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Special Termination of Minimal Model Program

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Abstract

This paper presents a self-contained proof of Special Termination of MMP (Minimal Model Program). By refining the assumptions and simplifying the argument, it offers a more accessible approach compared to the original proof in BCHM (Birkar-Cascini-Hacon-McKernan).

Keywords

Special Termination, Minimal Model Program, Birational Geometry, Algebraic Geometry

1. Introduction

The main purpose of this note is to provide a complete presentation on the special termination of MMP. We begin by extracting the positivity condition on the boundary from [1], which we define as the BCHM condition. This condition asserts that the boundaries of a Kawamata log terminal (klt for short) or divisorial log terminal (dlt for short) pair (X,B) include an ample divisor. A key advantage of the BCHM condition is that it remains preserved under restriction and throughout the Minimal Model Program (MMP) after appropriate boundary modifications, enabling us to establish the existence of pl-flips by induction on dimension.

2. Preliminaries

Let k be an algebraically closed field of characteristic zero fixed throughout the paper. A divisor means a $\mathbb{R}$-Cartier $\mathbb{R}$-Weil divisor. A divisor D over a normal variety X is a divisor on a birational model of X. A birational map $X \dashrightarrow Y$ is a *birational contraction* if its inverse map contracts no divisor.

Pairs. A *pair* $(X/U,B)$ consists of normal quasi-projective varieties X, Z, a $\mathbb{R}$-divisor B on X with coefficients in $[0,1]$ such that K_X+B is $\mathbb{R}$-Cartier and a projective morphism $X \to U$. If U is a point or U is unambiguous in the context, then we simply denote a pair by (X,B). For a prime divisor D on some birational model of X with a nonempty centre on X, $a(D,X,B)$ denotes the log discrepancy. For definitions and standard results on singularities of pairs, we refer to [2].

Log Minimal Models. A projective pair $(Y/U,B_Y)$ is a *log birational model* of a projective pair $(X/U,B)$ if we are given a birational map $\phi: X \dashrightarrow Y$ and $B_Y = B^{\sim} + E$ where $B^{\sim}$ is the birational transform of B and E is the reduced exceptional divisor of ϕ^{-1}, that is, $E=\sum E_j$ where E_j are the exceptional/X prime divisors on Y. A log birational model $(Y/U,B_Y)$ is a *weak log canonical* (*weak lc for short*) *model* of $(X/U,B)$ if

- $K_Y + B_Y$ is nef/U, and
- for any prime divisor D on X, which is exceptional/Y, we have

$$a(D,X,B) \le a(D,Y,B_Y)$$

A weak lc model $(Y/U,B_Y)$ is a *log minimal model* of $(X/U,B)$ if

- $(Y/U,B_Y)$ is $\mathbb{Q}$-factorial dlt,
- the above inequality on log discrepancies is strict.

A log minimal model $(Y/U,B_Y)$ is *good* if $K_Y + B_Y$ is semi-ample/U.

Ample Models and Log Canonical Models. Let D be a divisor on a normal variety X over Z. A normal variety T is the *ample model*/Z of D if we are given a rational map $\phi: X \dashrightarrow T$ such that there exists a resolution $X \overset{p}{\leftarrow} X' \overset{q}{\rightarrow} T$ with

- q being a contraction,
- $p^*D \sim_{\mathbb{R}} q^*D_T + E$ where D_T is an ample/Z divisor and $E \ge 0$, and
- for every divisor $B \in |p^*D/Z|_{\mathbb{R}}$, then $B \ge E$.

Note that the ample model is unique if it exists. The existence of the ample model is equivalent to saying that the divisorial ring $R(D)$ is a finitely generated $\mathcal{O}_Z$-algebra when $D \ge 0$ is $\mathbb{Q}$-Cartier.

BCHM Condition.

1) X is n-dimensional $\mathbb{Q}$-factorial normal algebraic variety, $\pi: X \to U$ is a projective morphism of normal quasi-projective varieties.

2) (X,B) is dlt pair with $S=\lfloor B \rfloor$ or (X,B) is a klt pair.

3) There exists a relatively ample $\mathbb{R}$-divisor A over U such that $B \ge A$.

3. Special Termination Theorem

Firstly, we recall some classical theorem

Theorem 3.1 (*Basepoint-free theorem*). *Let (X,Δ) be a projective klt pair. Let D be a nef Cartier divisor such that $aD-(K_X+\Delta)$ is ample for some $a>0$. Then, there is a positive integer b_0 such that $|bD|$ has no base points for every $b \ge b_0$.*

Theorem 3.2 (*Rationality theorem*). *Let (X,Δ) be a projective klt pair such that $K_X+\Delta$ is not nef. Let $a>0$ be an integer such that $a(K_X+\Delta)$ is Cartier. Let H be an ample Cartier divisor. We define*

$$r = \max\{t \in \mathbb{R} \mid H + t(K_X+\Delta) \text{ is nef}\}$$

Then r is a rational number of the form u/ v, where u and v are integers with

$$0 < v \le a(\dim X + 1)$$

The final theorem is the cone and contraction theorem.

Theorem 3.3 (*Cone and contraction theorem*). *Let* (X, Δ) *be a projective klt pair. Then, we have the following properties.*

1) There are (countably many possibly singular) rational curves $C_j \subset X$ such that

$$\overline{NE}(X) = \overline{NE}(X)_{(K_X+\Delta)\ge 0} + \sum \mathbb{R}_{\ge 0}[C_j]$$

2) Let $R \subset \overline{NE}(X)$ be a $(K_X + \Delta)$-negative extremal ray. Then, there is a unique morphism $\varphi_R : X \to Z$ to a projective variety Z such that $(\varphi_R)_* \mathcal{O}_X \simeq \mathcal{O}_Z$ and an irreducible curve $C \subset X$ is mapped to a point by φ_R if and only if $[C] \in R$.

Assume that we are given an LMMP with scaling, which consists of only a sequence $X_i \dashrightarrow X_{i+1}/Z_i$ of log flips, and that $(X_1/Z, B_1)$ is $\mathbb{Q}$-factorial dlt. Assume $\lfloor B_1 \rfloor \ne 0$ and pick a component S_1 of $\lfloor B_1 \rfloor$. Let $S_i \subset X_i$ be the birational transform of S_1 and T_i the normalisation of the image of S_i in Z_i. Using standard special termination arguments, we will see that termination of the LMMP near S_1 is reduced to termination in lower dimensions. It is well-known that the induced map $S_i \dashrightarrow S_{i+1}/T_i$ is an isomorphism in codimension one if $i \gg 0$. So, we could assume that these maps are all isomorphisms in codimension one. Put $K_{S_i} + B_{S_i} := (K_{X_i} + B_i)|_{S_i}$. In general, $S_i \dashrightarrow S_{i+1}/T_i$ is not a $(K_{S_i} + B_{S_i})$-flip. To apply induction, we note that $S_i \dashrightarrow S_{i+1}/T_i$ can be connected by a sequence of $(K_{S_i} + B_{S_i})$-flips [3].

Theorem 3.4 *Let* $\pi : X \to U$ *be a projective morphism of normal quasi-projective varieties. Let* $(X, B+C)$ *be a* $\mathbb{Q}$*-factorial dlt pair with* $S = \lfloor B \rfloor$, *such that* $K_X + B + C$ *is nef over U and* (X, B) *satisfy BCHM condition. Let* $\alpha_i : X_i \dashrightarrow X_{i+1}$ *be a sequence of flips and divisorial contractions over U for the* $(K_X + B)$*-MMP with a scaling of C over U.*

Then, there exists an integer $i > 0$ such that for all $j \ge i, \alpha_j$ is an isomorphism on a neighborhood of *S*.

Proof. Given that (X, B) satisfies the BCHM condition, we can write $(X, B) = (X, S + A + E)$ where $E \ge 0$ and *A* is an ample $\mathbb{Q}$-divisor. Fix any irreducible component S_1 of $S = \lfloor B \rfloor$.

For any rational number $0 < \epsilon \ll 1$, the divisor $A + \epsilon(S - S_1)$ is an ample $\mathbb{Q}$-divisor. Now, take a sufficiently general effective **R**-divisor $A_1 \equiv_U A + \epsilon(S - S_1)$. Then $(X, A_1 + E + (1-\epsilon)(S - S_1) + S_1)$ is plt and $(X, A_1 + E + (1-\epsilon)(S - S_1) + S_1 + C)$ is dlt. Since $B \equiv_U A_1 + E + (1-\epsilon)(S - S_1) + S_1$, after replacing *B* and *B*', we may assume that (X, B) is plt.

Let S_i, A_i, E_i, B_i and C_i be the strict transforms of S, A, E, B and *C* on X_i. Notice that $(X_i, S_i + A_i + E_i)$ is plt and $S_i + A_i + E_i = S_i$, then S_i is normal.

By adjunction formula, we can write

$$\left(K_{X_i}+B_{X_i}\right)\Big|_{S_i}=K_{S_i}+B_{S_i}$$

where $\left(S_i,B_{S_i}\right)$ is klt. Define the set $\mathcal{B}\subset[0,1]$ as the following:

$$\mathcal{B}=\left\{x=1-\frac{1}{m}+\sum_{i\in I}\frac{r_ib_i}{m}\middle|b_i\in\operatorname{Coeff}(B),m\in\mathbb{Z}_{>0},r_i\in\mathbb{Z}_{\geq0}\right\}$$

We have $\operatorname{Coeff}\left(B_{S_i}\right)\subset\mathcal{B}$ for any *i*. Since $\mathcal{B}\cap[0,1-\epsilon]$ is a finite set for any $\epsilon>0$ and $\left(S_i,B_i\right)$ is klt, we define an integer $0\leq d_{\mathcal{B}}<\infty$ by

$$d_{\mathcal{B}}\left(S,B_S\right):=\sum_{\beta\in\mathcal{B}}\#\left\{E\mid a\left(E;S,B_S\right)<-\beta\right\}$$

Moreover, we have that $a\left(E;S_i,B_{S_i}\right)\leq a\left(E;S_{i+1},B_{S_{i+1}}\right)$ for any divisor E over S. Thus,

$$d_{\mathcal{B}}\left(S_i,B_{S_i}\right)\geq d_{\mathcal{B}}\left(S_{i+1},B_{S_{i+1}}\right). \tag{1}$$

We claim that $\alpha_i:S_i\dashrightarrow S_{i+1}$ is an isomorphic in codimension 1 for $i\gg0$. Suppose there is a divisor $P\subseteq S_{i+1}$ and $\left(\alpha_i^{-1}\right)_*P=0$. Then inequality (1) is strict, as $-a\left(P;S_{i+1},B_{S_{i+1}}\right)\in\mathcal{B}$. Notice that $\mathcal{B}\cap[0,1-\epsilon]$ and picard number is finite, so α_i is isomorphic in codimension 1 after deleting finitely many steps.

By the above argument, we only need to consider the flip $X_m\dashrightarrow X_{m+1}$ over Z_m. Since (X,B) containing ample divisor A and $\left(K_{X_m}+B_m\right)\Big|_{S_m}=K_{S_m}+B_{S_m}$, the pair $\left(S_m,B_{S_m}\right)$ satisfy BCHM condition.

Now, take $h:\tilde{S}_m\to S_m$ to be a $\mathbb{Q}$-factorialization. Then we have $K_{\tilde{S}_m}+B_{\tilde{S}_m}=h^*\left(K_{S_m}+B_{S_m}\right)$ and $\left(\tilde{S}_m,\tilde{B}_{S_m}\right)$ satisfies the BCHM condition.

Next, run an MMP with scaling $\tilde{C}_m$ over T_m, where $\tilde{C}_m$ is the birational transform of C_m, and T_m is the normalisation of the image of S_m in Z_m [4]. As a result, we obtain a minimal model $\left(\tilde{S}_{m+1},\tilde{B}_{S_m+1}\right)$, which factors through S_{m+1}, as S_{m+1} is the log canonical model of S_m.

For the same reason, we can inductively construct a sequence of MMP on $\left(\tilde{S}_m,\tilde{B}_{S_m}\right)$ over T_m:

$$\tilde{S}_m=S_{m,0}\to S_{m,1}\to\cdots\to S_{m,l}=\tilde{S}_{m+1}$$

Since $K_{X_m}+B_m+t_mC_m$ is numerically trivial over Z_m, the induced divisors $K_{S_{m,i}}+B_{m,i}+t_mC_{m,i}\equiv0$ over Z_m. Hence, it is straightforward to verify that the sequence

$$\tilde{S}=\tilde{S}_0=S_{0,0}\dashrightarrow S_{0,1}\dashrightarrow\cdots\dashrightarrow S_{0,l}=\tilde{S}_1=S_{1,0}\dashrightarrow S_{1,1}\dashrightarrow\cdots$$

is, in fact, an MMP on $\tilde{f}:\left(\tilde{S},B_{\tilde{S}}\right)\to Z$ with scaling of $\tilde{C}$.

By inductive hypothesis, this MMP terminates. This means that, after finitely many steps, $K_{\tilde{S}_m}+\tilde{B}_{S,m}$ becomes nef over T_m, and consequently, $K_{S_m}+B_{S_m}$ is also nef over T_m. On the other hand, $-\left(K_{S_m}+B_{S_m}\right)$ is ample over T_m, so $S_m\to T_m$ does not contract any curve on Z_m. Similarly, $S_{m+1}\to T_m$ does not contract any curve on S_{m+1}. If S_m intersects $Exc\left(\alpha_m\right)$, then S_m as a divisor on X_m ample over T_m. Hence, $-S_{m+1}$ is ample over T_m, which contradicts to the fact that $S_{m+1}\to T_m$ does not contract any curve on S_{m+1}. This implies that the original MMP terminates in a neighborhood of S.

To construct log minimal model in dimension n assuming non-vanishing, one needs the special termination with scaling in dimension n, which is reduced to termination with scaling in lower dimension. More precisely, let $(X/U, B+C)$ be a klt pair of dimension n and satisfy BCHM condition. Run MMP on $K_X + B$ over U with scaling C, we need to prove this MMP terminates. By the definition of such MMP, $K_{X_i} + B_i + \lambda_i C_i$ is nef over U and $\lambda_1 \geq \lambda_2 \geq \cdots$. Then $(X_i, B_i + \lambda_i C_i)$ is a minimal model of $(X, B + \lambda_i C_i)$ over U.

The critical aspect of proving termination with scaling lies in demonstrating that there are only finitely many possible minimal models [5]. Specifically, if the MMP does not terminate, it implies the existence of infinitely many distinct minimal models, which would contradict the boundedness condition imposed by the special termination.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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