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Dynamical Properties of a Discrete Lesley-Gower Prey-Predator Model with Holling-II Type Functional Response

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Abstract

In this paper, we will study a class of discrete Leslie-Gower prey-predator models, which is a discretization of the continuous model proposed by Leslie and Gower in 1960. First, we find all fixed points, use hyperbolic and non-hyperbolic conditions to give the types of fixed points, and then analyze the bifurcation properties of non-hyperbolic fixed points. The generating conditions of Flip bifurcation and Neimark-Sacker bifurcation at fixed points are studied. Finally, numerical simulations of Flip bifurcation and Neimark-Sacker bifurcation are given.

Keywords

Leslie-Gower Prey-Predator Model, Non-Hyperbolic Fixed Point, Flip Bifurcation, Neimark-Sacker Bifurcation

1. Introduction

In a predator-predator model, the predator functional response is the most important factor determining the dynamic behavior of the model. The earliest model of ratio dependence was proposed by Leslie. In this model, changes in predators are assumed to be logical increases in the carrying capacity of a variable resource (prey). Such models are based on the assumption that the decrease in predator numbers is inversely related to the per capita availability of its preferred food. The dynamic nature of the predator of this model has been discussed by Leslie and Gower [1]. In the presence of a severe shortage of favorite prey, predators may switch to less preferred foods in order to survive. The resulting model no longer depends on ratios. This should be taken into account in the modified Leslie Gower model, for which Alaoui and Okiye [2] considered the degree of protection provided by the environment to enable prey to escape predation, giving the following model with Holling type II functional responses.

$$\begin{cases} \dfrac{\mathrm{d}X}{\mathrm{d}T} = X\left(a_1 - bX - \dfrac{c_1 Y}{X + k_1}\right) \\ \dfrac{\mathrm{d}Y}{\mathrm{d}T} = Y\left(a_2 - \dfrac{c_2 Y}{X + k_2}\right) \end{cases}, \tag{1}$$

where, the population densities of prey and predators are denoted by X and Y, respectively, all relevant parameters are positive, and their biological significance is as follows: a_1 is the logistic growth rate of prey, b is the intensity of interspecific competition within the prey population, c_1 represents the consumption of prey by predator, k_1 represents the degree of protection of the environment to predator, and the other constant a_2 represents the logistic growth rate of predator, c_2 represents the crowding effect among predators, the parameter k_2 provides a range of alternative predation options in an environment other than X.

To give the dimensionless form of system (1), we use the following scaling transformation:

$$X = a_1 x/b, \quad Y = a_1^2 y/(bc_1), \quad T = t/a_1,$$

$$a = b_1 k_1/a_1, \quad q = c_2/c_1, \quad p = a_2/a_1, \quad r = bk_2/a_1.$$

Then system (1) becomes

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = x(1-x) - \dfrac{xy}{x+a} \\ \dfrac{\mathrm{d}y}{\mathrm{d}t} = y\left(p - \dfrac{qy}{x+r}\right) \end{cases}. \tag{2}$$

The existence and persistence of positive solutions, the global stability of solutions, the existence of periodic and quasi-periodic solutions, and the bifurcation and chaos properties of system (2) have been deeply studied by many scholars, refer to the literature [2]-[11].

Almost all of these studies were on continuous systems (2). Continuous systems and corresponding discrete systems have many similar dynamic properties. For example, in bifurcation theory, the folding bifurcation and Hopf bifurcation in the one-parameter continuous case correspond to the folding bifurcation and Neimark-Sacker bifurcation in the discrete case. However, as pointed out in the literature [12], discrete cases may have richer properties than continuous systems, for example, period 3 can produce chaotic phenomena, as well as period-doubling bifurcations in bifurcation problems. Therefore, inspired by the above literature, in this paper, we consider the discrete case of system (2).

It is well known that the fixed point of a system is actually its singular solution. The orbit of the system at the normal point (non-fixed point) is locally structurally stable, while the orbit at the fixed point may be locally structurally unstable and may produce singular changes. Therefore, the properties of the fixed point and the orbit near the fixed point become more complex, and its research significance is greater, especially for the actual biological model (see [9]-[12]).

The discrete case corresponding to system (2) is

$$\begin{cases} x_{n+1} = x_n\left(2 - x_n\right) - \dfrac{x_n y_n}{x_n + a} \\ y_{n+1} = y_n\left(p - \dfrac{q y_n}{x_n + r}\right) + y_n \end{cases} . \tag{3}$$

The purpose of this paper is to study the dynamic properties of system (3). Firstly, all fixed points are obtained, and the types of fixed points are given by using hyperbolic and non-hyperbolic conditions. Then, the bifurcation properties of non-hyperbolic fixed points are analyzed, and the generation conditions of Flip bifurcation and Neimark-Sacker bifurcation are investigated. Finally, the numerical simulation of Flip bifurcation and Neimark-Sacker bifurcation are given.

For details on the conditions for discriminating between hyperbolic and non-hyperbolic fixed points and their significance for stability and bifurcation analysis, please refer to the literature [13] [14].

2. The Types of Fixed Points

In this section, we will discuss the hyperbolic and non-hyperbolic properties of fixed points and determine the types of fixed points. We change system (3) to a plane map $F: R^2 \mapsto R^2$:

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x\left(2-x\right) - \dfrac{xy}{x+a} \\ y\left(p+1-\dfrac{qy}{x+r}\right) \end{pmatrix} \tag{4}$$

Obviously, the system has fixed points.

$$E_0 = \left(0,0\right),\quad E_1 = \left(1,0\right),\quad E_2 = \left(0,\frac{pr}{q}\right),\quad E_3 = \left(x^*,y^*\right)$$

where

$$x^* = \frac{-\left(p-q+aq\right) + \sqrt{\left(p-q+aq\right)^2 - 4q\left(pr-aq\right)}}{2q},$$

$$y^* = \frac{p\left(x^*+r\right)}{q}.$$

Theorem 2.1 (A) Fixed point E_0 is the unstable node of the system; (B) Fixed point E_1 is the saddle point of the system.

Proof (A) Taylor's expansion of the map F at a fixed point E_0 is

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 2x - x^2 - \dfrac{xy}{a} + O\left(\left\|(x,y)\right\|^3\right) \\ \left(p+1\right)y - \dfrac{qy^2}{r} + O\left(\left\|(x,y)\right\|^3\right) \end{pmatrix}, \tag{5}$$

The Jacobian matrix is

$$J_F(E_0)=\begin{pmatrix}2 & 0\\ 0 & p+1\end{pmatrix}$$

And its eigenvalues are

$$\lambda_1=2,\quad \lambda_2=p+1.$$

Because of $p>0$, we have $\lambda_1>1$, $\lambda_2>1$. Then, the fixed point E_0 is the unstable node.

(B) similar to the proof of (A), we know that the eigenvalues of fixed point E_1 are

$$\lambda_3=0,\quad \lambda_4=p+1.$$

Then the conclusions can be drawn. The specific process is omitted.

Theorem 2.2 (A) Fixed point E_2 is non-hyperbolic if and only if (p,q) is located on the following four lines:

$$A_1:\left\{(p,q)\in\mathbb{R}^2 \middle| p=0\right\},$$

$$A_2:\left\{(p,q)\in\mathbb{R}^2 \middle| p=2\right\},$$

$$A_3:\left\{(p,q)\in\mathbb{R}^2 \middle| p=\frac{aq}{r}, p,q>0\right\},$$

$$A_4:\left\{(p,q)\in\mathbb{R}^2 \middle| p=\frac{3aq}{r}, p,q>0\right\}.$$

(B) If $2<\frac{aq}{r}$, the fixed point E_2 satisfies the following types shown in **Table 1**.

Table 1. Types of fixed point E_2.

Case	condition	eigenvalue	type
B_1	$0<p<2$	$\lambda_5>1$, $-1<\lambda_6<1$	saddle point
B_2	$\frac{aq}{r}<p<\frac{3aq}{r}$	$-1<\lambda_5<1$, $\lambda_6<-1$	saddle point
B_3	$2<p<\frac{aq}{r}$	$\lambda_5>1$, $\lambda_6<-1$	unstable node
B_4	$p>\frac{3aq}{r}$	$\lambda_5,\lambda_6<-1$	unstable node

(C) If $\frac{aq}{r}<2<\frac{3aq}{r}$, the fixed point E_2 satisfies the following types shown in **Table 2**.

Table 2. Types of fixed point E_2.

Case	condition	eigenvalue	type
C_1	$0<p<\frac{aq}{r}$	$\lambda_5>1$, $-1<\lambda_6<1$	saddle point
C_2	$2<p<\frac{3aq}{r}$	$-1<\lambda_5<1$, $\lambda_6<1$	saddle point

Continued

C_3	$p>\frac{3aq}{r}$	$\lambda_5,\lambda_6<-1$	unstable node
C_4	$\frac{aq}{r}<p<2$	$-1<\lambda_5,\lambda_6<1$	stable node

(D) If $\frac{3aq}{r}<2$, the fixed point E_2 satisfies the following types shown in **Table 3**.

Table 3. Types of fixed point E_2.

Case	condition	eigenvalue	type
D_1	$0<p<\frac{aq}{r}$	$\lambda_5>1$, $-1<\lambda_6<1$	saddle point
D_2	$\frac{3aq}{r}<p<2$	$\lambda_5<-1$, $-1<\lambda_6<1$	saddle point
D_3	$p>2$	$\lambda_5,\lambda_6<-1$	unstable node
D_4	$\frac{aq}{r}<p<\frac{3aq}{r}$	$-1<\lambda_5,\lambda_6<1$	stable node

Proof By performing the following coordinate transformation

$$w=x,\quad m=y-\frac{pr}{q},$$

we transform the fixed point E_2 to $E_2^0=(0,0)$. The transformed map is $\tilde{F}$:

$$\begin{pmatrix} w \\ m \end{pmatrix} \mapsto \begin{pmatrix} \left(2-\frac{pr}{aq}\right)w+\left(\frac{pr}{a^2q}-1\right)w^2-\frac{wm}{a}+O\left(\left\|(w,m)\right\|^3\right) \\ \frac{wp^2}{q}+(1-p)m-\frac{w^2p^2}{qr}-\frac{qm^2}{r}+\frac{2wmp}{r}+O\left(\left\|(w,m)\right\|^3\right) \end{pmatrix}. \tag{6}$$

The Jacobian matrix of map $\tilde{F}$ at E_2^0 is

$$D\tilde{F}(0,0)=\begin{pmatrix} 2-\frac{pr}{aq} & 0 \\ \frac{p^2}{q} & 1-p \end{pmatrix},$$

and its eigenvalues are

$$\lambda_5=2-\frac{pr}{aq},\lambda_6=1-p.$$

(A) If $\lambda_5=1$ or $\lambda_6=1$, then $p=\frac{aq}{r}$ or $p=0$; If $\lambda_5=-1$ or $\lambda_6=-1$, then $p=\frac{3aq}{r}$ or $p=2$. By the non-hyperbolic property of the fixed point, we know that fixed point E_2 is non-hyperbolic if and only if (p,q) is located on the following four lines A_1, A_2, A_3 and A_4.

Since E_2 is hyperbolic if and only if its eigenvalues are not on the unit circle, so, we need discuss the eigenvalues for three cases

(B) $2<\frac{aq}{r}$, (C) $\frac{aq}{r}<2<\frac{3aq}{r}$, (D) $\frac{3aq}{r}<2$.

(B) If $2<\frac{aq}{r}$, when $0<p<\frac{aq}{r}$, we have $\lambda_5>1$ and $-1<\lambda_6<1$, then E_2 is saddle point. We proved the Case B_1. Similarly, we can prove the Cases of B_2, B_3 and B_4.

According to the method of case (B), we can discuss the types of cases of (C) and (D).

3. The Flip Bifurcation of Fixed Point E_2

In this section, we choose p as the bifurcation parameter and consider the flip bifurcation of fixed point E_2 for the case $(p,q)\in A_2$.

Theorem 3.1 Suppose $3aq\neq 2r$ and $aq\neq 2r$, if $(p,q)\in\alpha_2$, then the system (4) has a flip bifurcation at a fixed point E_2. More precisely, when $p<2$, the fixed point E_2 is stable, when $p>2$, the fixed point E_2 becomes unstable, and the system bifurcates a stable 2-period orbit.

Proof We write the map $\tilde{F}$ as $\tilde{F}_p$ to strengthen the dependence on the parameter p, then

$$D\tilde{F}_p\left((0,0)\right)=\begin{pmatrix} 2-\dfrac{pr}{aq} & 0 \\ \dfrac{p^2}{q} & 1-p \end{pmatrix}.$$

Since $3aq\neq 2r$ and $aq\neq 2r$, if $(p,q)\in A_2$, we obtain $\lambda_5\neq 1,-1$, $\lambda_6=-1$ and corresponding eigenvectors are

$$\left(\frac{aq-pr+apq}{ap^2},1\right)^{\mathrm{T}},(0,1)^{\mathrm{T}}. \tag{7}$$

By using the eigenvector set (7), we obtain the following transformation

$$\begin{pmatrix} w \\ m \end{pmatrix}\mapsto\begin{pmatrix} \dfrac{aq-pr+apq}{ap^2} & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \bar{w} \\ \bar{m} \end{pmatrix}. \tag{8}$$

By selecting $\delta=2-p$ as a parameter, the system (7) can be changed into the following parametric suspension system by using the above transformation (8).

$$\begin{pmatrix} \bar{w} \\ \bar{m} \\ \delta \end{pmatrix}\mapsto\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \bar{w} \\ \bar{m} \\ \delta \end{pmatrix}+\begin{pmatrix} f_{11}\bar{w}^2-\dfrac{\bar{m}\bar{w}}{a}+O\left(\left\|(\bar{w},\bar{m})\right\|^3\right) \\ \delta\bar{m}-\dfrac{q}{r}\bar{m}^2-f_{12}\bar{w}^2+f_{13}\bar{m}\bar{w}+O\left(\left\|(\bar{w},\bar{m})\right\|^3\right) \\ 0 \end{pmatrix} \tag{9}$$

where

$$f_{11}=\frac{r+p}{a^2p}-\frac{r^2}{a^3q}-\frac{aq}{ap^2}+\frac{r-p-aq}{ap},$$

$$f_{12}=\frac{r+p}{a^2p}-\frac{r^2+ar}{a^3q}-\frac{aq}{ap^2}+\frac{aq(2-r)+r(r-p+2)}{apr}-\frac{q+2pq}{p^2r},$$

$$f_{13}=\frac{2q}{pr}-\frac{1}{a}.$$

According to the central manifold existence theorem (see [13], p. 246), we can obtain the local central manifold of the map (9):

$$W_{loc}^{c}(0,0)=\left\{(\bar{w},\bar{m},\delta)\in\mathbb{R}^3\,\middle|\,\bar{w}=h(\bar{m},\delta),h(0,0)=0,Dh(0,0)=0,|\bar{m}|<\varepsilon,|\delta|<\varepsilon\right\}$$

where ε is a small enough positive value. We assume

$$\bar{w}=h(\bar{m},\delta)=A\bar{m}^2+B\bar{m}\delta+C\delta^2+O\left(\left\|(\bar{w},\bar{m},\delta)\right\|^3\right).$$

The above central manifold must be satisfied:

$$N\left(h(\bar{m},\delta)\right)=h\left(-\bar{m}+\delta\bar{m}-\frac{q}{r}\bar{m}^2-f_{12}h(\bar{m},\delta)^2+f_{13}\bar{m}h(\bar{m},\delta),\delta\right)$$
$$-\left(2-\frac{pr}{aq}\right)h(\bar{m},\delta)-f_{11}h(\bar{m},\delta)^2+\frac{\bar{m}h(\bar{m},\delta)}{a}=0.$$

By comparing the coefficients, we get $A=B=C=0$, then the central manifold is:

$$\bar{w}=h(\bar{m},\delta)=O\left(\left\|(\bar{w},\bar{m},\delta)\right\|^3\right).$$

By substituting this into the map (9), we can get a one-dimensional map on the central manifold:

$$\bar{m}\mapsto\chi_1(\bar{m})=(\delta-1)\bar{m}-\frac{q}{r}\bar{m}^2+O\left(\left\|(\bar{w},\bar{m},\delta)\right\|^3\right).$$

We can verify the transversal and non-degenerate conditions for flip bifurcation (see [14], p. 127):

$$\left.\frac{\partial^2\chi_1}{\partial\bar{m}\partial\delta}\right|_{(\bar{m},\delta)=(0,0)}=1,$$

$$\left.\left[\frac{1}{2}\left(\frac{\partial^2\chi_1}{\partial\bar{m}^2}\right)^2+\frac{1}{3}\left(\frac{\partial^3\chi_1}{\partial\bar{m}^3}\right)\right]\right|_{(\bar{m},\delta)=(0,0)}=\frac{2q^2}{r^2}>0.$$

Thus, the system has a flip bifurcation at a fixed point E_2.

4. The Neimark-Sacker Bifurcation of Fixed Points E_3

In this section, we consider the Neimark-Sacker bifurcation of system (3) at the fixed point E_3. Because of the complexity of the calculation, we only discuss the possible conditions under which the Neimark-Sacker bifurcation occurs.

Using the following coordinate transformation

$$X=x-x^*,\quad Y=y-y^*,$$

we transform the fixed point E_3 to $E_3^0=(0,0)$. The transformed map is $\tilde{F}$:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \mapsto \begin{pmatrix} \left(X+x^*\right)\left(2-X-x^*\right)-x^*-\dfrac{\left(X+x^*\right)\left(Y+y^*\right)}{x^*+a} \\ (p+1)Y+py^*-\dfrac{qY^2+2qYy^*+q\left(y^*\right)^2}{x^*+r} \end{pmatrix}$$

$$+\begin{pmatrix} \dfrac{X\left(Yx^*+Xy^*+x^*y^*\right)}{\left(x^*+a\right)^2}-\dfrac{X^2x^*y^*}{\left(x^*+a\right)^3}+O\left(\left\|(X,Y)\right\|^3\right) \\ \dfrac{2qXYy^*+qX\left(y^*\right)^2}{\left(x^*+r\right)^2}-\dfrac{qX^2\left(y^*\right)^2}{\left(x^*+r\right)^3}+O\left(\left\|(X,Y)\right\|^3\right) \end{pmatrix}.$$

The Jacobian matrix of the map $\tilde{F}$ is:

$$D\tilde{F}(0,0)=\begin{pmatrix} 2-2x^*-\dfrac{ap\left(x^*+r\right)}{q\left(x^*+a\right)^2} & -\dfrac{x^*}{x^*+a} \\ \dfrac{p^2}{q} & 1-p \end{pmatrix},$$

and the characteristic equation is:

$$\lambda^2-\left[3-2x^*-p-\frac{ap\left(x^*+r\right)}{q\left(x^*+a\right)^2}\right]\lambda+\left[\left(2-2x^*-\frac{ap\left(x^*+r\right)}{q\left(x^*+a\right)^2}\right)(1-p)+\frac{x^*p^2}{q\left(x^*+a\right)}\right]=0.$$

Let

$$T=3-2x^*-p-\frac{ap\left(x^*+r\right)}{q\left(x^*+a\right)^2},$$

$$D=\left(2-2x^*-\frac{ap\left(x^*+r\right)}{q\left(x^*+a\right)^2}\right)(1-p)+\frac{x^*p^2}{q\left(x^*+a\right)}.$$

Then, when the following conditions are satisfied

(E) $T>0$, $T^2-4D<0$ and $D=1$,

the system has conjugate complex roots λ_7,λ_8 and $|\lambda_7|=|\lambda_8|=1$. In this case, the system may generate a Neimark-Sacker bifurcation at this fixed point. A numerical simulation of this phenomenon is given in the following section.

5. Numerical Simulation

In this section, we're going to give the numerical simulation of flip bifurcation at a fixed point E_2 and of Neimark-Sacker bifurcation at a fixed point E_3.

Simulation 1 Let $a=0.875$, $q=0.854$, $r=0.944$, choose p as a variation.

Parameter, initial value $(x_0,y_0)=(0.174,0.486)$, then the flip bifurcation diagram of system (3) at a fixed point E_2 is shown in **Figure 1**.

Simulation 2 Let $a=0.315$, $q=0.204$, $r=0.006$ and initial value $(x_0,y_0)=(0.274,0.154)$.

(a) when $p = 0.418$, we obtain by calculation $T \approx 0.748 > 0$, $D \approx 0.987$, the fixed point E_3 is stable (see **Figure 2(a)**).

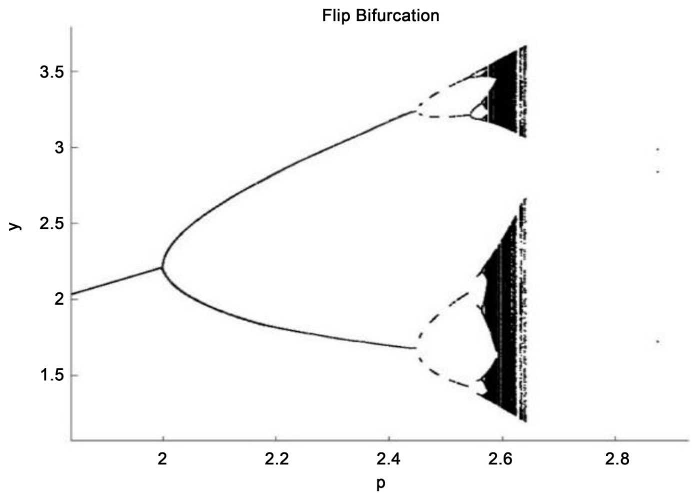

Figure 1. The flip bifurcation diagram of system (3).

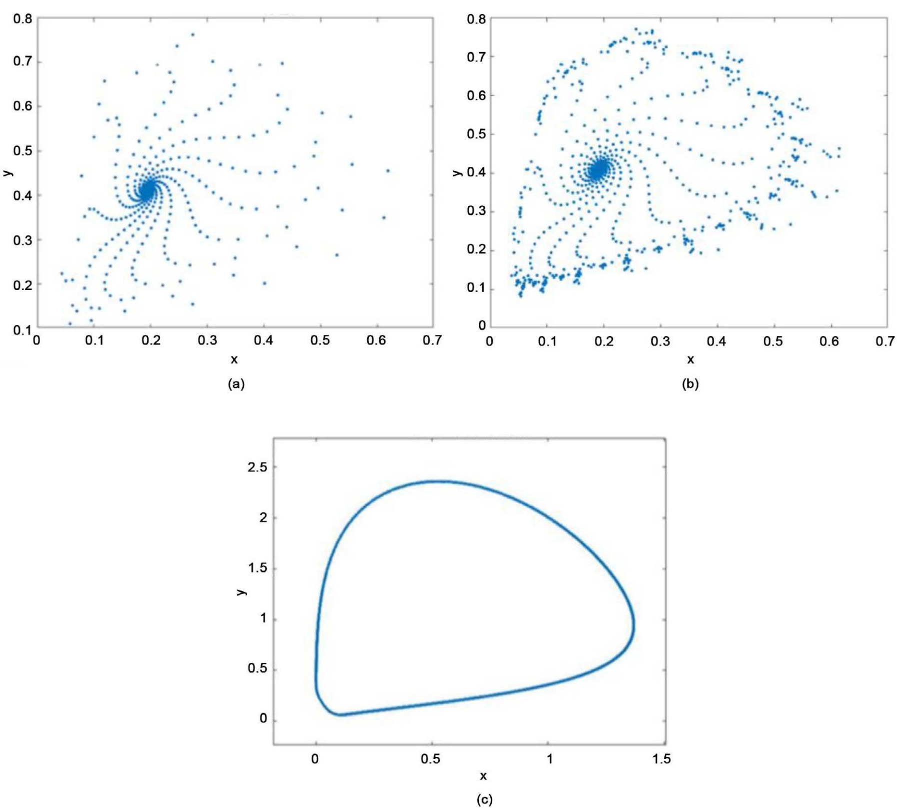

Figure 2. (a) $p = 0.418$, the fixed point E_3 is stable; (b) $p = 0.425$, the fixed point E_3 is stable; (c) $p = 0.455$, a stable limit cycle appears nearby E_3.

(b) when $p = 0.425$, we obtain by calculation $T \approx 0.739 > 0$, $D \approx 0.998$, the fixed point E_3 is also stable (see **Figure 2(b)**);

(c) when $p = 0.455$, we obtain by calculation $T \approx 0.727 > 0$, $D \approx 1.008$, the fixed point E_3 loses stability, and a stable limit cycle appears nearby (see **Figure 2(c)**).

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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