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Non-Spectrality of Certain Self-Affine Measures on the Generalized Spatial Sierpinski Gasket

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Abstract

Let $\mu_{M,D}$ be a self-affine measure associated with an expanding integer matrix $M=[p_1,0,0;p_4,p_2,0;p_5,0,p_3]$ and the digit set $D=\{0,e_1,e_2,e_3\}$ in the space $\mathbf{R}^3$, where $p_1,p_2,p_3\in\mathbf{Z}\setminus\{0,\pm1\}$, $p_4,p_5\in\mathbf{Z}$ and e_1,e_2,e_3 are the standard basis of unit column vectors in $\mathbf{R}^3$. In this paper, we mainly consider the case $p_1,p_2,p_3\in 2\mathbf{Z}+1,\ p_2\neq p_3,\ p_4=l(p_1-p_2),\ p_5=l(p_3-p_1)$, where $l\in 2\mathbf{Z}$. We prove that $\mu_{M,D}$ is a non-spectral measure, and there are at most 4-element $\mu_{M,D}$-orthogonal exponentials, and the number 4 is the best. The results here generalize the known results.

Keywords

Sierpinski Gasket, Non-Spectrality, Orthogonal Exponentials, Digit Set

1. Introduction

Let $M\in M_n(\mathbf{Z})$ be an expanding integer matrix (that is, all the eigenvalues $|\lambda_i(M)|>1$) and $D\subset\mathbf{Z}^n$ be a finite subset of cardinality $|D|$. The unique probability measure $\mu:=\mu_{M,D}$ satisfying the self-affine identity

$$\mu=\frac{1}{|D|}\sum_{d\in D}\mu\circ\phi_d^{-1} \tag{1.1}$$

with equal weight, where $\phi_d(\xi)=M^{-1}(\xi+d)\ (\xi\in\mathbf{R}^n)$ forms an affine iterated function system (IFS) $\{\phi_d\}_{d\in D}$. $\mu_{M,D}$ is also called invariant measure or self-similar measure. Such μ is supported on an invariant set $T(M,D)$ (see [1]), which is a unique nonempty compact set satisfying

$$MT=\bigcup_{d\in D}(T+d).$$

If there exists a set $\Lambda\subset\mathbf{R}^n$ such that $E(\Lambda):=\{e^{2\pi i\langle\lambda,x\rangle}:\lambda\in\Lambda\}$ forms an orthogonal basis (Fourier basis) for the Hilbert space $L^2(\mu_{M,D})$, we call $\mu_{M,D}$ a *spectral measure*. The set Λ is then called a *spectrum* for $\mu_{M,D}$ and the pair (μ,Λ) is called a *spectrum pair*. The study of spectral measures dates back to the work of Fuglede [2], whose famous spectral-tiling conjecture relation is difficult to establish in most cases. The research on the spectrality or non-spectrality of $\mu_{M,D}$ become a hot topic, which has its origin in number theory, harmonic analysis, fractal geometry and dynamical systems [3]-[6].

Jorgensen and Pedersen found the first non-atomic, singular continuous spectral measure, which showed that the Fourier transform theory can be applied to certain classes of fractals [3] [7]. In all these studies, the Fourier transform $\hat{\mu}_{M,D}$ of $\mu_{M,D}$ plays an important role. From (1.1), $\hat{\mu}_{M,D}(\xi)$ is given by

$$\hat{\mu}_{M,D}(\xi)=\int e^{2\pi i\langle x,\xi\rangle}\mathrm{d}\mu_{M,D}(x)=\prod_{j=1}^{\infty}m_D\left(M^{*-j}\xi\right),\quad \left(\xi\in\mathbf{R}^n\right)$$

where M^* denotes the transposed conjugate of M and

$$m_D(\xi)=\frac{1}{|D|}\sum_{d\in D}e^{2\pi i\langle d,\xi\rangle},\ \left(\xi\in\mathbf{R}^n\right)$$

denotes the symbol function of D. It is known that there are several methods to deal with the spectrality of self-affine measure (see [8]-[11] and references cited therein). Compared with spectral self-affine measures, there are a lot of non-spectral self-affine measures.

Usually, the orthogonal exponentials in $L^2(\mu_{M,D})$ is called $\mu_{M,D}$-orthogonal exponentials. Generally, the non-spectral problem can be divided into the following two classes.

Class (I): There are at most a finite number of orthogonal exponentials in $L^2(\mu_{M,D})$ (but the supremum of these finite numbers may be infinite, see [12] [13] and references cited therein), that is, $\mu_{M,D}$-orthogonal exponentials contain at most finite elements. The main questions here are to estimate the number of orthogonal exponentials in $L^2(\mu_{M,D})$ and to find them (see [14] [15]).

Class (II): There are natural infinite families of orthogonal exponentials in $L^2(\mu_{M,D})$, but none of them forms an orthogonal basis in $L^2(\mu_{M,D})$. The main question here is whether some of these families can be combined to form larger collections of orthogonal exponentials (see [16]-[18]).

For the generalized spatial Sierpinski gasket corresponding to

$$M=\begin{bmatrix}p_1&0&0\\p_4&p_2&0\\p_5&p_6&p_3\end{bmatrix}\text{ and }D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\},\qquad(1.2)$$

where $p_1,p_2,p_3\in\mathbf{Z}\setminus\{0,\pm1\}$ and $p_4,p_5,p_6\in\mathbf{Z}$. The previous results on the non-spectrality of self-affine measures can be summarized as the following.

Theorem 1.1. For self-affine measure $\mu_{M,D}$ given by (1.2) and $p_4=p_5=p_6=0$, the following spectrality and non-spectrality hold (see [12] [19]-[22]):

1) If $p_1,p_2,p_3\in(2\mathbf{Z}+1)\setminus\{\pm1\}$, then $\mu_{M,D}$ is a non-spectral measure, and

there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 4 is the best;

2) If $p_1 \in 2\mathbf{Z}\setminus\{0\}$ and $p_2 = p_3 \in (2\mathbf{Z}+1)\setminus\{\pm 1\}$, then there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, and the number 4 is the best;

3) If $p_1 \in 2\mathbf{Z}\setminus\{0\}$ and $p_2 = -p_3 \in (2\mathbf{Z}+1)\setminus\{\pm 1\}$, then there exist at most 8 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 8 is the best possible;

4) If $p_1 \in 2\mathbf{Z}\setminus\{0\}$ and $p_2 \neq \pm p_3 \in (2\mathbf{Z}+1)\setminus\{\pm 1\}$, then for any $l \in \mathbf{N}$, there exist $2l+6$ mutually orthogonal exponential functions in $L^2(\mu_{M,D})$.

Theorem 1.2. [23] Let $\mu_{M,D}$ corresponding to (1.2), if
$p_1, p_2, p_3 \in (2\mathbf{Z}+1)\setminus\{\pm 1\}$ and

$$p_4 = p_1 - p_2,\ \gamma p_6 = \eta(p_3 - p_2),\ \gamma(p_5 - p_1 + p_3) = \eta(p_2 - p_1) \tag{1.3}$$

for some $\beta, \gamma \in 2\mathbf{Z}+1$ and $\eta \in 2\mathbf{Z}$, then $\mu_{M^*,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M^*,D})$, where the number 4 is the best.

Theorem 1.3. [24] *L*et $\mu_{M,D}$ corresponding to (1.2) and $p_6 = 0$, if

$$p_1,\ p_2 = p_3 \in (2\mathbf{Z}+1)\setminus\{0, \pm 1\}, \tag{1.4}$$

then $\mu_{M,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best.

Theorem 1.4. [25] Let $\mu_{M,D}$ corresponding to (1.2) and $p_4 = 0$,
$p_5 = -p_6 = \xi \in R$, if

$$p_1 = p_2, p_3 \in \left\{\frac{p}{q} : p, q \in 2\mathbf{Z}+1\right\}, \tag{1.5}$$

then $\mu_{M^*,D}$ is a non-spectral measure, and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M^*,D})$, where the number 4 is the best.

In the case $p_6 = 0$, one can rewrite (1.3) as follows

$$p_4 = p_1 - p_2,\ p_5 = p_1 - p_3. \tag{1.6}$$

From Theorem 1.2, we know that $\mu_{M,D}$ is a non-spectral measure in the case (1.6), and there exist at most 4 mutually orthogonal exponential functions in $L^2(\mu_{M,D})$, where the number 4 is the best. In Theorem 1.3, Yuan is given the exact number of $\mu_{M,D}$-orthogonal exponentials in the case (1.5). This naturally leads to an open problem: How about the spectrality or non-spectrality of $\mu_{M,D}$ in the case

$$p_1, p_2, p_3 \in (2\mathbf{Z}+1)\setminus\{0, \pm 1\},\ p_2 \neq p_3? \tag{1.7}$$

Motivated by this, in the present paper, we resolve the above question in some cases. The main result of the paper is the following.

Theorem 1.5. Let the self-affine measure $\mu_{M,D}$ corresponding to (1.2) (1.7) and $p_6 = 0$, if

$$p_4 = l(p_1 - p_2),\quad p_5 = l(p_3 - p_1), \tag{1.8}$$

where $l\in 2\mathbf{Z}$, then $\mu_{M,D}$ is a non-spectral measure, there are at most 4-element $\mu_{M,D}$-orthogonal exponentials, where the number 4 is the best.

2. The Zero Set $Z\left(\hat{\mu}_{M,D}\right)$ of $\mu_{M,D}$

For the given digit set D in (1.2), it is known [19] that the zero set

$$Z\left(m_D\right):=\left\{x\in\mathbf{R}^3:m_D\left(x\right)=0\right\}=Z_1\cup Z_2\cup Z_3,$$

where

$$Z_1=\left\{\begin{pmatrix}\frac{1}{2}+k_1\\ a+k_2\\ \frac{1}{2}+a+k_3\end{pmatrix}:a\in\mathbf{R},k_1,k_2,k_3\in\mathbf{Z}\right\}\subset\mathbf{R}^3;$$

$$Z_2=\left\{\begin{pmatrix}\frac{1}{2}+a+k_1\\ \frac{1}{2}+k_2\\ a+k_3\end{pmatrix}:a\in\mathbf{R},k_1,k_2,k_3\in\mathbf{Z}\right\}\subset\mathbf{R}^3;$$

$$Z_3=\left\{\begin{pmatrix}a+k_1\\ \frac{1}{2}+a+k_2\\ \frac{1}{2}+k_3\end{pmatrix}:a\in\mathbf{R},k_1,k_2,k_3\in\mathbf{Z}\right\}\subset\mathbf{R}^3.$$

Then the zero set of Fourier transform $\hat{\mu}_{M,D}$ can be presented as

$$Z\left(\hat{\mu}_{M,D}\right)=\bigcup_{j=1}^{\infty}M^{*j}Z\left(m_D\right):=A_1\cup A_2\cup A_3, \tag{2.1}$$

where

$$A_1=\bigcup_{j=1}^{\infty}\left\{\begin{pmatrix}m\\ \left(a+k_2\right)p_2^j\\ \left(\frac{1}{2}+a+k_3\right)p_3^j\end{pmatrix}:a\in\mathbf{R},\ k_1,k_2,k_3\in\mathbf{Z}\right\}\subset\mathbf{R}^3; \tag{2.2}$$

$$A_2=\bigcup_{j=1}^{\infty}\left\{\begin{pmatrix}n\\ \left(\frac{1}{2}+k_2\right)p_2^j\\ \left(a+k_3\right)p_3^j\end{pmatrix}:a\in\mathbf{R},\ k_1,k_2,k_3\in\mathbf{Z}\right\}\subset\mathbf{R}^3; \tag{2.3}$$

$$A_3=\bigcup_{j=1}^{\infty}\left\{\begin{pmatrix}p\\ \left(\frac{1}{2}+a+k_2\right)p_2^j\\ \left(\frac{1}{2}+k_3\right)p_3^j\end{pmatrix}:a\in\mathbf{R},\ k_1,k_2,k_3\in\mathbf{Z}\right\}\subset\mathbf{R}^3 \tag{2.4}$$

and

$$m=\left(\frac{1}{2}+k_1\right)p_1^j+\left(a+k_2\right)\sum_{i=0}^{j-1}p_1^ip_2^{j-1-i}p_4+\left(\frac{1}{2}+a+k_3\right)\sum_{i=0}^{j-1}p_1^ip_3^{j-1-i}p_5;$$

$$n=\left(\frac{1}{2}+a+k_1\right)p_1^j+\left(\frac{1}{2}+k_2\right)\sum_{i=0}^{j-1}p_1^ip_2^{j-1-i}p_4+\left(a+k_3\right)\sum_{i=0}^{j-1}p_1^ip_3^{j-1-i}p_5;$$

$$p=\left(a+k_1\right)p_1^j+\left(\frac{1}{2}+a+k_2\right)\sum_{i=0}^{j-1}p_1^ip_2^{j-1-i}p_4+\left(\frac{1}{2}+k_3\right)\sum_{i=0}^{j-1}p_1^ip_3^{j-1-i}p_5.$$

Combined with (1.7) and (1.8), one can verify the following propositions hold.

Proposition 2.1. The sets A_1, A_2, A_3 given by (2.2), (2.3) and (2.4) hold the following statements:

(i) $\zeta\in A_j$ if and only if $-\zeta\in A_j$, $j=1,2,3$;

(ii) $Z\left(\hat{\mu}_{M,D}\right)\cap\mathbf{Z}^3=Z\left(\hat{\mu}_{M,D}\right)\cap\left\{(x,y,z)^t:x,y,z\in\frac{1}{2}+\mathbf{Z}\right\}=\varnothing$;

(ii) If $\zeta=\left(\zeta_1,\zeta_2,\zeta_3\right)^t\in A_1$, then $\zeta_1+l\zeta_2-l\zeta_3\in\frac{1}{2}+\mathbf{Z}$;

(iv) If $\zeta=\left(\zeta_1,\zeta_2,\zeta_3\right)^t\in A_2$, then $\zeta_2\in\frac{1}{2}+\mathbf{Z}$;

(v) If $\zeta=\left(\zeta_1,\zeta_2,\zeta_3\right)^t\in A_3$, then $\zeta_3\in\frac{1}{2}+\mathbf{Z}$.

Proposition 2.2. Let $\zeta=\left(\zeta_1,\zeta_2,\zeta_3\right)^t\in Z\left(\hat{\mu}_{M,D}\right)$. Then the following statements hold:

(a) If $\zeta\in A_1\pm A_1$, then $\zeta\in A_2\cup A_3$ and $\zeta_1+l\zeta_2-l\zeta_3\in\mathbf{Z}$;

(b) If $\zeta\in A_2\pm A_2$, then $\zeta\in A_1\cup A_3$ and $\zeta_2\in\mathbf{Z},\zeta_3\notin\mathbf{Z}$;

(c) If $\zeta\in A_3\pm A_3$, then $\zeta\in A_1\cup A_2$ and $\zeta_3\in\mathbf{Z},\zeta_2\notin\mathbf{Z}$;

(d) If $\zeta\in A_1$ and $\zeta_2\in\mathbf{Z}$, then $\zeta_3\notin\mathbf{Z}$; if $\zeta\in A_1$ and $\zeta_3\in\mathbf{Z}$, then $\zeta_2\notin\mathbf{Z}$;

(e) If $\zeta\in A_2$ and $\zeta_3\in\mathbf{Z}$, then $\zeta_1\notin\mathbf{Z}$;

(f) If $\zeta\in A_3$ and $\zeta_2\in\mathbf{Z}$, then $\zeta_1\notin\mathbf{Z}$.

3. Proof of Theorem 1.5

Assume that if $\lambda_j\left(j=1,2,\cdots,5\right)\in\mathbf{R}^3$ are such that the five functions

$$e^{2\pi i\langle\lambda_1,x\rangle},e^{2\pi i\langle\lambda_2,x\rangle},e^{2\pi i\langle\lambda_3,x\rangle},e^{2\pi i\langle\lambda_4,x\rangle},e^{2\pi i\langle\lambda_5,x\rangle}$$

are mutually orthogonal in $L^2\left(\mu_{M,D}\right)$, then

$$\lambda_i-\lambda_j\in Z\left(\hat{\mu}_{M,D}\right)=A_1\cup A_2\cup A_3,\ \left(1\le i\ne j\le 5\right).$$

Equivalently, the following 10 differences

$$\lambda_2-\lambda_1,\lambda_3-\lambda_1,\lambda_4-\lambda_1,\lambda_5-\lambda_1;$$
$$\lambda_3-\lambda_2,\lambda_4-\lambda_2,\lambda_5-\lambda_2;$$
$$\lambda_4-\lambda_3,\lambda_5-\lambda_3;$$
$$\lambda_5-\lambda_4$$

belong to $A_1\cup A_2\cup A_3$. In particular, we have

$$\lambda_2-\lambda_1,\lambda_3-\lambda_1,\lambda_4-\lambda_1,\lambda_5-\lambda_1\in A_1\cup A_2\cup A_3. \tag{3.1}$$

For convenience, define

$$\lambda_j - \lambda_k = \left(x_{j,k}, y_{j,k}, z_{j,k}\right)^t \in \mathbf{R}^3 \text{ for } j,k \geq 1 \text{ and } j \neq k.$$

From Proposition 2.1 and 2.2, one can obtain the following claims.

Claim 3.1. The set A_1 (or A_2 or A_3) can not contain three differences of the form $\lambda_{j_1}-\lambda_j, \lambda_{j_2}-\lambda_j, \lambda_{j_3}-\lambda_j$, where $j_1, j_2, j_3 \in \{1,2,\cdots,5\}\setminus\{j\}$ are three different numbers.

In fact, if

$$\lambda_{j_1}-\lambda_j, \lambda_{j_2}-\lambda_j, \lambda_{j_3}-\lambda_j \in A_1, \tag{3.2}$$

then $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_2} \in A_2 \cup A_3$ (by Proposition 2.2 (a)). If $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_2} \in A_2$, then (by Proposition 2.1 (iv)), $y_{j_2,j_1}, y_{j_3,j_1} \in \frac{1}{2}+\mathbf{Z}$ and $y_{j_3,j_2} \in \mathbf{Z}$, a contradiction. The same reason illustrates that $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_2}$ can not in the set A_3 simultaneously. Without loss of generality, we may assume that $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1} \in A_2, \lambda_{j_3}-\lambda_{j_2} \in A_3$. By Proposition 2.1 (iv), Proposition 2.2 (a) and (3.2), we have $y_{j_2,j_1}, y_{j_3,j_1}, z_{j_3,j_2} \in \frac{1}{2}+\mathbf{Z}$ and $x_{j_3,j_2}+ly_{j_3,j_2}-lz_{j_3,j_2} \in \mathbf{Z}$, then $x_{j_3,j_2}, y_{j_3,j_2} \in \mathbf{Z}$, a contradiction (by Proposition 2.2 (f)). Hence, the set A_1 can not contain three differences of the form $\lambda_{j_1}-\lambda_j, \lambda_{j_2}-\lambda_j, \lambda_{j_3}-\lambda_j$. If

$$\lambda_{j_1}-\lambda_j, \lambda_{j_2}-\lambda_j, \lambda_{j_3}-\lambda_j \in A_2, \tag{3.3}$$

then $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_2} \in A_1 \cup A_3$ (by Proposition 2.2 (b)). If $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_2} \in A_1$, then (by Proposition 2.1 (iii))

$$x_{j_2,j_1}+ly_{j_2,j_1}-lz_{j_2,j_1}, x_{j_3,j_1}+ly_{j_3,j_1}-lz_{j_3,j_1} \in \frac{1}{2}+\mathbf{Z}$$
$$\text{and } x_{j_3,j_2}+ly_{j_3,j_2}-lz_{j_3,j_2} \in \mathbf{Z},$$

a contradiction of Proposition 2.1 (iii). If $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_2} \in A_3$, then $z_{j_2,j_1}, z_{j_3,j_1} \in \frac{1}{2}+\mathbf{Z}$ and $z_{j_3,j_2} \in \mathbf{Z}$ (by Proposition 2.1 (v)), a contradiction. Without loss of generality, let

$$\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1} \in A_1, \lambda_{j_3}-\lambda_{j_2} \in A_3. \tag{3.4}$$

Then from Proposition 2.1 (v), Proposition 2.2 (b) and (3.3) (3.4), we have $y_{j_3,j_2} \in \mathbf{Z}$, $z_{j_3,j_2} \in \frac{1}{2}+\mathbf{Z}$ and $x_{j_2,j_1}+ly_{j_2,j_1}-lz_{j_2,j_1}$, $x_{j_3,j_1}+ly_{j_3,j_1}-lz_{j_3,j_1} \in \frac{1}{2}+\mathbf{Z}$, which yields $x_{j_3,j_2} \in \mathbf{Z}$, a contradiction of Proposition 2.2 (f). If $\lambda_{j_2}-\lambda_{j_1}, \lambda_{j_3}-\lambda_{j_1} \in A_3$, $\lambda_{j_3}-\lambda_{j_2} \in A_1$, from Proposition 2.1 (v), Proposition 2.2 (b) and (3.3), we have $z_{j_2,j_1}, z_{j_3,j_1} \in \frac{1}{2}+\mathbf{Z}$, then $y_{j_3,j_2}, z_{j_3,j_2} \in \mathbf{Z}$, a contradiction of Proposition 2.2 (d). Hence, the set A_2 can not contain three differences of the form $\lambda_{j_1}-\lambda_j, \lambda_{j_2}-\lambda_j, \lambda_{j_3}-\lambda_j$. This proof is also suitable for the set A_3.

Based on Claim 3.1, we only need to consider the following four cases:

Case 1. 2-2-0 (2-0-2) distribution. In this case, we may assume (without loss of generality) that

$$\lambda_2-\lambda_1, \lambda_3-\lambda_1 \in A_1,\ \lambda_4-\lambda_1, \lambda_5-\lambda_1 \in A_2. \tag{3.5}$$

Then (by Proposition 2.1 (iv) and Proposition 2.2 (a) (b))

$$x_{3,2}+ly_{3,2}-lz_{3,2} \in \mathbf{Z},\ y_{4,1}, y_{5,1} \in \frac{1}{2}+\mathbf{Z} \tag{3.6}$$

and

$$\lambda_3-\lambda_2 \in A_2 \cup A_3, \lambda_5-\lambda_4 \in A_1 \cup A_3. \tag{3.7}$$

We can divide (3.7) into four cases.

Case 1.1. $\lambda_3-\lambda_2 \in A_2, \lambda_5-\lambda_4 \in A_1$. In this case, from Proposition 2.1 (iii) (iv), we have

$$y_{3,2} \in \frac{1}{2}+\mathbf{Z}, x_{5,4}+ly_{5,4}-lz_{5,4} \in \frac{1}{2}+\mathbf{Z}. \tag{3.8}$$

If $\lambda_5-\lambda_2 \in A_1$, then (by Proposition 2.1 (iii) and (3.6) (3.8))

$$x_{5,2}+ly_{5,2}-lz_{5,2} \in \frac{1}{2}+\mathbf{Z} \text{ and } x_{4,2}+ly_{4,2}-lz_{4,2}, x_{4,3}+ly_{4,3}-lz_{4,3} \in \mathbf{Z}, \tag{3.9}$$

which yields $\lambda_4-\lambda_2, \lambda_4-\lambda_3 \in A_2 \cup A_3$. In the case $\lambda_4-\lambda_2 \in A_2$, from (3.8) and Proposition 2.2 (b), we have $y_{4,3} \in \mathbf{Z}$. Hence, $\lambda_4-\lambda_3 \in A_3$ and $z_{4,3} \in \frac{1}{2}+\mathbf{Z}$. Combined with (3.9), we get $x_{4,3} \in \mathbf{Z}$, a contradiction. In the case $\lambda_4-\lambda_2 \in A_3$, from Proposition 2.1 (v), we have

$$z_{4,2} \in \frac{1}{2}+\mathbf{Z}. \tag{3.10}$$

If $\lambda_4-\lambda_3 \in A_2$, then $y_{4,2}, x_{4,2} \in \mathbf{Z}$ (by Proposition 2.2 (b) and (3.8) (3.9)), a contradiction. If $\lambda_4-\lambda_3 \in A_3$, then $z_{3,2}, x_{3,2} \in \mathbf{Z}$ (by Proposition 2.2 (c) and (3.8) (3.10)), a contradiction. Hence, $\lambda_5-\lambda_2 \notin A_1$. The same reason illustrates that $\lambda_5-\lambda_3, \lambda_4-\lambda_2, \lambda_4-\lambda_3 \notin A_1$. Hence

$$\lambda_4-\lambda_2, \lambda_4-\lambda_3, \lambda_5-\lambda_2, \lambda_5-\lambda_3 \in A_2 \cup A_3.$$

Based on Claim 3.1, we only need to consider the following three cases.

Case 1.1.1 $\lambda_4-\lambda_2 \in A_3, \lambda_5-\lambda_2 \in A_3$. In this case, from Proposition 2.2 (c) and (3.6), we have $y_{5,4}, z_{5,4} \in \mathbf{Z}$ and $\lambda_5-\lambda_4 \in A_1$, a contradiction (by Proposition 2.2 (d)).

Case 1.1.2 $\lambda_4-\lambda_2 \in A_2, \lambda_5-\lambda_2 \in A_3$. In this case, from Proposition 2.2 (b) and (3.6), we have $y_{2,1} \in \mathbf{Z}$, $y_{5,2} \in \frac{1}{2}+\mathbf{Z}$. Combined with $\lambda_3-\lambda_2 \in A_2$, we get $y_{5,3} \in \mathbf{Z}$ and $\lambda_5-\lambda_3 \in A_3$. Now, $\lambda_3-\lambda_2=(\lambda_5-\lambda_2)-(\lambda_5-\lambda_3) \in A_3-A_3$, we have $x_{3,2}, z_{3,2} \in \mathbf{Z}$, a contradiction.

Case 1.1.3 $\lambda_4-\lambda_2 \in A_3, \lambda_5-\lambda_3 \in A_2$. This case can be proved similarly to Case 1.1.2.

Case 1.2. $\lambda_3-\lambda_2 \in A_3, \lambda_5-\lambda_4 \in A_1$. This case can be proved similarly to Case 1.1.

Case 1.3. $\lambda_3 - \lambda_2, \lambda_5 - \lambda_4 \in A_3$. In this case, by Proposition 2.1 (v) and (3.6), we have

$$z_{3,2}, z_{5,4} \in \frac{1}{2} + \mathbf{Z}, y_{5,4} \in \mathbf{Z}. \tag{3.11}$$

We first show that the following Claim holds.

Claim 3.2. The set $\{\lambda_4 - \lambda_2, \lambda_4 - \lambda_3, \lambda_5 - \lambda_2, \lambda_5 - \lambda_3\}$ has at least one element in A_1.

In fact, if

$$\lambda_4 - \lambda_2, \lambda_4 - \lambda_3, \lambda_5 - \lambda_2, \lambda_5 - \lambda_3 \in A_2 \cup A_3, \tag{3.12}$$

then, from Claim 3.1, we need to divide (3.12) into the following three cases.

Case A: $\lambda_4 - \lambda_2 \in A_2, \lambda_5 - \lambda_2 \in A_2$. In this case, from (3.6) Proposition 2.1 (iii) and Proposition 2.2 (b), we have

$$y_{4,2}, y_{5,2} \in \frac{1}{2} + \mathbf{Z} \text{ and } y_{2,1} \in \mathbf{Z}. \tag{3.13}$$

In the case $\lambda_4 - \lambda_3 \in A_2$, from (3.6) (3.11) (3.13) and Proposition 2.2 (b), we have $x_{3,2}, y_{3,2} \in \mathbf{Z}$, a contradiction. In the case $\lambda_4 - \lambda_3 \in A_3$, from (3.11) and (3.12), we have $\lambda_5 - \lambda_3 \in A_2$ and $y_{5,3} \in \frac{1}{2} + \mathbf{Z}$. Combined with (3.6) (3.11) (3.13), we get $x_{3,2}, y_{3,2} \in \mathbf{Z}$, a contradiction.

Case B: $\lambda_4 - \lambda_2 \in A_2, \lambda_5 - \lambda_2 \in A_3$. In this case, from Proposition 2.2 (b) and (3.6), we have $y_{2,1} \in \mathbf{Z}$ and $y_{5,2} \in \frac{1}{2} + \mathbf{Z}$. For $\lambda_5 - \lambda_3 = (\lambda_5 - \lambda_2) - (\lambda_3 - \lambda_2) \in A_3 - A_3$ and (3.12) (3.13), we get $y_{5,3} \in \frac{1}{2} + \mathbf{Z}$ and $y_{3,2} \in \mathbf{Z}$. Combined with (3.6) and (3.9), we have $x_{3,2} \in \mathbf{Z}$, a contradiction.

Case C: $\lambda_4 - \lambda_2 \in A_3, \lambda_5 - \lambda_2 \in A_2$. This case can be proved similarly to Case B.

Therefore, the set $\{\lambda_4 - \lambda_2, \lambda_4 - \lambda_3, \lambda_5 - \lambda_2, \lambda_5 - \lambda_3\}$ has at least one element in A_1.

From Claim 3.2, without loss of generality, we may assume that $\lambda_4 - \lambda_2 \in A_1$, then

$$x_{4,2} + l y_{4,2} - l z_{4,2} \in \frac{1}{2} + \mathbf{Z}. \tag{3.14}$$

By Proposition 2.2 (a) (f) and (3.11), we get that $\lambda_5 - \lambda_2 \notin A_1$ and $\lambda_5 - \lambda_3 \notin A_1$. Then $\lambda_5 - \lambda_2, \lambda_5 - \lambda_3 \in A_2 \cup A_3$. Based on Claim 3.1, we need to consider the following three cases.

Case 1.3.1 $\lambda_5 - \lambda_2 \in A_2, \lambda_5 - \lambda_3 \in A_2$. In this case, from (3.6) (3.11) and Proposition 2.2 (b), we have $x_{3,2}, y_{3,2} \in \mathbf{Z}$, a contradiction.

Case 1.3.2 $\lambda_5 - \lambda_2 \in A_2, \lambda_5 - \lambda_3 \in A_3$. In this case, for $\lambda_5 - \lambda_2 = (\lambda_5 - \lambda_3) - (\lambda_2 - \lambda_3) \in A_3 - A_3$ and Proposition 2.2 (b), we get $y_{2,1}, z_{5,2} \in \mathbf{Z}$. Combined with (3.6) (3.11) (3.14), we have $x_{4,2}, y_{4,2}, z_{4,2} \in \frac{1}{2} + \mathbf{Z}$, a contradiction of Proposition 2.1 (ii).

Case 1.3.3 $\lambda_5 - \lambda_2 \in A_3, \lambda_5 - \lambda_3 \in A_2$. This case can be proved similarly to Case 1.3.2.

Case 1.4 $\lambda_3-\lambda_2\in A_2,\lambda_5-\lambda_4\in A_3$. In this case, from (3.6) and Proposition 2.1 (iv) (v), we have

$$y_{3,2},z_{5,4}\in\frac{1}{2}+\mathbf{Z}. \tag{3.15}$$

With the same method as Claim 3.2, we can prove that the set $\{\lambda_4-\lambda_2,\lambda_4-\lambda_3,\lambda_5-\lambda_2,\lambda_5-\lambda_3\}$ has at least one element in A_3. Without loss of generality, we may assume that $\lambda_4-\lambda_2\in A_3$, then (by Proposition 2.1 (v))

$$z_{4,2}\in\frac{1}{2}+\mathbf{Z}. \tag{3.16}$$

By Proposition 2.2 (c) and (3.15), we get $z_{5,2}\in\mathbf{Z}$ and $\lambda_5-\lambda_2\in A_1\cup A_2$.

If $\lambda_5-\lambda_2\in A_1$, then $x_{5,1}+ly_{5,1}-lz_{5,1}\in\mathbf{Z}$ (by Proposition 2.2 (a) and (3.6)). In the case $\lambda_4-\lambda_3\in A_1$, from Proposition 2.2 (a) and (3.6), we have $x_{4,1}+ly_{4,1}-lz_{4,1}\in\mathbf{Z}$, $x_{5,4}+ly_{5,4}-lz_{5,4}\in\mathbf{Z}$. Combined with (3.6) (3.15), we get $x_{5,4}\in\mathbf{Z}$, a contradiction. In the case $\lambda_4-\lambda_3\in A_2$, from Proposition 2.1 (iv) and (3.6) (3.15), we have $y_{4,3}\in\frac{1}{2}+\mathbf{Z}$ and $y_{4,2}\in\mathbf{Z}$. Combined with (3.15) (3.16), we have $z_{5,2}\in\mathbf{Z}$, a contradiction. In the case $\lambda_4-\lambda_3\in A_3$, for $\lambda_3-\lambda_2=(\lambda_4-\lambda_2)-(\lambda_4-\lambda_3)\in A_3-A_3$ and (3.6), we get $x_{3,2},z_{3,2}\in\mathbf{Z}$, a contradiction.

If $\lambda_5-\lambda_2\in A_2$, then $y_{2,1}\in\mathbf{Z}$ and $y_{4,2}\in\frac{1}{2}+\mathbf{Z}$ (by Proposition 2.2 (b) and (3.6)). In the case $\lambda_4-\lambda_3\in A_1$, from Proposition 2.2 (a) and (3.5), we get

$$x_{4,1}+ly_{4,1}-lz_{4,1}\in\mathbf{Z}. \tag{3.17}$$

By (3.5) (3.16) and (3.17), we have $x_{4,2}+ly_{4,2}-lz_{4,2}\in\frac{1}{2}+\mathbf{Z}$ and $x_{4,2}\in\frac{1}{2}+\mathbf{Z}$, a contradiction of Proposition 2.1 (ii). In the case $\lambda_4-\lambda_3\in A_2$, from Proposition 2.1 (iv) and (3.15), we have $y_{2,1}\in\frac{1}{2}+\mathbf{Z}$ and $y_{5,2}\in\mathbf{Z}$, a contradiction. In the case $\lambda_4-\lambda_3\in A_3$, from Proposition 2.1 (v) and (3.6) (3.15) (3.16), we get $x_{3,2},z_{3,2}\in\mathbf{Z}$, a contradiction.

Similarly, we can prove 2-0-2 distribution does not hold.

Case 2 2-1-1 distribution. In this case, we may assume that

$$\lambda_2-\lambda_1,\lambda_3-\lambda_1\in A_1,\lambda_4-\lambda_1\in A_2,\lambda_5-\lambda_1\in A_3, \tag{3.18}$$

then (by Proposition 2.1 (iv)(v) and Proposition 2.2 (a))

$$y_{4,1},z_{5,1}\in\frac{1}{2}+\mathbf{Z}\text{ and }x_{3,2}+ly_{3,2}-lz_{3,2}\in\mathbf{Z} \tag{3.19}$$

and

$$\lambda_3-\lambda_2\in A_2\cup A_3. \tag{3.20}$$

We can divide (3.20) into two cases.

Case 2.1. $\lambda_3-\lambda_2\in A_2$. In this case, we have (by Proposition 2.1 (iv))

$$y_{3,2}\in\frac{1}{2}+\mathbf{Z}. \tag{3.21}$$

With the same method as Claim 3.2, we can prove that the set $\{\lambda_4-\lambda_2, \lambda_4-\lambda_3, \lambda_5-\lambda_2, \lambda_5-\lambda_3\}$ has at least one element in A_1. Without loss of generality, we may assume that $\lambda_4-\lambda_2 \in A_1$, then (by (3.6) and Proposition 2.2 (a))

$$x_{4,1}+ly_{4,1}-lz_{4,1} \in \mathbf{Z}. \tag{3.22}$$

If $\lambda_5-\lambda_2 \in A_1$, then (by Proposition 2.2 (a) and (3.6)),

$$x_{5,1}+ly_{5,1}-lz_{5,1} \in \mathbf{Z}. \tag{3.23}$$

By Proposition 2.1 (iii) and (3.22) (3.23), we get $\lambda_5-\lambda_4 \in A_2 \cup A_3$. In the case $\lambda_5-\lambda_4 \in A_2$, from Proposition 2.2 (b) and (3.19) (3.23), we have $x_{5,1}, y_{5,1} \in \mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_4 \in A_3$, from Proposition 2.2 (c) and (3.19) (3.22), we have $x_{4,1}, z_{4,1} \in \mathbf{Z}$, a contradiction. Hence, $\lambda_5-\lambda_2 \notin A_1$. The same reason illustrates that $\lambda_5-\lambda_3 \notin A_1$. Then

$$\lambda_5-\lambda_2, \lambda_5-\lambda_3 \in A_2 \cup A_3.$$

Based on Claim 3.1, we need to consider the following three cases.

Case 2.1.1 $\lambda_5-\lambda_2 \in A_3, \lambda_5-\lambda_3 \in A_3$. From Proposition 2.2 (c), we have $\lambda_3-\lambda_2=(\lambda_5-\lambda_2)-(\lambda_5-\lambda_3) \in A_3-A_3$. Combined with (3.19) (3.21), we have $x_{3,2}, z_{3,2} \in \mathbf{Z}$, a contradiction.

Case 2.1.2 $\lambda_5-\lambda_2 \in A_2, \lambda_5-\lambda_3 \in A_3$. From Proposition 2.1 (v) Proposition 2.2 (c) and (3.21), we have

$$z_{3,1}, y_{5,3} \in \mathbf{Z}, z_{5,3} \in \frac{1}{2}+\mathbf{Z}. \tag{3.24}$$

By Claim 3.1 and (3.22) (3.33), we know that $\lambda_5-\lambda_4 \in A_2 \cup A_3$. In the case $\lambda_5-\lambda_4 \in A_2$, from Proposition 2.2 (b) and (3.19) (3.23), we have $x_{5,1}, y_{5,1} \in \mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_4 \in A_3$, from Proposition 2.2 (c) and (3.19) (3.22), we have $x_{4,1}, y_{4,1} \in \mathbf{Z}$, a contradiction.

Case 2.1.3 $\lambda_5-\lambda_2 \in A_3, \lambda_5-\lambda_3 \in A_2$. This case can be proved similarly to Case 2.1.2.

Case 2.2. $\lambda_3-\lambda_2 \in A_3$. This case can be proved similarly to Case 2.1.

Case 3 0-2-2 distribution. In this case, we may assume that

$$\lambda_2-\lambda_1, \lambda_3-\lambda_1 \in A_2, \lambda_4-\lambda_1, \lambda_5-\lambda_1 \in A_3, \tag{3.25}$$

then (by Proposition 2.1 (iv) (v) and Proposition 2.2 (b) (c))

$$y_{2,1}, y_{3,1}, z_{4,1}, z_{5,1} \in \frac{1}{2}+\mathbf{Z} \text{ and } y_{3,2}, z_{5,4} \in \mathbf{Z} \tag{3.26}$$

and

$$\lambda_3-\lambda_2 \in A_1 \cup A_3 \text{ and } \lambda_5-\lambda_4 \in A_1 \cup A_2. \tag{3.27}$$

We can divide (3.27) into four cases.

Case 3.1. $\lambda_3-\lambda_2 \in A_1, \lambda_5-\lambda_4 \in A_1$. In this case, we have

$$x_{3,2}+ly_{3,2}-lz_{3,2}, x_{5,4}+ly_{5,4}-lz_{5,4} \in \frac{1}{2}+\mathbf{Z}. \tag{3.28}$$

With the same method as Claim 3.2, we can prove that the set $\{\lambda_4-\lambda_2, \lambda_4-\lambda_3, \lambda_5-\lambda_2, \lambda_5-\lambda_3\}$ has at least one element in A_1. Without loss of generality, we may assume that $\lambda_4 - \lambda_2 \in A_1$, then

$$x_{4,2} + ly_{4,2} - lz_{4,2} \in \frac{1}{2} + \mathbf{Z}. \quad (3.29)$$

From Claim 3.1 and (3.28) (3.29), we have $\lambda_5 - \lambda_2, \lambda_4 - \lambda_3 \in A_2 \cup A_3$.

In the case $\lambda_5 - \lambda_2 \in A_2, \lambda_4 - \lambda_3 \in A_2$, from Proposition 2.2 (b) and (3.26) (3.28), we have $y_{4,1}, y_{5,1} \in \mathbf{Z}$ and $y_{5,4} \in \mathbf{Z}$, a contradiction.

In the case $\lambda_5 - \lambda_2 \in A_3, \lambda_4 - \lambda_3 \in A_3$, from Proposition 2.2 (c) and (3.26), we have $z_{2,1}, z_{3,1} \in \mathbf{Z}$ and $y_{3,2}, z_{3,2} \in \mathbf{Z}$, a contradiction.

In the case $\lambda_5 - \lambda_2 \in A_3, \lambda_4 - \lambda_3 \in A_2$, from Proposition 2.2 (b) (c) and (3.26) (3.29), we have $z_{2,1}, y_{4,1} \in \mathbf{Z}$ and $x_{4,2}, y_{4,2}, z_{4,2} \in \frac{1}{2} + \mathbf{Z}$, a contradiction.

In the case $\lambda_5 - \lambda_2 \in A_2, \lambda_4 - \lambda_3 \in A_3$, from Proposition 2.2 (b) (c) and (3.26) (3.28) (3.29), we have $z_{3,1}, y_{5,1}, x_{5,2} + ly_{5,2} - lz_{5,2} \in \mathbf{Z}$ and $x_{5,3} + ly_{5,3} - lz_{5,3} \in \frac{1}{2} + \mathbf{Z}$. Combined with (3.26), we get $x_{5,3}, y_{5,3}, z_{5,3} \in \frac{1}{2} + \mathbf{Z}$, a contradiction.

Case 3.2. $\lambda_3 - \lambda_2 \in A_1, \lambda_5 - \lambda_4 \in A_2$. In this case, from Proposition 2.1 (iii) (iv), we have

$$y_{5,4} \in \frac{1}{2} + \mathbf{Z}, x_{3,2} + ly_{3,2} - lz_{3,2} \in \frac{1}{2} + \mathbf{Z}. \quad (3.30)$$

With the same method as Claim 3.2, we can prove that the set $\{\lambda_4 - \lambda_2, \lambda_4 - \lambda_3, \lambda_5 - \lambda_2, \lambda_5 - \lambda_3\}$ has at least one element in A_1. Without loss of generality, we may assume that $\lambda_4 - \lambda_2 \in A_1$, then (by Proposition 2.1 (iii))

$$x_{4,2} + ly_{4,2} - lz_{4,2} \in \frac{1}{2} + \mathbf{Z}. \quad (3.31)$$

From Claim 3.1 and Proposition 2.2 (a), we have $\lambda_5 - \lambda_2, \lambda_4 - \lambda_3 \in A_2 \cup A_3$. In the case $\lambda_5 - \lambda_2 \in A_2$, $\lambda_4 - \lambda_3 \in A_2$, from Proposition 2.2 (b) and (3.26) (3.30), we have $y_{4,1}, y_{5,1} \in \mathbf{Z}$ and $y_{5,4} \in \mathbf{Z}$, a contradiction. In the case $\lambda_5 - \lambda_2 \in A_3$, $\lambda_4 - \lambda_3 \in A_3$, from Proposition 2.2 (c) and (3.26), we have $z_{2,1}, z_{3,1} \in \mathbf{Z}$ and $y_{3,2}, z_{3,2} \in \mathbf{Z}$, a contradiction. In the case $\lambda_5 - \lambda_2 \in A_3$, $\lambda_4 - \lambda_3 \in A_2$, from Proposition 2.2 (b) (c) and (3.26) (3.31), we have $z_{2,1}, y_{4,1} \in \mathbf{Z}$ and $x_{4,2}, y_{4,2}, z_{4,2} \in \frac{1}{2} + \mathbf{Z}$, a contradiction. In the case $\lambda_5 - \lambda_2 \in A_2$, $\lambda_4 - \lambda_3 \in A_3$, from $\lambda_4 - \lambda_2 = \lambda_5 - \lambda_2 - (\lambda_5 - \lambda_4) \in A_2 - A_2$ and (3.26), we have $y_{4,2} \in \mathbf{Z}$, $y_{4,1} \in \frac{1}{2} + \mathbf{Z}$ and $y_{4,3} \in \mathbf{Z}$. Combined with $\lambda_4 - \lambda_3 = \lambda_4 - \lambda_2 - (\lambda_3 - \lambda_2) \in A_1 - A_1$ and Proposition 2.2 (a), we get $x_{4,3} \in \mathbf{Z}$, a contradiction.

Case 3.3. $\lambda_3 - \lambda_2 \in A_3, \lambda_5 - \lambda_4 \in A_1$. This case can be proved similarly to Case 3.2.

Case 3.4. $\lambda_3 - \lambda_2 \in A_3, \lambda_5 - \lambda_4 \in A_2$. In this case, from Proposition 2.1 (iv) (v), we have

$$z_{3,2}, y_{5,4} \in \frac{1}{2} + \mathbf{Z}. \quad (3.32)$$

With the same method as Claim 3.2, we can prove that the set $\{\lambda_4-\lambda_2,\lambda_4-\lambda_3,\lambda_5-\lambda_2,\lambda_5-\lambda_3\}$ has at least one element in A_1. Without loss of generality, we may assume that $\lambda_4-\lambda_2\in A_1$, then

$$x_{4,2}+ly_{4,2}-lz_{4,2}\in\frac{1}{2}+\mathbf{Z}. \tag{3.33}$$

From Claim 3.1 and Proposition 2.2 (a), we have $\lambda_5-\lambda_2,\lambda_4-\lambda_3\in A_2\cup A_3$. In the case $\lambda_5-\lambda_2\in A_2$, $\lambda_4-\lambda_3\in A_2$, from Proposition 2.2 (b) and (3.26), we have $y_{4,1},y_{5,1}\in\mathbf{Z}$ and $y_{5,4}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_2\in A_2$, $\lambda_4-\lambda_3\in A_3$, from Proposition 2.2 (b) (c) and (3.32), we have $y_{4,2},z_{4,2}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_2\in A_3$, $\lambda_4-\lambda_3\in A_2$, from Proposition 2.2 (b) (c) and (3.32), we have $y_{5,3},z_{5,3}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_2\in A_3$, $\lambda_4-\lambda_3\in A_3$, from Proposition 2.2 (c) and (3.25), we have $z_{3,1},z_{2,1}\in\mathbf{Z}$, a contradiction of (3.32).

Case 4. 1-2-1 distribution. In this case, we may assume that

$$\lambda_2-\lambda_1\in A_1,\lambda_3-\lambda_1,\lambda_4-\lambda_1\in A_2,\lambda_5-\lambda_1\in A_3, \tag{3.34}$$

then (by Proposition 2.1 (iii) (iv) (v))

$$y_{3,1},y_{4,1},z_{5,1},x_{2,1}+ly_{2,1}-lz_{2,1}\in\frac{1}{2}+\mathbf{Z} \tag{3.35}$$

and

$$\lambda_4-\lambda_3\in A_1\cup A_3. \tag{3.36}$$

We can divide (3.36) into two cases.

Case 4.1. $\lambda_4-\lambda_3\in A_1$. In this case, we have

$$x_{4,3}+ly_{4,3}-lz_{4,3}\in\frac{1}{2}+\mathbf{Z}. \tag{3.37}$$

We can prove the following Claim hold.

Claim 3.3. The set $\{\lambda_3-\lambda_2,\lambda_4-\lambda_2\}$ has at least one element in A_1.

In fact, if

$$\lambda_3-\lambda_2,\lambda_4-\lambda_2\in A_2\cup A_3.$$

Then, based on Claim 3.1, we need to consider the following three cases.

Case (a): $\lambda_3-\lambda_2,\lambda_4-\lambda_2\in A_3$. In this case, from (3.35) and Proposition 2.2 (c), we have $y_{4,3},z_{4,3}\in\mathbf{Z}$, a contradiction.

Case (b): $\lambda_3-\lambda_2,\lambda_4-\lambda_2\in A_2$. In this case, from (3.35) and Proposition 2.2 (b), we have

$$y_{2,1}\in\mathbf{Z}. \tag{3.38}$$

If $\lambda_5-\lambda_3,\lambda_5-\lambda_4\in A_2$, then $y_{5,1},y_{5,2}\in\mathbf{Z}$ and $\lambda_5-\lambda_2\in A_1\cup A_3$ (by Proposition 2.1 (iv) and (3.38)). In the case $\lambda_5-\lambda_2\in A_1$, from Proposition 2.2 (a) and (3.35), we have $x_{5,1}+ly_{5,1}-lz_{5,1}\in\mathbf{Z}$ and $x_{5,1}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_2\in A_3$, from Proposition 2.2 (c) and (3.38), we get $y_{2,1},z_{2,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_3,\lambda_5-\lambda_4\in A_3$, then (by Proposition 2.2 (c) and (3.38)) $y_{4,3},z_{4,3}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_3\in A_2$, $\lambda_5-\lambda_4\in A_3$, then, $y_{5,1},y_{5,2}\in\mathbf{Z}$ and $\lambda_5-\lambda_2\in A_1\cup A_3$ (by Proposition 2.2 (c) and (3.38)). In the case $\lambda_5-\lambda_2\in A_1$, from Proposition 2.2 (a) and (3.35), we have $x_{5,1}+ly_{5,1}-lz_{5,1}\in\mathbf{Z}$ and $x_{5,1}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_2\in A_3$, from Proposition 2.2 (c) and (3.38), we get $y_{2,1}, z_{2,1}\in\mathbf{Z}$, a contradiction. Hence the set $\{\lambda_5-\lambda_3, \lambda_5-\lambda_4\}$ has at least one element in A_1. In this case, without loss of generality, we may assume that

$$\lambda_5-\lambda_3\in A_1. \tag{3.39}$$

It follows from Claim 3.1 that $\lambda_5-\lambda_2\in A_1\cup A_3$. If $\lambda_5-\lambda_2\in A_1$, then (by Proposition 2.2 (a) and (3.35) (3.38))

$$x_{5,1}+ly_{5,1}-lz_{5,1}\in\mathbf{Z}. \tag{3.40}$$

From Claim 3.1, we know that $\lambda_5-\lambda_4\in A_2\cup A_3$. In the case $\lambda_5-\lambda_4\in A_2$, from Proposition 2.2 (b) and (3.35) (3.40), we have $x_{5,1}, y_{5,1}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_4\in A_3$, from (3.37) (3.39) (3.40) and Proposition 2.2 (c), we get $x_{4,1}+ly_{4,1}-lz_{4,1}\in\mathbf{Z}$ and $x_{4,1}, z_{4,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_2\in A_3$, then $y_{2,1}, z_{2,1}\in\mathbf{Z}$ (by Proposition 2.2 (c) and (3.35) (3.38)), a contradiction.

Case (c): $\lambda_3-\lambda_2\in A_2, \lambda_4-\lambda_2\in A_3$. In this case, from (3.35) and Proposition 2.2 (b), we have

$$y_{2,1}\in\mathbf{Z} \text{ and } y_{4,2}, z_{4,2}\in\frac{1}{2}+\mathbf{Z}. \tag{3.41}$$

With the same method as Case (b), we can prove that the set $\{\lambda_5-\lambda_3, \lambda_5-\lambda_4\}$ has at least one element in A_1. In this case, without loss of generality, we may assume that

$$\lambda_5-\lambda_3\in A_1. \tag{3.42}$$

Combined with (3.37), we have $\lambda_5-\lambda_4\in A_2\cup A_3$. If $\lambda_5-\lambda_4\in A_2$, from Proposition 2.2 (b) and (3.35) (3.41), we have

$$y_{5,1}, y_{5,2}\in\mathbf{Z} \text{ and } \lambda_5-\lambda_2\in A_1\cup A_3. \tag{3.43}$$

In the case $\lambda_5-\lambda_2\in A_1$, from Proposition 2.2 (a) and (3.35) (3.43), we have $x_{5,1}+ly_{5,1}-lz_{5,1}\in\mathbf{Z}$ and $x_{5,1}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_2\in A_3$, from Proposition 2.2 (c) and (3.41), we get $y_{2,1}, z_{2,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_4\in A_3$, then (by Proposition 2.2 (c) and (3.35) (3.37) (3.42))

$$z_{4,1}, z_{5,2}\in\mathbf{Z}, z_{5,4}, z_{2,1}\in\frac{1}{2}+\mathbf{Z} \text{ and } x_{5,4}+ly_{5,4}-lz_{5,4}\in\mathbf{Z} \tag{3.44}$$

and $\lambda_5-\lambda_2\in A_1\cup A_2$. In the case $\lambda_5-\lambda_2\in A_1$, from Proposition 2.2 (a) and (3.35) (3.44), we have $x_{5,1}+ly_{5,1}-lz_{5,1}$, $x_{4,1}+ly_{4,1}-lz_{4,1}\in\mathbf{Z}$ and $x_{4,1}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_2\in A_2$, from Proposition 2.1 (iv) and (3.35) (3.41) (3.44), we have $y_{5,1}\in\frac{1}{2}+\mathbf{Z}$ and $x_{5,4}, y_{5,4}\in\mathbf{Z}$, a contradiction.

Hence the set $\{\lambda_3-\lambda_2, \lambda_4-\lambda_2\}$ has at least one element in A_1. In this case, without loss of generality, we may assume that $\lambda_3-\lambda_2\in A_1$, then (by Proposition 2.2 (a) and (3.37))

$$x_{3,1}+ly_{3,1}-lz_{3,1}\in\mathbf{Z} \text{ and } x_{4,2}+ly_{4,2}-lz_{4,2}\in\mathbf{Z} \tag{3.45}$$

and $\lambda_4-\lambda_2\in A_2\cup A_3$.

If $\lambda_4-\lambda_2\in A_2$, then

$$y_{2,1}\in\mathbf{Z}. \tag{3.46}$$

If $\lambda_5-\lambda_2\in A_1$, then (by Proposition 2.2 (a) and (3.35) (3.45))

$$x_{5,1}+ly_{5,1}-lz_{5,1},\ x_{5,3}+ly_{5,3}-lz_{5,3}\in\mathbf{Z} \tag{3.47}$$

and $\lambda_5-\lambda_3\in A_2\cup A_3$. In the case $\lambda_5-\lambda_3\in A_2$, from Proposition 2.2 (b) and (3.35) (3.47), we get $x_{5,1},y_{5,1}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_3\in A_3$, from Proposition 2.2 (c) and (3.35) (3.45), we have $x_{3,1},z_{3,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_2\in A_2$, then (by (3.35) (3.46))

$$y_{5,1}\in\frac{1}{2}+\mathbf{Z},\ y_{5,3}\in\mathbf{Z} \tag{3.48}$$

and $\lambda_5-\lambda_3\in A_1\cup A_3$. In the case $\lambda_5-\lambda_3\in A_1$, from Proposition 2.1 (iii) and (3.35) (3.37) (3.45) (3.48), we have

$$x_{5,4}+ly_{5,4}-lz_{5,4},\ y_{5,4}\in\mathbf{Z}. \tag{3.49}$$

Hence, $\lambda_5-\lambda_4\in A_3$ and $z_{5,4}\in\frac{1}{2}+\mathbf{Z}$. Combined with (3.49), we get $x_{5,4}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_3\in A_3$, from (3.35) (3.45), we have $x_{3,1},z_{3,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_2\in A_3$, then (by Proposition 2.2 (c) and (3.46)) $z_{2,1}\in\mathbf{Z}$, a contradiction.

If $\lambda_4-\lambda_2\in A_3$, then (by Proposition 2.1 (v) and (3.35) (3.45))

$$z_{4,2}\in\frac{1}{2}+\mathbf{Z},\ x_{4,2}+ly_{4,2}-lz_{4,2}\in\mathbf{Z}\text{ and }x_{4,1}+ly_{4,1}-lz_{4,1}\in\frac{1}{2}+\mathbf{Z}. \tag{3.50}$$

If $\lambda_5-\lambda_2\in A_1$, then (by Proposition 2.2 (a) and (3.35) (3.45))

$$x_{5,1}+ly_{5,1}-lz_{5,1}\in\mathbf{Z} \tag{3.51}$$

and $\lambda_5-\lambda_3\in A_2\cup A_3$. In the case $\lambda_5-\lambda_3\in A_2$, from Proposition 2.2 (b) and (3.35) (3.51), we have $x_{5,1},y_{5,1}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_3\in A_3$, from Proposition 2.2 (c) and (3.35) (3.45), we get $x_{3,1},z_{3,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_2\in A_2$, then (by Proposition 2.1 (iv))

$$y_{5,2}\in\frac{1}{2}+\mathbf{Z}. \tag{3.52}$$

If $\lambda_5-\lambda_3\in A_1$, then (by Proposition 2.2 (a) and (3.37)) $x_{5,4}+ly_{5,4}-lz_{5,4}\in\mathbf{Z}$ and $\lambda_5-\lambda_4\in A_2\cup A_3$. In the case $\lambda_5-\lambda_4\in A_2$, from Proposition 2.2 (b) and (3.50) (3.52), we get $x_{4,2},y_{4,2}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_4\in A_3$, from Proposition 2.2 (c) (3.50) (3.52) and $\lambda_5-\lambda_2=(\lambda_5-\lambda_3)-(\lambda_2-\lambda_3)\in A_1-A_1$, we have $z_{4,1}\in\mathbf{Z},\ z_{2,1}\in\frac{1}{2}+\mathbf{Z}$ and $x_{5,2},z_{5,2}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_3\in A_2$, then (by Proposition 2.2 (b) and (3.35) (3.52)) $y_{3,2}\in\mathbf{Z}$ and $y_{2,1}\in\frac{1}{2}+\mathbf{Z}$. Combined with (3.35) and (3.50), we have $x_{4,2},y_{4,2}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_3\in A_3$, then (by Proposition 2.2 (c) and (3.35) (3.45)) $x_{3,1},z_{3,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_2\in A_3$, then (by Proposition 2.2 (c) and (3.35)) $z_{2,1}\in\mathbf{Z}$. Combined with (3.35) and (3.50), we have $x_{4,1},y_{4,1},z_{4,1}\in\frac{1}{2}+\mathbf{Z}$, a contradiction.

Case 4.2. $\lambda_4 - \lambda_3 \in A_3$. In this case, we have

$$y_{4,3} \in \mathbf{Z},\ z_{4,3} \in \frac{1}{2} + \mathbf{Z}. \tag{3.53}$$

With the same method as Claim 3.3, we can prove that the set $\{\lambda_3 - \lambda_2, \lambda_4 - \lambda_2\}$ has at least one element in A_1. Without loss of generality, we may assume that $\lambda_3 - \lambda_2 \in A_1$, then (by Proposition 2.2 (a) and (3.35))

$$x_{3,1} + ly_{3,1} - lz_{3,1} \in \mathbf{Z}. \tag{3.54}$$

If $\lambda_4 - \lambda_2 \in A_1$, then (by Proposition 2.2 (a) and (3.35) (3.53) (3.54)) $x_{4,1} + ly_{4,1} - lz_{4,1}$, $x_{4,3} + ly_{4,3} - lz_{4,3} \in \mathbf{Z}$ and $x_{4,3} \in \mathbf{Z}$, a contradiction.

If $\lambda_4 - \lambda_2 \in A_2$, then (by Proposition 2.2 (b) and (3.35))

$$y_{2,1} \in \mathbf{Z}. \tag{3.55}$$

If $\lambda_5 - \lambda_2 \in A_1$, then (by Proposition 2.2 (a) and (3.35))

$$x_{5,1} + ly_{5,1} - lz_{5,1} \in \mathbf{Z}. \tag{3.56}$$

Combined with (3.54) and (3.55), we have $x_{5,3} + ly_{5,3} - lz_{5,3} \in \mathbf{Z}$ and $\lambda_5 - \lambda_3 \in A_2 \cup A_3$. In the case $\lambda_5 - \lambda_3 \in A_2$, from Proposition 2.2 (b) and (3.35) (3.56), we get $x_{5,1}, y_{5,1} \in \mathbf{Z}$, a contradiction. In the case $\lambda_5 - \lambda_3 \in A_3$, from Proposition 2.2 (c) and (3.35) (3.54), we have $x_{3,1}, z_{3,1} \in \mathbf{Z}$, a contradiction. If $\lambda_5 - \lambda_2 \in A_2$, then (by Proposition 2.1 (iv) and (3.35) (3.55))

$$y_{5,1} \in \frac{1}{2} + \mathbf{Z},\ y_{5,3} \in \mathbf{Z} \tag{3.57}$$

and $\lambda_5 - \lambda_3 \in A_1 \cup A_3$. In the case $\lambda_5 - \lambda_3 \in A_1$, from Proposition 2.1 (iii) and (3.54), we get

$$x_{5,1} + ly_{5,1} - lz_{5,1} \in \frac{1}{2} + \mathbf{Z}. \tag{3.58}$$

Combined with (3.35) and (3.57), we have $x_{5,1} \in \frac{1}{2} + \mathbf{Z}$, a contradiction. In the case $\lambda_5 - \lambda_3 \in A_3$, from Proposition 2.2 (c) and (3.54), we get $x_{3,1}, z_{3,1} \in \mathbf{Z}$, a contradiction. If $\lambda_5 - \lambda_2 \in A_3$, from Proposition 2.2 (c) and (3.55), we have $y_{2,1}, z_{2,1} \in \mathbf{Z}$, a contradiction.

If $\lambda_4 - \lambda_2 \in A_3$, then (by Proposition 2.1 (v))

$$z_{4,2} \in \frac{1}{2} + \mathbf{Z}. \tag{3.59}$$

Combined with (3.53), we have

$$z_{3,2} \in \mathbf{Z}. \tag{3.60}$$

If $\lambda_5 - \lambda_2 \in A_1$, then (by Proposition 2.2 (a) and (3.35) (3.54))

$$x_{5,1} + ly_{5,1} - lz_{5,1} \in \mathbf{Z} \text{ and } x_{5,3} + ly_{5,3} - lz_{5,3} \in \mathbf{Z}. \tag{3.61}$$

Hence, $\lambda_5 - \lambda_3 \in A_2 \cup A_3$. In the case $\lambda_5 - \lambda_3 \in A_2$, from Proposition 2.2 (b) and (3.35) (3.61), we have $x_{5,1}, y_{5,1} \in \mathbf{Z}$, a contradiction. In the case $\lambda_5 - \lambda_3 \in A_3$, from Proposition 2.2 (c) and (3.35) (3.54), we get $x_{3,1}, z_{3,1} \in \mathbf{Z}$, a contradiction. If

$\lambda_5-\lambda_2\in A_2$, then (by Proposition 2.1 (iv))

$$y_{5,2}\in\frac{1}{2}+\mathbf{Z}. \tag{3.62}$$

In the case $\lambda_5-\lambda_3\in A_1$, for $\lambda_4-\lambda_2,\lambda_4-\lambda_3\in A_3$, Proposition 2.1 (i) and Claim 3.1, we have $\lambda_5-\lambda_4\in A_1\cup A_2$. If $\lambda_5-\lambda_4\in A_1$, then, for $\lambda_4-\lambda_3=(\lambda_5-\lambda_3)-(\lambda_5-\lambda_4)\in A_1-A_1$ and (3.53), we get $x_{4,3},y_{4,3}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_4\in A_2$, then (by Proposition 2.2 (b) and (3.35) (3.60) (3.62)), $y_{2,1}\in\frac{1}{2}+\mathbf{Z}$ and $y_{5,1},y_{3,2},z_{3,2}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_3\in A_2$, from Proposition 2.2 (b) and (3.35) (3.60) (3.62), we get $y_{3,2},z_{3,2}\in\mathbf{Z}$, a contradiction. In the case $\lambda_5-\lambda_3\in A_3$, from (3.35) and (3.54), we have $x_{3,1},z_{3,1}\in\mathbf{Z}$, a contradiction. If $\lambda_5-\lambda_2\in A_3$, then (by Proposition 2.2 (c) and (3.35) (3.59)) $z_{2,1}\in\mathbf{Z},z_{4,1}\in\frac{1}{2}+\mathbf{Z}$. Combined with (3.53) and (3.54), we have $x_{3,1},z_{3,1}\in\mathbf{Z}$, a contradiction.

The above discussion shows that $L^2(\mu_{M,D})$ can not contain five mutually orthogonal exponential functions. Take

$$\Lambda=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}\frac{(1-l)p_1^j+lp_3^j}{2}\\0\\\frac{p_3^j}{2}\end{pmatrix},\begin{pmatrix}\frac{(1-l)p_1^j-lp_2^j}{2}\\\frac{p_2^j}{2}\\0\end{pmatrix},\begin{pmatrix}\frac{l(p_2^j+p_3^j)}{2}\\\frac{-p_2^j}{2}\\\frac{p_3^j}{2}\end{pmatrix}\right\},$$

where $p_1,p_2,p_3\in(2\mathbf{Z}+1)\setminus\{0,\pm1\},l\in2\mathbf{Z}$. We can verify that $E(\Lambda)$ is a 4-element $\mu_{M,D}$-orthogonal exponentials, which yields the number 4 is the best.

Corollary 3.4. For the self-affine measure $\mu_{M,D}$ corresponding to

$$M=\begin{bmatrix}P_1&0&0\\P_4&P_2&0\\P_5&0&P_3\end{bmatrix}\text{ and }D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}d\\0\\0\end{pmatrix},\begin{pmatrix}0\\d\\0\end{pmatrix},\begin{pmatrix}0\\0\\d\end{pmatrix}\right\},$$

if $p_j\in(2\mathbf{Z}+1)\setminus\{0,\pm1\}(j=1,2,3),p_2\neq p_3,d\neq0$ and

$$p_4=l(p_1-p_2),\quad p_5=l(p_3-p_1),$$

where $l\in2\mathbf{Z}$, then there are at most 4-element $\mu_{M,D}$-orthogonal exponentials, and the number 4 is the best.

Example 3.5. Let

$$M=\begin{bmatrix}5&0&0\\-52&31&0\\36&0&23\end{bmatrix},\quad D=\left\{\begin{pmatrix}0\\0\\0\end{pmatrix},\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\1\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\},$$

then there are at most 4-element $\mu_{M,D}$-orthogonal exponentials, and the number 4 is the best.

Take $l=2\in2\mathbf{Z}$, one can verify that $p_1=5$, $p_2=31$, $p_3=23\in(2\mathbf{Z}+1)\setminus\{0,\pm1\}$, $p_4=2(p_1-p_2)=-52$, $p_5=2(p_3-p_1)=36$, which shows that the condition (1.8) holds. Then there are at most 4-element $\mu_{M,D}$-orthogonal exponentials, and the number 4 is the best.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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