left(F_c - m\dot{v}\right)v = \alpha\dot{v}^2 \tag{8}$$

where $\alpha \equiv q^2 / \left(6\pi\varepsilon_0 c^3\right)$. This has pathological behavior, namely, if the particle starts at rest, the acceleration is then also zero, and thus, nothing moves despite the application of a finite pushing force.

2.2.2. Abraham-Lorentz Version

On the other hand, for the Feynman power, we get the Abraham-Lorentz equation

$$F_c - ma = -\alpha\dot{a} \qquad \text{(Abraham-Lorentz)}$$

whose solution is

$$a = \frac{F_c}{m} + \mathbb{C}\exp\left(\frac{m}{\alpha}t\right) \tag{9}$$

where $\alpha \equiv q^2 / \left(6\pi\varepsilon_0 c^3\right)$, but this is non-pathological if the integration constant $\mathbb{C}$ is zero.

While the Larmor form for power can never be negative, this is not always the case for the Feynman form (this possibility was not mentioned in the original work by Feynman [5]), since if $\dot{\boldsymbol{a}}$ and $\boldsymbol{v}$ are in the same direction, the predicted Feynman power is negative, and since this is not possible, the Feynman form might be more correctly expressed as

$$P_F = \begin{cases} -\dfrac{q^2}{6\pi\varepsilon_0 c^3}\dot{\boldsymbol{a}}\cdot\boldsymbol{v} & \text{if } \dot{\boldsymbol{a}}\cdot\boldsymbol{v} < 0 \\ 0 & \text{if } \dot{\boldsymbol{a}}\cdot\boldsymbol{v} \geq 0 \end{cases} \qquad \text{(Feynman power)}$$

This ties into the run-away solution for the Abraham-Lorentz solution (Equation (9)), where the integration constant $\mathbb{C}$ must be chosen to be zero on physical grounds, and hence one has zero Feynman force when $\dot{\boldsymbol{a}}$ is in the same direction as $\boldsymbol{v}$, namely no negative power is a compelled and consistent choice. One might be concerned that this exclusion will make a difference for sinusoidal motion, but this is not the case since when $x \sim \sin\omega t$ this leads to a Feynman power that goes like $\cos^2\omega t$ and thus has no negative component; however, for exponential motion ($x \sim \mathrm{e}^t$) the suppression mechanism comes heavily into effect, but the easiest way to avoid radiation while speeding up a charge is simply to maintain a constant acceleration.

2.3. Stationary Charge/Monopole System

Dirac [2] contemplated a charge/monopole that, despite being stationary, is perceived as having angular momentum $\boldsymbol{L}$ due to the electromagnetic momentum density $\boldsymbol{g}$ (see Equation (2)); this is illustrated in **Figure 1**.

However, indiscriminate use of the Poynting vector (those not excluding the constant part) can lead to various problems, namely a stationary electric charge together with a magnetic charge (monopole) will have an angular momentum that does *not* depend on their separation as originally tackled by Dirac [2] but more easily seen in the texts by Jackson [4] or Griffiths [6]. A stationary system having angular momentum is in itself strange, but even worse, if the charge and monopole are approaching each other, the angular momentum flips sign as they pass, and thus violating the conservation of angular momentum.

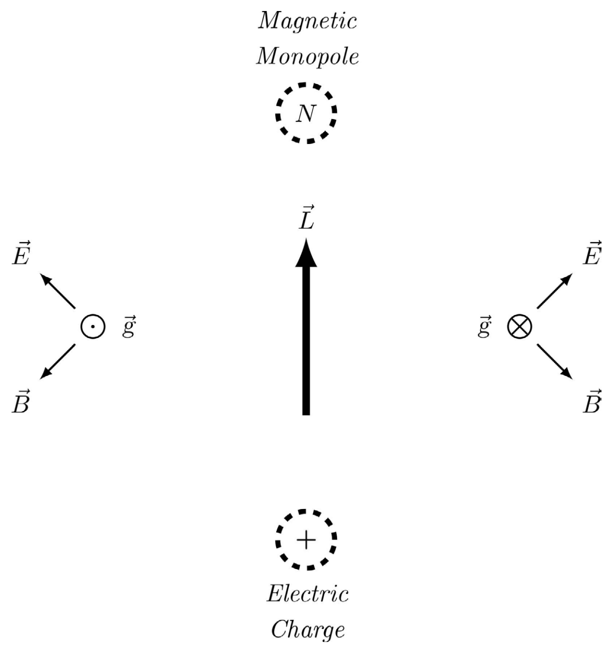

Figure 1. Fields for a stationary charge/monopole system.

These problems can be resolved once it is recognized that a time-invariant $\boldsymbol{E}$ and $\boldsymbol{B}$ field represent photons with zero frequency, which, since they cannot exist, need to be removed from the original Poynting vector, so once it is recognized there is no momentum contribution from non-changing fields, the angular momentum that would have arisen is no longer present. In this way, both paradoxes cease to exist once the original Larmor formula for radiated power.

$$P_L = \alpha a^2$$

is replaced by the Feynman form

$$P_F = -\alpha \dot{\boldsymbol{a}} \cdot \boldsymbol{v}$$

where $\alpha \equiv q^2 \big/ \left(6\pi\varepsilon_0 c^3\right)$; q being the charge, $\boldsymbol{a}$ the acceleration and $\boldsymbol{v}$ the velocity.

3. Lenz's Law and Magnetic Monopoles

Continuing this contemplation of magnetic monopoles, one has Lenz's law as part of Faraday's law in integral form

$$\overbrace{\oint_{\partial S} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}}^{\text{electric voltage}} = -\frac{\mathrm{d}}{\mathrm{d}t} \overbrace{\iint_S \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{A}}^{\text{magnetic flux}} \qquad \text{(Faraday's law)}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the electric and magnetic fields, respectively. Faraday's law yields the generated EMF (voltage) for a closed loop that results from the changing magnetic flux through that loop. The minus sign, which is Lenz's law, plays a very important role, namely, the conservation of energy; without it the induced current would add energy to the system.

Now look at the corresponding equation for magnetism, namely Ampère's law (c being the speed of light)

$$\oint_{\partial S} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = +\frac{1}{c^2}\frac{\mathrm{d}}{\mathrm{d}t}\iint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{A} \qquad \text{(Ampère's law)}$$

The issue here is the plus sign; if magnetic charges existed, there would be an induced magnetic voltage and current, but the lack of a minus sign in the corresponding magnetic Lenz's law would imply a violation of the conservation of energy (as detailed below). This might be taken as a reason why magnetic monopoles are not found in Nature.

The Energy Argument

Start with a solenoid being turned on and feeding into a conducting ring followed by a test coil, as illustrated in **Figure 2**.

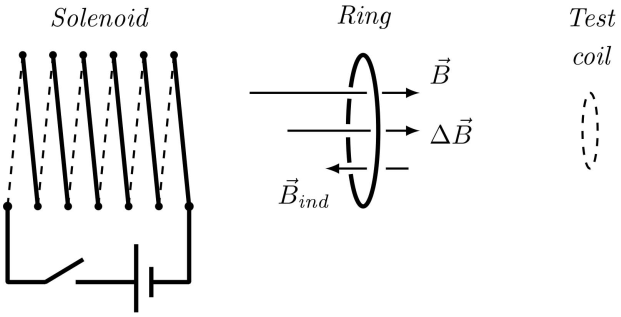

Figure 2. Lenz's law and conservation of energy.

One can now see that, due to the negative of Lenz's law (the induced magnetic field *opposes* the change of the original magnetic flux) that the original field $\boldsymbol{B}$ is weakened by the induced field $\boldsymbol{B}_{ind}$. Using the example of an exponentially increasing current makes conceptualizing the mechanism easier since the derivative of an exponential is again an exponential, albeit of differing strength, so all fields vary in sync. The test coil on the right experiences a reduced induced voltage due to the intervening ring since Lenz's law has a negative; if it was positive, energy would not be conserved in this counter-example.

4. Conclusion

The Poynting vector, as usually used, includes time-independent contributions that do not propagate and, therefore, should not be included. Once these contributions are removed, a long-standing gravitational paradox is resolved, and some strange features of the charge/monopole system are found to be no longer present. Furthermore, magnetic monopoles, if they existed, would be accompanied by Lenz's law, which defies conservation of energy, suggesting that magnetic charges might not be found in Nature.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

The accelerating charge paradox

According to Larmor's formula, a non-relativistic accelerating point charge radiates with a power given by

$$P_L = \frac{q^2}{6\pi\varepsilon_0 c^3} a^2 \qquad \text{(Larmor power)}$$

A paradox arises because, by the equivalence principle, a charge at rest in a gravitational field is equivalent to one accelerating, and so would radiate, but in the gravitational case, there is no energy source to power this radiation. Worse still, the equations of motion that follow from $P = \boldsymbol{F} \cdot \boldsymbol{v}$ are pathological [1]. By leaving out the zero-frequency photons, one is led to the Feynman form for radiated power

$$P_F = \begin{cases} -\dfrac{q^2}{6\pi\varepsilon_0 c^3} \dot{\boldsymbol{a}} \cdot \boldsymbol{v} & \text{if } \dot{\boldsymbol{a}} \cdot \boldsymbol{v} < 0 \\ 0 & \text{if } \dot{\boldsymbol{a}} \cdot \boldsymbol{v} \geq 0 \end{cases} \qquad \text{(Feynman power)}$$

which not only resolves the paradox (a uniformly accelerating charge does not radiate), but also gives rise to non-pathological equations of motion as shown in the main text Section 2.2.