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On the Degree Resistance Distance of Unicyclic Graphs

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Abstract

Let G be a connected graph with vertex set $V(G)$. Then the degree resistance distance of G is defined as $D_R(G)=\sum_{\{u,v\}\subseteq V(G)}(d(u)+d(v))R(u,v)$, where $d(u)$ is the degree of the vertex u, and $R(u,v)$ is the degree resistance distance between u and v in graph G. A unicyclic graph is a connected graph with a unique cycle. In this paper, we characterize the unique graph with the third-maximum degree resistance distance among all unicyclic graphs with n vertices.

Keywords

Degree Resistance Distance, Unicyclic Graph, Resistance Distance, Kirchhoff Index

1. Introduction

In this paper, a graph means a simple undirected graph. Let $G=(V,E)$ be a graph with n vertices and m edges. The degree of v in G, denoted by $d_G(v)$, is equal to the number of vertices adjacent to v in G. The distance between two vertices u and v of G, denoted by $d_G(u,v)$, is the length of a shortest path joining u and v in G. For notations and terminologies undefined here, the readers may refer to [1].

In theoretical chemistry, to correlate molecular structures with the physicochemical and bilogical properties of molecules, many topological indices have been introduced and investigated over the years. One of the most studied problems in chemical graph theory is to characterize extremal graphs with respect to certain topological indices among the set of all trees, unicyclic graphs, bicyclic graphs, tricyclic graphs, etc. The Wiener index, as the oldest and one of the most popular molecular structure descriptors [2] [3], is well correlated with many physical and chemical properties of a variety of classes of chemical compounds. This topological index was introduced in 1947 [4] and is defined as the sum of distances between all pairs of vertices, namely that

$$W(G)=\sum_{\{u,v\}\subseteq V(G)} d(u,v).$$

For details on its mathematical properties, see the survey [5].

A modified version of the Wiener index, introduced by Dobrynin, Kochetova and Gutman [6], is the degree distance defined as

$$D(G)=\sum_{\{u,v\}\subseteq V(G)} \left(d(u)+d(v)\right)d(u,v).$$

In 2006, Yuan and An [7] presented the maximum value of degree distance for unicyclic graphs. Later, Tomescu [8] determined the unicyclic and bicyclic graphs with minimum degree distance. Some further mathematical results on degree distance can be found in [9] [10]-[13].

Based on the theory of electrical networks, Klein and Randić [14] 1993 introduced a new distance function named resistance distance. They viewed a graph G as an electric network N such that each edge of G is assumed as a unit resistor. The resistance distance between two vertices u and v of G, denoted by $R(u,v)=R(u,v|G)$, is defined as the effective resistance between the nodes u and v in N. This new kind of distance between vertices of a graph was eventually studied in detail [15]-[20]. By replacing the ordinary distance with resistance distance in the expression for the Wiener index, we can arrive at the Kirchhoff index

$$Kf(G)=\sum_{\{u,v\}\subseteq V(G)} R(u,v),$$

which has been widely studied [19] [21]-[25].

In analogy with the degree distance of a graph, the degree resistance distance of a graph G was first proposed by Gutman, Feng and Yu [26] as

$$D_R(G)=\sum_{\{u,v\}\subseteq V(G)} \left(d_G(u)+d_G(v)\right)R_G(u,v).$$

Palacios [27] named the same graph invariant "*additive degree-Kirchhoff index*". In [26], some properties of D_R-index are given, and the unicyclic graphs with minimum and second minimum D_R-value are determined. Later, the unicyclic graphs with the maximum and the second-maximum D_R-value are characterized in [28] [29]. Qi *et al.* [30] considered the maximum D_R-value of n-vertex unicyclic graphs with a given maximum degree. The first three minimum D_R-values among all cacti with n vertices and t cycles are completely determined in [31] [32].

To the best of our knowledge, the third-maximum value of degree resistance distance for unicyclic graphs has not been considered so far. We will solve it in this paper. Recall that a unicyclic graph is a connected graph with the same number of vertices and edges. Denote by $\mathcal{U}(n)$ the set of all unicyclic graphs of order n. Let P_n, C_n and S_n be the path, the cycle and the star on n vertices, respectively.

This paper is organized as follows. In Section 2, we introduce some transformations to compare the degree resistance distance of two graphs. In Section 3, we determine the unique graph with the third-maximum degree resistance distance among unicyclic graphs in $\mathcal{U}(n)$.

2. Lemmas

Let $R_G(u,v)$ denote the resistance distance between u and v in the graph G. It is known that $R_G(u,v)=R_G(v,u)$ and $R_G(u,v)\geq 0$ with equality if and only if $u=v$. For a vertex v in G, we define $Kf_v(G)=\sum_{u\in V(G)}R_G(u,v)$ and $D_v(G)=\sum_{u\in V(G)}d_G(u)R_G(u,v)$.

For the sake of brevity, in the whole of our context, for any two vertices u,v of G (or G'), we let $d(v)=d_G(v)$ (or $d'(v)=d_{G'}(v)$) and $R(u,v)=R_G(u,v)$ (or $R'(u,v)=R_{G'}(u,v)$). In the following, we give some necessary lemmas that will be used to prove our main results.

Lemma 1 ([14]) *Let G be a graph, x be a cut vertex of G and let u,v be vertices belonging to different components which arise upon deletion of x. Then* $R_G(u,v)=R_G(u,x)+R_G(x,v)$.

Gutman *et al.* [26] presented the following exact formulas for cycles.

Lemma 2 ([26]) *Let C_k be a cycle with length k and $v\in C_k$. Then*

$$Kf(C_k)=\frac{k^3-k}{12},\quad D_R(C_k)=\frac{k^3-k}{3} \text{ and } Kf_v(C_k)=\frac{k^2-1}{6}.$$

Definition 3 ([26]). Let v be a vertex of degree $p+1$ in a graph G, such that $vv_1, vv_2, \cdots, vv_p$ are pendant edges incident with v, and u is the neighbor of v distinct from $v_1, v_2, \cdots, v_p$. We form a graph $G'=\sigma(G,v)$ by deleting the edges $vv_1, vv_2, \cdots, vv_p$ and adding new edges $uv_1, uv_2, \cdots, uv_p$. We say that G' is a σ-transform of the graph G (see **Figure 1**).

Lemma 4 ([26]) *Let $G'=\sigma(G,v)$ be a σ-transform of the graph G, $d_G(u)\geq 1$. Then $D_R(G)\geq D_R(G')$. Equality holds if and only if G is a star with v as its center.*

Lemma 5 ([28]) *Let G_0 be a connected graph with $m_0>1$ edges, and $u,v\in V(G_0)$ be two distinct vertices with the degree at least 3 in G_0 such that $R(u,v)=l$. Let $P_s=u_1u_2\cdots u_s$ and $P_t=v_1v_2\cdots v_t$ be two paths of order $s\geq 1$ and $t\geq 1$, respectively. Let $G_{s,t}$ be the graph obtained from G_0, P_s and P_t by adding edges uu_1, vv_1. Suppose that $G_{s-1,t+1}=G_{s,t}-u_su_{s-1}+v_tu_s$ and $G_{s+1,t-1}=G_{s,t}-v_{t-1}v_t+u_sv_t$. Then either $D_R(G_{s,t})<D_R(G_{s-1,t+1})$ or $D_R(G_{s,t})<D_R(G_{s+1,t-1})$.*

Remark 6. From the proof of Lemma 5, we know that the conditions $d_{G_0}(u), d_{G_0}(v)\geq 3$ are not necessary. As long as G_0 is not a path, or G_0 is a path with $\max\{d_{G_0}(u), d_{G_0}(v)\}\geq 2$, the result in Lemma 5 always holds.

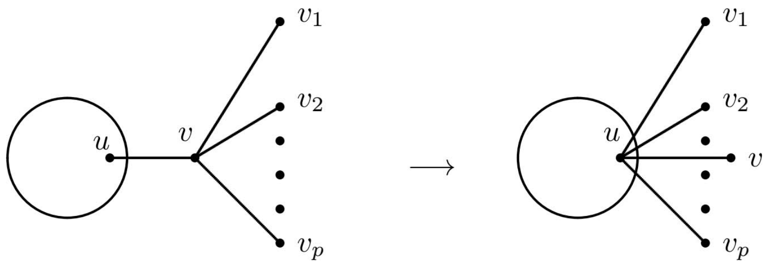

Figure 1. The σ-transform of the graph G.

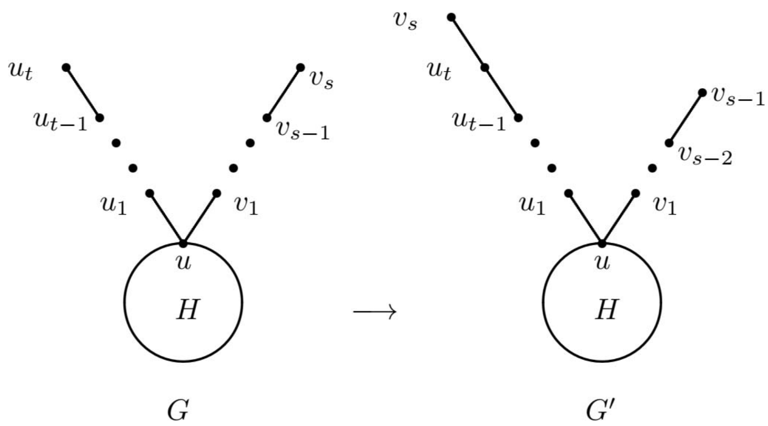

Figure 2. Graphs G and G' in Lemma 7.

We show in the following lemma that the result in Lemma 5 still holds if u and v are the same.

Lemma 7. *Let G be a connected graph with a cut vertex u and $d_G(u) \geq 3$. Let the paths $P_u = u_1u_2\cdots u_t$ and $P_v = v_1v_2\cdots v_s$ $(1 \leq s \leq t)$ be two connected components of $G-u$. Let $G' = G - v_{s-1}v_s + u_tv_s$ (see Figure 2), where $v_0 = u$ if $s=1$. Then $D_R(G) < D_R(G')$.*

Proof. Let $d(v_{s-1}) = a$. Then $a \geq 2$. Let $A = V(G) - \{v_{s-1}, v_s, u_t\}$, $B = \{v_{s-1}, v_s, u_t\}$, $C = \{u_1, u_2, \cdots, u_{t-1}\}$, $D = \{v_1, v_2, \cdots, v_{s-2}\}$. For the transformation from G to G', one has $R(x,y) = R'(x,y)$ for any $x, y \in A$, $d(x) = d'(x)$, $R'(x, v_{s-1}) = R(x, v_{s-1})$ and $R'(x,u_t) = R(x,u_t)$ for any $x \in A$. By the definition of the degree resistance distance, we get

$$D_1 = \sum_{\{x,y\}\subseteq A} \left[\left(d'(x)+d'(y)\right)R'(x,y) - \left(d(x)+d(y)\right)R(x,y)\right] = 0,$$

$$\begin{aligned}
D_2 &= \sum_{\{x,y\}\subseteq B} \left[\left(d'(x)+d'(y)\right)R'(x,y) - \left(d(x)+d(y)\right)R(x,y)\right] \\
&= \left(d'(v_{s-1})+d'(v_s)\right)R'(v_{s-1},v_s) - \left(d(v_{s-1})+d(v_s)\right)R(v_{s-1},v_s) \\
&\quad + \left(d'(v_{s-1})+d'(u_t)\right)R'(v_{s-1},u_t) - \left(d(v_{s-1})+d(u_t)\right)R(v_{s-1},u_t) \\
&\quad + \left(d'(v_s)+d'(u_t)\right)R'(v_s,u_t) - \left(d(v_s)+d(u_t)\right)R(v_s,u_t) \\
&= a(t+s) - (a+1) + (a+1)(s+t-1) - (a+1)(s+t-1) + 3 - 2(t+s) \\
&= (a-2)(t+s-1) \\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
D_3 &= \sum_{x\in A, y\in B}\left[\left(d'(x)+d'(y)\right)R'(x,y)-\left(d(x)+d(y)\right)R(x,y)\right] \\
&= \sum_{x\in A}\left[\left(d'(x)+d'(v_{s-1})\right)R'(x,v_{s-1})-\left(d(x)+d(v_{s-1})\right)R(x,v_{s-1})\right] \\
&\quad+\sum_{x\in A}\left[\left(d'(x)+d'(v_s)\right)R'(x,v_s)-\left(d(x)+d(v_s)\right)R(x,v_s)\right] \\
&\quad+\sum_{x\in A}\left[\left(d'(x)+d'(u_t)\right)R'(x,u_t)-\left(d(x)+d(u_t)\right)R(x,u_t)\right] \\
&= \sum_{x\in A}\left[\left(d(x)+a-1\right)R(x,v_{s-1})-\left(d(x)+a\right)R(x,v_{s-1})\right] \\
&\quad+\sum_{x\in A}\left[\left(d(x)+1\right)R'(x,v_s)-\left(d(x)+1\right)R(x,v_s)\right] \\
&\quad+\sum_{x\in A}\left[\left(d(x)+2\right)R(x,u_t)-\left(d(x)+1\right)R(x,u_t)\right] \\
&= \sum_{x\in A}R(x,u_t)-\sum_{x\in A}R(x,v_{s-1})+\sum_{x\in A}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&> \sum_{x\in A}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&:= \Gamma.
\end{aligned}$$

If $s\geq 2$, then $C\neq\varnothing$ and

$$\begin{aligned}
\Gamma &= \sum_{x\in C}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right)+\sum_{x\in D}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&\quad+\sum_{x\in A-(C\cup D)}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&= 3\left(\frac{(t-1)(2+t)}{2}-\frac{(t-1)(2s+t)}{2}\right)+3\left(\frac{(s-2)(2t+s+1)}{2}-\frac{(s-2)(s+1)}{2}\right) \\
&\quad+\sum_{x\in A-(C\cup D)}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&= 3s-3t-3+\sum_{x\in A-(C\cup D)}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&> 3s-3t-3+4(t+1-s) \\
&= t-s+1 \\
&> 0.
\end{aligned}$$

If $s=1$, then $D=\varnothing$ and

$$\begin{aligned}
\Gamma &= \sum_{x\in C}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right)+\sum_{x\in A-C}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&= 3\left(\frac{(t-1)(2+t)}{2}-\frac{(t-1)(2+t)}{2}\right)+\sum_{x\in A-C}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&= \sum_{x\in A-C}\left(d(x)+1\right)\left(R'(x,v_s)-R(x,v_s)\right) \\
&> 2(t+1-s) \\
&> 0.
\end{aligned}$$

Since $D_R(G')-D_R(G)=D_1+D_2+D_3$, we have $D_R(G')-D_R(G)>0$.

Let $\mathcal{U}(n,k)$ be the set of all unicyclic graphs with order n and girth k, and P_n^k denote the graph obtained by identifying one end-vertex of P_{n-k+1} with any vertex of C_k. Let $T(k,i,1)$ be the graph constructed from $P_{k-1}=v_1v_2\cdots v_{k-1}$ by adding one pendant edge to the vertex $v_i\,(1\le i\le k-2)$ and $U\left(C_l;T(k,i,1)\right)$ $(l\ge 3, 1\le i\le k-2, k+l=n+1)$ be the unicyclic graph obtained from a cycle C_l by identifying v_1 of $T(k,i,1)$ with a vertex of C_l. **Figure 3** illustrates the graphs P_n^3 and $U\left(C_3;T(n-2,n-4,1)\right)$.

Chen *et al.* [28] characterized the unicyclic graphs with the maximum and the second-maximum D_R-value. They also obtained the unique graph with maximum D_R-value among $\mathcal{U}(n,l)$.

Theorem 8 ([28]) *Let* $G\in\mathcal{U}(n,l)$, *then*

$$D_R(G)\le l^3-\frac{1}{3}(4n+3)l^2+nl+\frac{2}{3}n^3-\frac{1}{3}n,$$ *the equality holds if and only if* $G\cong P_n^l$.

Theorem 9 ([28] [29]) *Let* $G\in\mathcal{U}(n)$ *be an arbitrary unicyclic graph, then*

$$D_R(G)\le\frac{2n^3}{3}-\frac{28n}{3}+18,$$ *with the equality holds if and only if* $G\cong P_n^3$.

Theorem 10 (28) *Let* $G\in\mathcal{U}(n)(n\ge 6)$ *be an arbitrary unicyclic graph,* $G\ncong P_n^3$. *Then* $$D_R(G)\le\frac{2n^3}{3}-\frac{40n}{3}+28,$$ *with the equality holds if and only if* $G\cong U\left(C_3;T(n-2,n-4,1)\right)$.

Remark 11. From the proof of Theorem 8, we have $\max\limits_{4\le l\le n}\left\{D_R\left(P_n^l\right)\right\}=D_R\left(P_n^4\right)$.

3. The Third-Maximum Degree Resistance Distance of Unicyclic Graphs

In this section, we will determine the unique graph with the third-maximum degree resistance distance among $\mathcal{U}(n)$. Let P_n^4, G^* and $U\left(C_3;T(n-2,i,1)\right)$ $(1\le i\le n-5)$ be depicted in **Figure 4**.

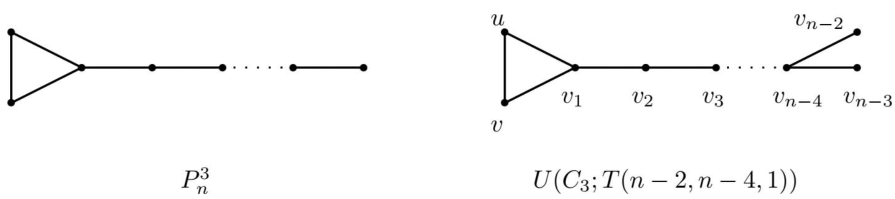

Figure 3. Graphs P_n^3 and $U\left(C_3;T(n-2,n-4,1)\right)$.

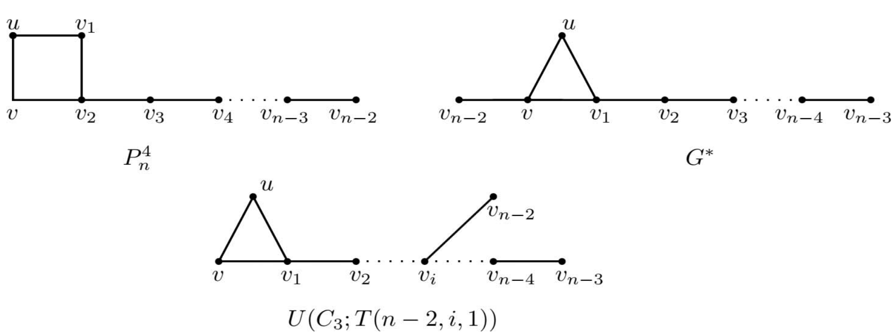

Figure 4. Graphs P_n^4, G^* and $U\left(C_3;T(n-2,i,1)\right)$.

Lemma 12. *Suppose* G *is a graph in* $\mathcal{U}(n)$ *having the third-maximum degree resistance distance, where* $n\ge 6$. *Then* $G\cong P_n^4$, *or* G^*, *or* $U(C_3;T(n-2,i,1))$ *for some* $1\le i\le n-5$.

Proof. Let l be the girth of G. By Theorem 8 and Remark 11, $G\cong P_n^4$ if $l\ge 4$. Suppose $l=3$ and $C=uvv_1$ be the cycle of G. We show that $G\cong G^*$, or $U(C_3;T(n-2,i,1))$, where $1\le i\le n-5$.

Let T_v, T_u and T_{v_1} be the components of $G-\{vu,vv_1,uv_1\}$ containing v, u and v_1, respectively. If $\min\{|V(T_u)|,|V(T_v)|,|V(T_{v_1})|\}\ge 2$, by Lemma 7, T_v (T_{v_1}, T_u, resp.) is a path such that v (v_1, u, resp.) is a leaf in T_v (T_{v_1}, T_u, resp.). By Lemma 5 and Remark 6, there is a graph $G'\in\mathcal{U}(n)\setminus\{P_n^3,U(C_3;T(n-2,n-4,1))\}$ with $D_R(G)<D_R(G')$, a contradiction. Thus, $\min\{|V(T_u)|,|V(T_v)|,|V(T_{v_1})|\}=1$. Without loss of generality, we suppose $|V(T_u)|=1$, *i.e.*, $V(T_u)=\{u\}$.

By repeating the use of Lemmas 7, 5 and Remark 6, we have $G\cong U(C_3;T(n-2,i,1))$ for some $1\le i\le n-5$ if $\min\{|V(T_v)|,|V(T_{v_1})|\}=1$, and $G\cong G^*$ if $\min\{|V(T_v)|,|V(T_{v_1})|\}\ge 2$.

Now we give our main result as follows.

Theorem 13. *For graphs in* $\mathcal{U}(n)\setminus\{P_n^3,U(C_3;T(n-2,n-4,1))\}$ $(n\ge 6)$, G^* *is the unique graph with maximum* D_R*-value, where* G^* *is depicted in* **Figure 4** *and* $D_R(G^*)=\frac{2n^3}{3}-\frac{50n}{3}+\frac{142}{3}$.

Proof. By Lemma 12, in order to prove this theorem, it suffices to show that $D_R(G^*)>D_R(P_n^4)$ and $D_R(G^*)>D_R(U(C_3;T(n-2,i,1)))$ for each $1\le i\le n-5$. Let $G=U(C_3;T(n-2,n-4,1))$ be depicted in **Figure 3**.

(1) By Theorems 10 and 8,

$$D_R(G)-D_R(P_n^4)=\left(\frac{2n^3}{3}-\frac{40n}{3}+28\right)-\left(\frac{2n^3}{3}-\frac{53n}{3}+48\right)=\frac{13n}{3}-20.$$

(2) Let $A=\{u,v_1,v_2,\cdots,v_{n-5},v_{n-3}\}$, $B=\{v,v_{n-4},v_{n-2}\}$. For the transformation from G to $G'=G^*$, one has $R(x,y)=R'(x,y)$ for any $x,y\in A$, $d(x)=d'(x)$, $R'(x,v_{n-4})=R(x,v_{n-4})$ and $R'(x,v)=R(x,v)$ for any $x\in A$. By the definition of the degree resistance distance, we get

$$D_1:=\sum_{\{x,y\}\subseteq A}\left[(d(x)+d(y))R(x,y)-(d'(x)+d'(y))R'(x,y)\right]=0,$$

$$\begin{aligned}D_2:=&\sum_{\{x,y\}\subseteq B}\left[(d(x)+d(y))R(x,y)-(d'(x)+d'(y))R'(x,y)\right]\\&=(d(v)+d(v_{n-2}))R(v,v_{n-2})-(d'(v)+d'(v_{n-2}))R'(v,v_{n-2})\\&\quad+(d(v)+d(v_{n-4}))R(v,v_{n-4})-(d'(v)+(d'(v_{n-4}))R'(v,v_{n-4})\\&\quad+(d(v_{n-4})+d(v_{n-2}))R(v_{n-4},v_{n-2})-(d'(v_{n-4})+d'(v_{n-2}))R'(v_{n-4},v_{n-2})\end{aligned}$$

$$=3\left(n-4+\frac{2}{3}\right)-4+5\left(n-5+\frac{2}{3}\right)-5\left(n-5+\frac{2}{3}\right)+4-3\left(n-4+\frac{2}{3}\right)$$
$$=0$$

and

$$\begin{aligned}D_3 &:= \sum_{x\in A, y\in B}\left[\left(d(x)+d(y)\right)R(x,y)-\left(d'(x)+d'(y)\right)R'(x,y)\right]\\
&=\sum_{x\in A}\left[\left(d(x)+d(v)\right)R(x,v)-\left(d'(x)+d'(v)\right)R'(x,v)\right]\\
&\quad+\sum_{x\in A}\left[\left(d(x)+d(v_{n-4})\right)R(x,v_{n-4})-\left(d'(x)+d'(v_{n-4})\right)R'(x,v_{n-4})\right]\\
&\quad+\sum_{x\in A}\left[\left(d(x)+d(v_{n-2})\right)R(x,v_{n-2})-\left(d'(x)+d'(v_{n-2})\right)R'(x,v_{n-2})\right]\\
&=\sum_{x\in A}\left[\left(d(x)+2\right)R(x,v)-\left(d(x)+3\right)R(x,v)\right]\\
&\quad+\sum_{x\in A}\left[\left(d(x)+3\right)R(x,v_{n-4})-\left(d(x)+2\right)R(x,v_{n-4})\right]\\
&\quad+\sum_{x\in A}\left[\left(d(x)+1\right)R(x,v_{n-2})-\left(d(x)+1\right)R'(x,v_{n-2})\right]\\
&=-\sum_{x\in A}R(x,v)+\sum_{x\in A}R(x,v_{n-4})+\sum_{x\in A}\left(d(x)+1\right)\left(R(x,v_{n-2})-R'(x,v_{n-2})\right)\\
&=\left(-\frac{n^2}{2}+\frac{23n}{6}-9\right)+\left(\frac{n^2}{2}-\frac{7n}{2}+\frac{20}{3}\right)+\left[\left(\frac{3n^2}{2}-\frac{13n}{2}+5\right)-\left(\frac{3n^2}{2}-\frac{19n}{2}+22\right)\right]\\
&=\frac{10n}{3}-\frac{58}{3}.\end{aligned}$$

Thus, $D_R(G)-D_R(G^*)=D_1+D_2+D_3=\frac{10n}{3}-\frac{58}{3}$.

(3) Let $A=V(G)-\{v_i,v_{n-4},v_{n-2}\}$, $B=\{v_i,v_{n-4},v_{n-2}\}$. For the transformation from G to $G'=U(C_3;T(n-2,i,1))$, one has $R(x,y)=R'(x,y)$ for any $x,y\in A$, $d(x)=d'(x)$, $R'(x,v_{n-4})=R(x,v_{n-4})$ and $R'(x,v_i)=R(x,v_i)$ for any $x\in A$. If $i\neq 1$, then

$$D_1 := \sum_{\{x,y\}\subseteq A}\left[\left(d(x)+d(y)\right)R(x,y)-\left(d'(x)+d'(y)\right)R'(x,y)\right]=0,$$

$$\begin{aligned}D_2 &:= \sum_{\{x,y\}\subseteq B}\left[\left(d(x)+d(y)\right)R(x,y)-\left(d'(x)+d'(y)\right)R'(x,y)\right]\\
&=\left(d(v_i)+d(v_{n-2})\right)R(v_i,v_{n-2})-\left(d'(v_i)+d'(v_{n-2})\right)R'(v_i,v_{n-2})\\
&\quad+\left(d(v_i)+d(v_{n-4})\right)R(v_i,v_{n-4})-(d'(v_i)+\left(d'(v_{n-4})\right)R'(v_i,v_{n-4})\\
&\quad+\left(d(v_{n-4})+d(v_{n-2})\right)R(v_{n-4},v_{n-2})-\left(d'(v_{n-4})+d'(v_{n-2})\right)R'(v_{n-4},v_{n-2})\\
&=3(n-3-i)-4+5(n-4-i)-5(n-4-i)+4-3(n-3-i)\\
&=0,\end{aligned}$$

$$\begin{aligned}D_3 &:= \sum_{x\in A}\left[\left(d(x)+d(v_i)\right)R(x,v_i)-\left(d'(x)+d'(v_i)\right)R'(x,v_i)\right]\\
&=\sum_{x\in A}\left[\left(d(x)+2\right)R(x,v_i)-\left(d(x)+3\right)R(x,v_i)\right]\\
&=-\sum_{x\in A}R(x,v_i)\\
&=-i^2+(n-5)i-\frac{3n^2-21n+38}{6},\end{aligned}$$

$$
\begin{aligned}
D_4 &:= \sum_{x\in A}\left[\left(d(x)+d(v_{n-4})\right)R(x,v_{n-4})-\left(d'(x)+d'(v_{n-4})\right)R'(x,v_{n-4})\right]\\
&= \sum_{x\in A}\left[\left(d(x)+3\right)R(x,v_{n-4})-\left(d(x)+2\right)R(x,v_{n-4})\right]\\
&= \sum_{x\in A} R(x,v_{n-4})\\
&= \frac{n^2}{2}-\frac{7n}{2}+\frac{19}{3}+i
\end{aligned}
$$

and

$$
\begin{aligned}
D_5 &:= \sum_{x\in A}\left[\left(d(x)+d(v_{n-2})\right)R(x,v_{n-2})-\left(d'(x)+d'(v_{n-2})\right)R'(x,v_{n-2})\right]\\
&= \sum_{x\in A}\left[\left(d(x)+1\right)R(x,v_{n-2})-\left(d(x)+1\right)R'(x,v_{n-2})\right]\\
&= \sum_{x\in A}\left[\left(d(x)+1\right)\left(R(x,v_{n-2})-R'(x,v_{n-2})\right)\right]\\
&= \left(\frac{3n^2}{2}-\frac{13n}{2}+3i+4\right)-\left(\frac{3n^2}{2}+3i^2-3ni+17i+12-\frac{17n}{2}\right)\\
&= -3i^2-14i+3ni+2n-8.
\end{aligned}
$$

Similarly, if $i=1$, then $D_1=0$, $D_2=n-5$, $D_3=-\frac{3n^2-27n+74}{6}$, $D_4=\frac{n^2}{2}-\frac{7n}{2}+\frac{22}{3}$ and $D_5=4n-20$.

Let $f(x)=-4x^2+4nx-18x+2n-8$, $1\le i\le n-5$. Then

$$
\begin{aligned}
D_R(G)-D_R\left(U\left(C_3;T(n-2,i,1)\right)\right) &= D_1+D_2+D_3+D_4+D_5\\
&= f(i)\\
&\ge f(n-5)\\
&= 4n-18.
\end{aligned}
$$

Since $\min\left\{\frac{13n}{3}-20,\frac{10n}{3}-\frac{58}{3},4n-18\right\}=\frac{10n}{3}-\frac{58}{3}$ if $n\ge 6$, we get $D_R(G^*)>D_R(P_n^4)$ and $D_R(G^*)>D_R\left(U\left(C_3;T(n-2,i,1)\right)\right)$ for each $1\le i\le n-5$. Moreover, by (2) and Theorem 10,

$$
\begin{aligned}
D_R(G^*) &= D_R\left(U\left(C_3;T(n-2,n-4,1)\right)\right)-\left(\frac{10n}{3}-\frac{58}{3}\right)\\
&= \frac{2n^3}{3}-\frac{40n}{3}+28-\frac{10n}{3}+\frac{58}{3}=\frac{2n^3}{3}-\frac{50n}{3}+\frac{142}{3}.
\end{aligned}
$$

4. Conclusion

In this paper, we completely characterized the unicyclic graphs with the third-maximum degree resistance distance. Some transformations for comparing the degree resistance distance of two graphs were introduced. This may help determine the maximum or minimum degree resistance distance for other classes of graphs. We will investigate these problems in our future studies.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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