

Modification of Constants in the Schrödinger Equation Giving Access to Neutrino Behavior

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Abstract

The paper deals with a study of the Schrödinger equation with an original approach. Recalling the well-known relation: $p \rightarrow i\hbar\Delta_r$. It considers this equation for which the kinetic factor is $E_{kin} = \frac{p^2}{2M} = -\frac{\hbar^2}{2M}\Delta_r^2$. Making the kinetic factor $E_{kin} = -\Delta_r^2$ can be obtained if one defines a mass $M = \frac{\hbar^2}{2}$, very small and close to the accepted mass of a neutrino ν_e . The Schrödinger equation reduces to: $-\Delta_r^2\phi(r) = E\phi(r)$. The energy E is that given by Dirac (1927), (c being the light velocity), with his remark that two solutions exist $E = \pm\sqrt{p^2c^2 + M^2c^4}$. The body of this paper shows all solutions obtained when solving the simplified Schrödinger equation.

Keywords

Schrödinger Equation, Planck Constant, Particle Physics, Neutrino Physics

1. Introduction

This paper studies all combinations and solutions for which the kinetic operator

$E_{kin} = -\frac{\hbar^2}{2M}\Delta_r^2 = -\Delta_r^2 = -1$ that is: $M = \frac{\hbar^2}{2}$. Thus:

$$E_{kin} = -\Delta_r^2 \quad (1)$$

Applying this transformation to the Schrödinger equation: [1] For free particles as photons or neutrinos, it is correct to put $V = 0$, and separate variables r and time t .

$$i\hbar\frac{\partial}{\partial t}\Psi(r,t) = -\frac{\hbar^2}{2M}\Delta_r^2\Psi(r,t) = (E - V)\Psi(r,t) \quad (2)$$

$$i\hbar \frac{\partial}{\partial t} \Psi(r, t) = -\frac{\hbar^2}{2M} \Delta_r^2 \Psi(r, t) = E \Psi(r, t) \quad (3)$$

$$\Psi(r, t) = \phi(r) \times e^{-\frac{iEt}{\hbar}} \quad (4)$$

$$-\Delta_r^2 \phi(r) = \frac{2M}{\hbar^2} E \phi(r) \quad (5)$$

The phase factor $e^{-\frac{iEt}{\hbar}} = 1$ for time $t=0$ and $\Psi(r, t)$ reduces to $\phi(r)$.

It is important to note that the kinetic factor $-\Delta_r^2$ makes this equation a wave equation that propagate.

Defining the Planck constant as: $\hbar = 1.054571 \times 10^{-34} \text{ J}\cdot\text{s}$ and $h = 2\pi \times \hbar$, a neutrino has a velocity close to the light velocity c , thus it is possible and reasonable to write $V=0$.

If one considers a photon, the energy is $E = pc = h\nu = \hbar\omega$. ν is the frequency of the photon.

In a 1D approach, it is possible to write:

$$-\frac{\hbar^2}{2M} \phi''(x) = E \phi(x) \quad (6)$$

$$-\frac{\hbar^2}{2M} \phi''(x) = Mc^2 \phi(x) \quad (7)$$

$$\phi''(x) = -\frac{2M}{\hbar^2} E \phi(x) \quad (8)$$

Equation (8) is the expression of a wave equation, with very simple trigonometric solutions. E is positive and restricted to the kinetic energy of the neutrino.

If the neutrino has a small mass, inferior to $0.8 \frac{e^2}{c^2}$, this mass is accepted for the ν_e , [2] the electronic neutrino: mass of the neutrino

$$m_{\nu_e} = 1.425976 \times 10^{-36} \text{ kg}.$$

Dealing with relativity, the energy has to be changed according to the famous Albert Einstein formula:

$$E = Mc^2 \text{ or } \gamma \times Mc^2, \text{ where } \gamma = \frac{1}{\sqrt{1-\beta^2}} \text{ is the Lorentz factor and } \beta = \frac{v}{c}.$$

This relation requires $\gamma=1$ that is $v=0$, thus $M = M_0$ is the rest mass. The impulse p is: $p = \gamma M \times v$ combining these relations. The relativistic energy follows from: [3]

The theory shows that in a frame where the velocity of the particle is v , the energy E and the impulse $p = M \times v$ are given by:

$$E = \gamma Mc^2 \equiv Mc^2 / \sqrt{1-(v^2/c^2)}, \quad p = \gamma Mv \equiv Mv / \sqrt{1-(v^2/c^2)}.$$

Finally:

$$E^2 - p^2 c^2 = M^2 c^4.$$

Assuming a small but real mass M for the neutrino, it is correct to write:

$$E^2 = p^2 c^2 + M^2 c^4 \quad (9)$$

$$E = \pm \sqrt{p^2 c^2 + M^2 c^4} \quad (10)$$

Retaining the positive root that is: $E = \sqrt{p^2 c^2 + M^2 c^4}$.

2. Obtaining the Kinetic Term E

Setting $E = M \times c^2$ the solution of Equation (5) is: with a 1D approach Equation (8) admits the simple solution:

$$\phi''(x) = -\frac{2M}{\hbar^2} E \phi(x) \quad (11)$$

$$\phi(x) = c_2 \sin\left(\sqrt{2} \sqrt{\frac{M^2 c^2}{\hbar^2}} x\right) \quad (12)$$

Numerically:

$$\phi(x) = c_2 \sin\left(2.84278 \times 10^{42} Mx\right) \quad (13)$$

c_2 is an integration constant taken as $c_2 = 1$. It is possible to define a mass M so that this mass M in this equation is written as: $M = \frac{\hbar}{c \times x}$.

It has a correct dimension of a mass x being a length, taking $x = \sqrt{2} \frac{\pi}{2}$, quantity that is a length; thus, the dimension of M is correct. Numerically $M = 3.518 \times 10^{-42}$ kg gives a simple argument in the $\phi(x)$ and this is compatible with an accepted neutrino mass, leaving aside the three different neutrino masses. Because of the smallness of the neutrino mass, [2] it is possible to write as a first-order development:

$$E = pc \times \sqrt{1 + \frac{M^2 c^4}{p^2 c^2}} \approx pc \times \left(1 + \frac{M^2 c^4}{2p^2 c^2}\right) \quad (14)$$

The quantity $pc = hv = \frac{hc}{\lambda}$ is the energy of a photon, or a neutrino with no mass.

Setting $E = M \times c^2$ the mass M in this equation to $M = \frac{\hbar}{c \times x}$. It has a correct dimension of a mass x being a length, taking $x = \sqrt{2} \frac{\pi}{2}$, quantity that is a length; thus, the dimension of M is correct. Numerically $M = 3.518 \times 10^{-42}$ kg compatible with an accepted neutrino mass, leaving aside the three different neutrino masses. [4] [5] The neutrino estimated mass is less than $m_\nu = \frac{0.8 \text{ eV}}{c^2}$ giving a numerical estimate: $m_\nu = 1.42685 \times 10^{-36}$ kg. It appears that the sin function is oscillatory as a photon wave function (not normalizable).

Because the neutrino estimated mass is very small, $m_\nu \leq \frac{0.8 \text{ eV}}{c^2}$ choosing the development at first order in Equation (14).

$$\text{That can be written } \Delta E = \frac{M^2 c^4}{2hv}.$$

Replacing M^2 by the value $M^2 = \frac{\hbar^2}{c^2 \times x^2}$.

$$E = pc \times \sqrt{1 + \frac{M^2 c^4}{p^2 c^2}} \approx pc \times \left(1 + \frac{M^2 c^4}{2p^2 c^2}\right) \quad (15)$$

$$E = pc + \frac{M^2 c^4}{2pc} \quad (16)$$

$$\Delta E = \frac{M^2 c^4}{2pc} \quad (17)$$

$$M = \sqrt{\frac{\hbar^2}{c^2 x^2}} \quad (18)$$

$$E = pc + \frac{\hbar^2 c^4}{2c^2 pc \times x^2} \quad (19)$$

$$E = pc + \frac{\hbar^2 c}{2p \times x^2} \quad (20)$$

For a photon (massless particle) $E = pc = h\nu$ thus $p = \frac{h\nu}{c}$. Thus, it proves that the neutrino (when considering its mass) differs from the photon by an amount of energy:

$$\Delta E = + \frac{\hbar^2 c}{2p \times x^2} \quad (21)$$

Let's solve the correct equation easily obtained with Mathematica: $E > 0$ is the photon energy: $pc = h\nu = h \frac{c}{\lambda}$.

$$E = pc + \frac{M^2 c^4}{2pc} \quad (22)$$

The mass M is defined: $M = \frac{\hbar}{c}$.

Inserting the mass M in the energy equation $E = pc + \frac{M^2 c^4}{2pc}$ gives numerically:

$$E = \frac{1.66702 \times 10^{-60}}{p \times x^2} + 2.99792 \times 10^8 p \quad (23)$$

The solution to find with Mathematica is a little more complex:

$$M = \frac{\hbar}{c \times x} \quad (24)$$

$$E = pc + \frac{\hbar^2 c}{2p \times x^2} \quad (25)$$

$$-\phi''(x) = E \times \phi(x) \quad (26)$$

$$-\phi''(x) = \left(pc + \frac{\hbar^2 c}{2p \times x^2} \right) \times \phi(x) \quad (27)$$

¹Using the relation $E = \frac{1.34719 \times 10^{25} M^2}{p} + 2.99792 \times 10^8 p$, the Mathematica

solution is:

$$\phi(x) = \cos\left(\frac{2\sqrt{2.99792 \times 10^8 Px^2 - 1.34719 \times 10^{25} M^2 x}}{\sqrt{Px}}\right) \tag{28}$$

$$+ i \sin\left(\frac{2\sqrt{2.99792 \times 10^8 Px^2 - 1.34719 \times 10^{25} M^2 x}}{\sqrt{Px}} - 1\right) \tag{29}$$

$$\left(\cos\left(\frac{\sqrt{1.34719 \times 10^{25} M^2 + 2.99792 \times 10^8 Px^2 x}}{\sqrt{Px}}\right)\right) \tag{30}$$

$$- i \sin\left(\frac{\sqrt{1.34719 \times 10^{25} M^2 + 2.99792 \times 10^8 Px^2 x}}{\sqrt{Px}}\right) \tag{31}$$

Finally, taking the real part of $\phi(x)$ the expression reduces to:

$$\phi(x) = \cos\left(\frac{2\sqrt{2.99792 \times 10^8 Px^2 - 1.34719 \times 10^{25} M^2 x}}{\sqrt{Px}} - 1\right) \tag{32}$$

$$\left(\cos\left(\frac{\sqrt{1.34719 \times 10^{25} M^2 + 2.99792 \times 10^8 Px^2 x}}{\sqrt{Px}}\right)\right)$$

or else:

$$\phi(x) = \cos\left(\frac{x\sqrt{1.34719 \times 10^{25} M^2 + 2.99792 \times 10^8 Px^2}}{\sqrt{Px}}\right) \tag{33}$$

$$\left(\cos\left(\frac{2x\sqrt{2.99792 \times 10^8 Px^2 - 1.34719 \times 10^{25} M^2}}{\sqrt{Px}}\right) - 1\right)$$

To be closer to a full solution, it is possible to obtain the squared module of these functions:

$$|\phi(x)|^2 = \cos^2\left(\frac{\sqrt{1.34719 \times 10^{25} M^2 + 2.99792 \times 10^8 Px^2 x}}{\sqrt{Px}}\right) \tag{34}$$

$$\left(\cos\left(\frac{2x\sqrt{2.99792 \times 10^8 Px^2 - 1.34719 \times 10^{25} M^2}}{\sqrt{Px}}\right) - 1\right)^2$$

$$+ \sin^2\left(\frac{2x\sqrt{2.99792 \times 10^8 Px^2 - 1.34719 \times 10^{25} M^2}}{\sqrt{Px}}\right)$$

$$- \sin^2\left(\frac{\sqrt{1.34719 \times 10^{25} M^2 + 2.99792 \times 10^8 Px^2 x}}{\sqrt{Px}}\right)$$

¹Here, the light velocity is fixed $c = 2.99792 \times 10^8 \text{ m} \cdot \text{s}^{-1}$.

Using the relation $E = pc + \frac{\hbar^2 \times c}{2p \times x^2}$,

$$E = \frac{1.34719 \times 10^{25} M^2}{p} + 2.99792 \times 10^8 p \quad (35)$$

leads to another solution is:

$$\begin{aligned} \phi(x) = & \sqrt{x} (\text{Bessel}J[\frac{6.6681038 \times 10^{-60}}{2\sqrt{Px}} (17314.5\sqrt{Px}x)] \\ & + \text{Bessel}Y[\frac{6.6681038 \times 10^{-60}}{2\sqrt{Px}} (17314.5\sqrt{Px}x)] \end{aligned} \quad (36)$$

The relation $E = -\sqrt{p^2 c^2 + M^2 c^4}$ holds with the negative sign and gives another solution for $\phi(x)$ expressed as an exponential. Changing the sign of the energy, that is:

$$-\phi''(x) = -E \times \phi(x) \quad (37)$$

$$\phi''(x) - \frac{2M^2 c^2}{\hbar^2} \phi(x) = 0 \quad (38)$$

leads to other solutions: That is with M, Px, x variables:

$$\phi(x) = e^{\frac{x\sqrt{1.3471910^{25} M^2 + 2.9979210^8 Px^2}}{\sqrt{Px}}} - e^{-\frac{x\sqrt{1.3471910^{25} M^2 + 2.9979210^8 Px^2}}{\sqrt{Px}}} \quad (39)$$

$$\phi(x) = 2 \sinh\left(\frac{x\sqrt{1.34719 \times 10^{25} M^2 + 2.9979 \times 10^8 Px^2}}{\sqrt{Px}}\right) \quad (40)$$

That can be shown with several Plot3D functions with Mathematica:²

3. Conclusions

The main idea in this article is to change the kinetic term that makes the Schrödinger equation:

$$-\frac{\hbar^2}{2M} \Delta_r^2 \phi(r) = E \phi(r) \quad (41)$$

It is possible to change the factor $-\frac{\hbar^2}{2M} \Delta_r^2$, that is $M = \frac{\hbar^2}{2}$, then the wave equation is simpler:

$$-\Delta_r^2 \phi(r) = E \phi(r) \quad (42)$$

In a 1D approach, it reduces to:

$$-\frac{d^2}{dx^2} \phi(x) = E \phi(x) \quad (43)$$

$$-\phi''(x) = E \phi(x) \quad (44)$$

The wave solution is simpler. With such small mass $M = \frac{\hbar^2}{2} = 0.55 \times 10^{-68} \text{ kg}$, it can concern neutrinos with their maximal estimated mass:

²The integration constant is fixed to $c_1 = 1$.

$$M_\nu = 0.8 \frac{\text{eV}}{c^2} = 1.42598 \times 10^{-36} \text{ kg}.$$

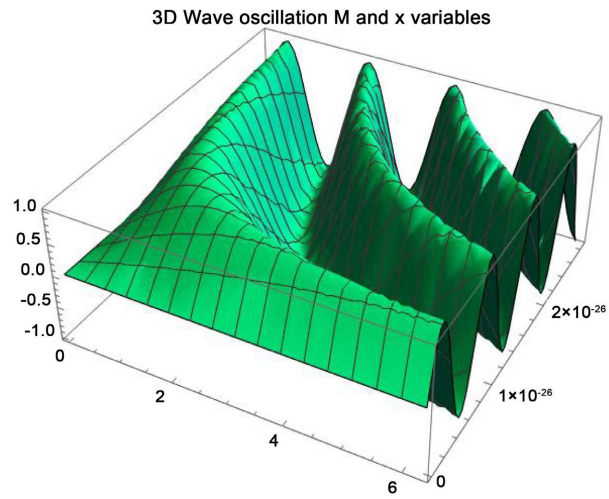


Figure 1. $E =$ simple oscillating 3D plot (Mass, x) with a $0 \leq M \leq 4.48 \times 10^{-26} \text{ kg}$ and $0 \leq x \leq 4\pi$ as a variable.

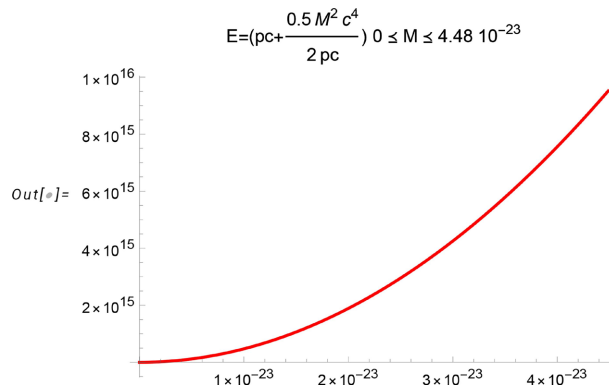


Figure 2. $E = pc + \frac{0.5c^4 M^2}{2pc}$ ($0 \leq M \leq 4.48 \times 10^{-23} \text{ kg}$) pc is the photon energy.

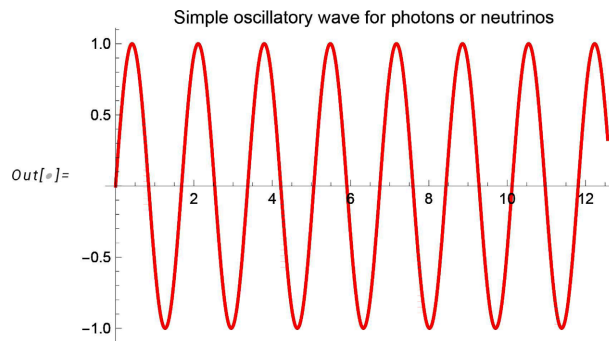


Figure 3. Simple oscillating solution free wave function for photon ($M=0$) and neutrinos with a small neutrino fixed mass $M = 2.84 \times 10^{-26} \text{ kg}$ with x as a variable.

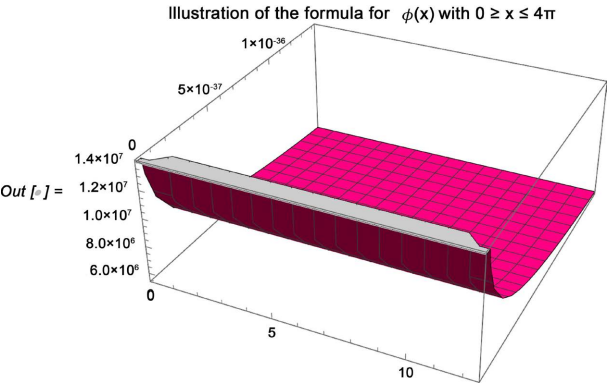


Figure 4. Exponential solution

$$\phi(x) = 2 \sinh\left(\frac{\sqrt{2.9979 \times 10^8 Px^2 + 1.6670 \times 10^{-60} x}}{\sqrt{Px}}\right)$$

the variable $0 \le Px \le 1.42598 \times 10^{-36}$ kg Px maximum value for the neutrino with $0 \le x \le 4\pi$ as a variable.

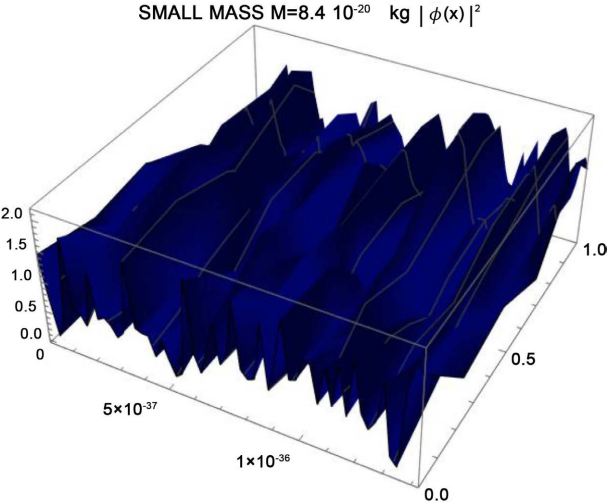


Figure 5. Solution $|\phi(x)|^2$ the mass M is fixed

$M = 8.4 \times 10^{-20}$ kg and $0 \le x \le 4\pi$ with Px (impulse) $\le 1.42598 \times 10^{-36}$ kg, for the neutrino with $0 \le x \le 4\pi$ as a variable.

Owing to the sign of the Einstein formula $E = \pm\sqrt{p^2 c^2 + M^2 c^4}$ two solutions exist, one is a wave equation that is the usual wave function of a photon or a neutrino propagating in space-time.

The second solution gives a new insight into neutrino theory. The complex wave function appears, a new phenomenon shown in this paper.

That can be shown with several Plot3D functions with Mathematica. (See **Figures 1-5**)

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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