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Stability of One Kind Complex-Valued System by Lyapunov Function with Impulsive Control Field

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Abstract

In this paper, we investigate one kind of complex-valued systems with an impulsive control field, where the complex-valued system is governed by the Schrödinger equation, which is used for quantum systems, etc. We study the convergence of the complex-valued system with impulsive control fields by one Lyapunov function based on the state distance and the invariant principle of impulsive systems. We propose new results for the mentioned complex-valued systems in the form of sufficient conditions and also present one numerical simulation to illustrate the effectiveness of the proposed control method.

Keywords

Complex-Valued System, Lyapunov Function, Stability

1. Introduction

Extending control to complex-valued systems, such as quantum systems, *i.e.*, physical systems whose behavior is not governed by classical laws but dominated by quantum effects, has become an important area of research recently [1]-[6] and references therein. The growing interest in the subject can be attributed both to theoretical and experimental breakthroughs that have made control of quantum phenomena an increasingly realistic objective, as well as the prerequisite for many exciting novel technologies, such as quantum chemistry, quantum information processing, quantum electronics, etc. One of the proposed techniques to control quantum systems is the Lyapunov method ([7]-[11]). Lyapunov Asymptotic Stability (LAS) deals with the behavior of a system within a sufficiently long (in principle infinite) time interval. The Hilbert-Schmidt state distance between an arbitrary initial state and an arbitrary target state is used as the Lyapunov function ([7] [12]).

As a matter of fact, practically, there has been increasing interest in the analysis and synthesis of impulsive systems, or impulsive control systems, due to their significance both in theory and applications, see [13]-[16] and the references therein. The switching control method has been applied to control problems in many systems ([17]-[20]) and is also available to complex-valued systems. Inspired by the switching control method, we develop the impulsive control method to drive a complex-valued system to a given target state.

In this paper, we develop the notion of impulsive stability for one kind of complex-valued systems based on the Lyapunov method and the invariant principle of impulsive systems. The further part of the paper goes ahead with notations and definitions of quantum systems (one kind of complex-valued systems) with impulsive control fields and introduces the invariant principle of impulsive systems. Section 3 provides one control field to drive quantum systems based on a Lyapunov function and analyzes the asymptotic stability of quantum systems with impulsive control fields. We justify the effectiveness of the proposed control field in one simulation experiment in Section 4.

2. Mathematical Preliminaries

Consider the impulsive dynamical system described by

$$\begin{cases} \dot{z}(t) = f_c(z(t)), & t \in (\tau_k, \tau_{k+1}); \\ \Delta z(t) = f_d(z(t)), & t = \tau_k. \end{cases} \tag{2.1}$$

where $z(t) \in \mathbb{R}^n$ denotes the system state, $f_c(z)$ is a continuous function from $\mathbb{R}^n$ to $\mathbb{R}^n$, the set $E = \{\tau_1, \tau_2, \cdots : \tau_1 < \tau_2 < \cdots\} \subset \mathbb{R}^+$ is an unbounded, closed, discrete subset of $\mathbb{R}^+$, which denotes the set of times when jumps occur and $f_d : \mathbb{R}^n \to \mathbb{R}^n$ denotes the incremental change of the state at the time τ_k. In the n-dimensional complex space $\mathbb{C}^n$, we choose the most common norm $\|z\| := \sqrt{z^* z}$, where z is represented as a column vector $(z_1, z_2, \cdots, z_n)^{\mathrm{T}}$, and z^* denotes its conjugate transpose. Denote by $M_n(\mathbb{C})$ the space of $n \times n$ complex matrices with an inner product $(\cdot,\cdot) : M_n(\mathbb{C}) \times M_n(\mathbb{C}) \to \mathbb{C}$,

$$(a,b) = Tr(ab),$$

and the norm $\|a\|^2 = (a,a)$.

Consider the following complex-valued system, which is a n-level quantum system with two control fields, and set the Plank constant $\hbar = 1$:

$$i|\dot{\alpha}(t)\rangle = \left(H_0 + f_1(t) H_1 + \sum_{k=1}^{\infty} f_2(t) H_2 \delta(t - \tau_k) \right) |\alpha(t)\rangle, \tag{2.2}$$

where the ket $|\alpha(t)\rangle \in \mathbb{C}^n$ represents the state vector of quantum systems, which is right continuous, and the state vector evolves on or in a sphere with radius one, and we denote the set of quantum states by VS_n, and $\delta(\cdot)$ is the Dirac impulse. Physically, two states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ that differ by a phase $\theta(t) \in R$, *i.e.*, $|\alpha_1\rangle = \exp(i\theta(t))|\alpha_2\rangle$, describe the same physical state in or on the sphere of $\mathbb{C}^n$.

We denote the bra associated with the ket $|\alpha(t)\rangle$ with $\langle\alpha(t)|$. When the quantum system evolves freely under its own internal dynamics, *i.e.*, there is no external field implemented on the system, just the free Hamiltonian H_0 is introduced. $H_j\ (j=1,2)$ represents the interaction energy between the system and the external classical control fields $f_j(t)(j=1,2)$, and are called interaction Hamiltonians. $H_j\ (j=0,1,2)$ are all $n\times n$ self-adjoint operators in the *n*-dimensional Hilbert space $\mathcal{H}$ and assumed to be time-independent. In this paper, we set the first control function $f_1(t)$ is continuous, the other one $f_2(t)$ only takes effect to quantum systems at the impulsive points *E*.

By choosing a special basis of Hilbert space, we can suppose H_0 diagonal, and set $H_0 = diag(a_1, a_2, \cdots, a_n)$, with $a_k \ge a_{k+1}$ and $w_{kl} = a_k - a_l \le 0$ when $k \le l$, and

$$H_1 := (h_{jk})_{n\times n}, h_{jk} = h_{kj}^*, j,k = 1,2,\cdots,n,$$

under this basis.

For quantum systems, the target state is usually an eigenstate of the free Hamiltonian, and suppose the target state $|\alpha_f\rangle$ satisfies:

$$H_0|\alpha_f\rangle = \lambda_f|\alpha_f\rangle,$$

where λ_f is the eigenvalue of H_0 corresponding to $|\alpha_f\rangle$.

By the same method in [18], we obtain that quantum systems (2.2) with impulsive control fields can be described as

$$\begin{cases} i|\dot{\alpha}(t)\rangle = (H_0 + f_1(t)H_1)|\alpha(t)\rangle, & t \ne \tau_k; \\ \Delta|\alpha\rangle = f_2(t)H_2|\alpha(\tau_k^-)\rangle, & t = \tau_k. \end{cases} \tag{2.3}$$

When taking non-trival geometry about states, we add a second control ω corresponding to $\theta(t)$ into consideration [10], then investigate the following quantum systems

$$i|\dot{\alpha}(t)\rangle = \left(H_0 + f_1(t)H_1 + \sum_{k=1}^{\infty} f_2(t)H_2\delta(t-\tau_k) + \omega I\right)|\alpha(t)\rangle, \tag{2.4}$$

where I is the identity matrix. If the control field $f_2(t)$ only takes effect at the impulsive point *E*, the quantum systems with impulsive control fields are

$$\begin{cases} i|\dot{\alpha}(t)\rangle = (H_0 + f_1(t)H_1 + \omega I)|\alpha(t)\rangle, & t \ne \tau_k; \\ \Delta|\alpha\rangle = f_2(t)H_2|\alpha(\tau_k^-)\rangle, & t = \tau_k. \end{cases} \tag{2.5}$$

In order to control quantum systems(2.2) or (2.4) to target states, we focus on finding control fields $f_1(t)$ and $f_2(\tau_k)$, such that the quantum systems with impulsive control fields (2.3) or (2.5) are driven to target states. Firstly, we introduce the invariant principle of impulsive systems.

Lemma 2.1. [21] *Consider the impulsive dynamical system* (2.1), *assume* $\mathcal{D}_c \subset \mathcal{D}$ *is a compact positively invariant set with respect to* (2.1), *and assume that there exists a* C^1 *function* $V : \mathcal{D}_c \to \mathbb{R}$ *such that*

1) $\dot{V}(x(t)) \le 0, x \in \mathcal{D}_c, t \ne \tau_k$;

2) $V(x(\tau_k^-) + f_d(x(\tau_k^-)) \le V(x(\tau_k^-)), x \in \mathcal{D}_c, t = \tau_k$;

Let

$G \triangleq \{x \in \mathcal{D}_c : t \ne \tau_k, \dot{V}(x(t)) = 0\} \cup \{x \in \mathcal{D}_c : t = \tau_k, V(x(\tau_k^-) + f_d(x(\tau_k^-)) = V(x(\tau_k^-))\}$ *, and let* $M \subset G$ *denote the largest invariant set contained in G. If* $x_0 \in \mathcal{D}_c$ *, then* $x(t) \to M$ *as* $t \to \infty$.

We also applied the invariant principle of complex-valued impulsive systems in [13].

3. Main Results

In this section, we shall establish the stability criteria for one kind of complex-valued systems. The Lyapunov function is given based on quantum state distance, which will reach 0 if the system state is driven to the target state. It is commonly used in control theory.

Theorem 1. *For the quantum system* (2.3), *if* H_0 *is non-degenerate, set control fields* $f_1(t) = K_1 g_1\left(Im\left(e^{i\angle\langle\alpha(t)|\alpha_f\rangle}\langle\alpha_f|H_1|\alpha(t)\rangle\right)\right)$ *and* $f_2(\tau_k) = K_2 g_2\left(Re\left(e^{i\angle\langle\alpha(\tau_k^-)|\alpha_f\rangle}\langle\alpha_f|H_2|\alpha(\tau_k^-)\rangle\right)\right)$ *where constants* $K_1, K_2 > 0$, *the image of function* $y_j = g_j(x_j)(j = 1,2)$ *passes the origin of plane* $x_j - y_j$ *monotonically and lies in quadrant I or III, then quantum systems with impulses* (2.3) *converge to the largest invariant set* $VS_n \cap E_1$, *where* $E_1 = \{|\alpha\rangle : \langle\alpha_f|H_1|\alpha\rangle = 0\}$. *If all the states in* E_1 *are equivalent to the target state* $|\alpha_f\rangle$, *then the systems will converge asymptotically to the target state* $|\alpha_f\rangle$.

Proof. Choosing a Lyapunov function

$$V(|\alpha(t)\rangle, t) = \frac{1}{2}\left(1 - |\langle\alpha_f|\alpha(t)\rangle|^2\right). \tag{3.1}$$

When $t \ne \tau_k$,

$$\begin{aligned}\dot{V} &= -f_1(t)Im(\langle\alpha_f|H_1|\alpha(t)\rangle\langle\alpha(t)|\alpha_f\rangle) \\ &= -f_1(t)|\langle\alpha(t)|\alpha_f\rangle| Im\left(e^{i\angle\langle\alpha(t)|\alpha_f\rangle}\langle\alpha_f|H_1|\alpha(t)\rangle\right),\end{aligned}$$

as discussed in [8], by the control field

$$f_1(t) = K_1 g_1\left(Im\left(e^{i\angle\langle\alpha(t)|\alpha_f\rangle}\langle\alpha_f|H_1|\alpha(t)\rangle\right)\right), \tag{3.2}$$

we have

$$\begin{aligned}\dot{V}(t) = &-K_1|\langle\alpha(t)|\alpha_f\rangle| Im\left(e^{i\angle\langle\alpha(t)|\alpha_f\rangle}\langle\alpha_f|H_1|\alpha(t)\rangle\right) g_1\left(Im\left(e^{i\angle\langle\alpha(t)|\alpha_f\rangle}\langle\alpha_f|H_1|\alpha(t)\rangle\right)\right) \\ &< 0(t \ne \tau_k).\end{aligned}$$

When $t = \tau_k$,

$$\begin{aligned} &V\left(\left|\alpha\left(\tau_k\right)\right\rangle,\tau_k\right)=V\left(\left|\alpha\left(\tau_k^+\right)\right\rangle,\tau_k^+\right)\\ &=\frac{1}{2}\left(1-\left\langle\alpha\left(\tau_k^-\right)\right|\left(I+f_2\left(\tau_k\right)H_2\right)\left|\alpha_f\right\rangle\left\langle\alpha_f\right|\left(I+f_2\left(\tau_k\right)H_2\right)\left|\alpha\left(\tau_k^-\right)\right\rangle\right)\\ &=V\left(\left|\alpha\left(\tau_k^-\right)\right\rangle,\tau_k^-\right)-f_2\left(\tau_k\right)\left|\left\langle\alpha\left(\tau_k^-\right)\middle|\alpha_f\right\rangle\right|Re\left(e^{i\angle\left\langle\alpha\left(\tau_k^-\right)\middle|\alpha_f\right\rangle}\left\langle\alpha_f\right|H_2\left|\alpha\left(\tau_k^-\right)\right\rangle\right)\\ &\quad-\frac{1}{2}f_2^2\left(\tau_k\right)\left\langle\alpha\left(\tau_k^-\right)\right|H_2\left|\alpha_f\right\rangle\left\langle\alpha_f\right|H_2\left|\alpha\left(\tau_k^-\right)\right\rangle, \end{aligned} \tag{3.3}$$

by the control field

$$f_2\left(\tau_k\right)=K_2g_2\left(Re\left(e^{i\angle\left\langle\alpha\left(\tau_k^-\right)\middle|\alpha_f\right\rangle}\left\langle\alpha_f\right|H_2\left|\alpha\left(\tau_k^-\right)\right\rangle\right)\right), \tag{3.4}$$

and $\left\langle\alpha\left(\tau_k^-\right)\right|H_2\left|\alpha_f\right\rangle\left\langle\alpha_f\right|H_2\left|\alpha\left(\tau_k^-\right)\right\rangle>0$, we have

$$V\left(\left|\alpha\left(\tau_k\right)\right\rangle,\tau_k\right)<V\left(\left|\alpha\left(\tau_k^-\right)\right\rangle,\tau_k^-\right), \tag{3.5}$$

where $K_j\left(j=1,2\right)$ can be chosen properly to adjust the control amplitude. And if $\left\langle\alpha\left(t\right)\middle|\alpha_f\right\rangle=0$, or $\left\langle\alpha\left(\tau_k^-\right)\middle|\alpha_f\right\rangle=0$, we set $\angle\left\langle\alpha\left(t\right)\middle|\alpha_f\right\rangle=0^\circ$, or $\angle\left\langle\alpha\left(\tau_k^-\right)\middle|\alpha_f\right\rangle=0^\circ$.

By the definition of the invariant set and properties of the limit point, if we choose the control field $f_1\left(t\right)$ (3.2), which is the same as that in [8], the largest invariant set of quantum systems with impulses (2.3) is $VS_n\cap E_1$, where $E_1=\left\{\left|\alpha\right\rangle:\left\langle\alpha_f\right|H_1\left|\alpha\right\rangle=0\right\}$. From the invariant principle Lemma 2.1, quantum systems with impulsive control fields (2.3) will converge to $VS_n\cap E_1$.

Thus, we complete the proof.

4. Illustrative Examples

In order to illustrate the effectiveness of the proposed method in this paper, one numerical simulation has been presented for one five-level quantum system, which is a complex-valued system, and the Fourth-order Runge-Kutta method is used to solve with time steps size 0.06.

Example 1. *Consider the five-level quantum system with internal Hamiltonian, the first control Hamiltonian* [4, 8], *and the second control Hamiltonian given as follows*:

$$H_0=\begin{pmatrix}1&0&0&0&0\\0&1.2&0&0&0\\0&0&1.3&0&0\\0&0&0&2&0\\0&0&0&0&2.15\end{pmatrix},H_1=\begin{pmatrix}0&0&0&1&1\\0&0&0&1&1\\0&0&0&1&1\\1&1&0&0&0\\1&1&0&0&0\end{pmatrix},H_2=\begin{pmatrix}0&0&1&0&0\\0&0&0&1&1\\1&0&0&0&0\\0&1&0&0&1\\0&1&0&1&0\end{pmatrix}.$$

Let the initial state and the target state be $\left|\alpha_0\right\rangle=\begin{pmatrix}1&0&0&0&0\end{pmatrix}^{\mathrm{T}}$ *and* $\left|\alpha_f\right\rangle=\begin{pmatrix}0&0&0&0&1\end{pmatrix}^{\mathrm{T}}$, *respectively. The parameters are chosen as* $K_1=0.15$, $K_2=0.001$. *Set the state* $\left|\alpha\left(t\right)\right\rangle=\begin{pmatrix}x_1&x_2&x_3&x_4&x_5\end{pmatrix}^{\mathrm{T}}$, *by the control fields.*

$$f_1(t) = K_1 Im\left(e^{i\angle\langle\alpha(t)|\alpha_f\rangle}\langle\alpha_f|H_1|\alpha(t)\rangle\right),$$

$$f_2(\tau_k) = K_2 Re\left(e^{i\angle\langle\alpha(\tau_k^-)|\alpha_f\rangle}\langle\alpha_f|H_2|\alpha(\tau_k^-)\rangle\right),$$

we have the simulation result shown in **Figure 1**. *The component* x_5 *increases to* 1 *as time is more than* 50, *and the other four components decrease to* 0, *especially, the components* x_1, x_2 and x_3 *when time is more than* 50. *It demonstrates the control performance with an impulsive control field* $f_2(\tau_k)$, *and the finial transition probability attains about* 0.94149, *which excels the one* (*about* 0.93785) *in* [8].

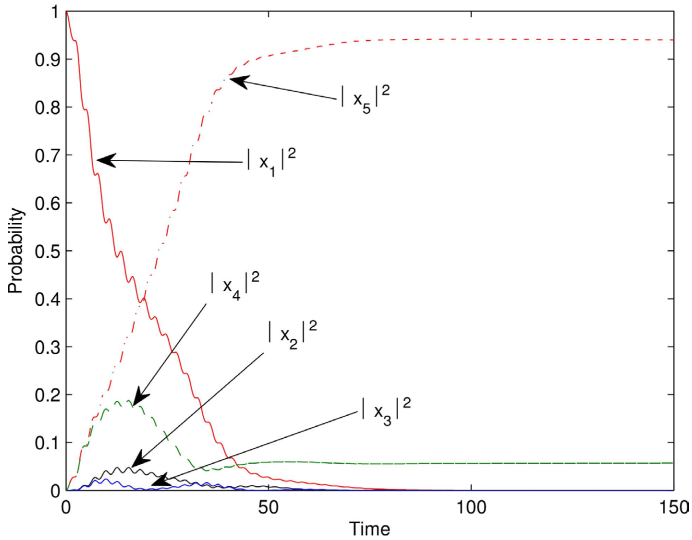

Figure 1. The population of the five-level system trajectory from $|\alpha_0\rangle$ by control fields f_1, f_2 in Example 1.

5. Conclusion

In this paper, the stability of a kind of complex-valued impulsive systems has been addressed. Taking advantage of the Lyapunov function based on system state distance in the complex fields, the stability criteria of a complex-valued impulsive system has been established, which not only generalised some known results in literature but also greatly reduced the complexity of analysis and computation. The theoretical results have been verified by a numerical simulation to illustrate the effectiveness and advantages of the proposed method compared with existing results.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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