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Propagation Dynamics of Forced Pulsating Waves for a Time Periodic Lotka-Volterra Cooperative System with Nonlocal Effects in Shifting Habitats

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Abstract

In this paper, we will concern the existence, asymptotic behaviors and stability of forced pulsating waves for a Lotka-Volterra cooperative system with nonlocal effects under shifting habitats. By using the alternatively-coupling upper-lower solution method, we establish the existence of forced pulsating waves, as long as the shifting speed falls in a finite interval where the endpoints are obtained from KPP-Fisher speeds. The asymptotic behaviors of the forced pulsating waves are derived. Finally, with proper initial, the stability of the forced pulsating waves is studied by the squeezing technique based on the comparison principle.

Keywords

Nonlocal Effects, Time Periodic Lotka-Volterra System, Forced Pulsating Waves, Shifting Habitats

1. Introduction

Climate change, such as global warming, is believed to be the greatest threat to biodiversity [1]. Global warming has caused the destruction of Marine species diversity near the equator, and species have shown a trend of migration to the north and south poles. In the past, the tropics provided ideal temperatures for many species. But as the equatorial waters get hotter, the outflow of the species that originally lived there accelerates. Ocean warming is causing large-scale changes in the latitudinal gradient of Marine biodiversity. At the same time, creatures that live on land would also move to the poles and colder elevations. Climate change drives the shifts in species range and distribution, see [2] [3]. This impact on ecological species has to be taken seriously.

For this phenomenon and its influence, many researchers have made very great scientific research results, see [4]-[12]. Berestycki *et al.* [13] proposed a reaction-diffusion equation under a shifting environment.

$$u_t(t,x) = du_{xx}(t,x) + g\left(x-ct, u(t,x)\right), t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{1}$$

Here, $u(t,x)$ denotes the population density at time *t* and location *x*. The function *g* represents the net effect of reproduction and mortality and $d>0$ is the diffusion rate for species. They have proved the existence of the forced traveling waves for Equation (1). In [14], Berestycki and Fang established the existence and nonexistence of forced waves for the Fisher-KPP equation in a shifting environment.

$$u_t(t,x) = du_{xx}(t,x) + u(t,x)\left[r(x-ct) - u(t,x)\right], t \in \mathbb{R}^+, x \in \mathbb{R}. \tag{2}$$

Wu *et al.* [15] were concerned with the existence and uniqueness of forced waves in a general reaction-diffusion equation with time delay under climate change. They showed that a nondecreasing and unique wavefront with a speed consistent with the habitat shifting speed exists for Equation (2).

Species interactions can influence the range sizes of populations. Both two species follow the Logistic growth rate, which is "on the move" to capture the key point that the environment is both heterogeneous and directionally shifting over time with a forced rate $c>0$. As is well known, there are usually more than one biological species sharing the same habitat and their typically interspecies relationships. Thus, there is a growing interest in the study of two species in shifting habitats, for example, competition [16] [17], cooperation [18] [19] and predator-prey [20] [21].

Subject to seasonal succession, climate change provides such a shifting and time-periodic environment for the species. Fang *et al.* in [22] studied the nonautonomous reaction-diffusion equation in a time-periodic shifting environment,

$$u_t(t,x) = u_{xx}(t,x) + u(t,x)\,g\left(t, x-ct, u(t,x)\right), t>0, x \in \mathbb{R}.$$

That is $g\left(x-ct, u(t,x)\right)$ in Equation (1) becomes $g\left(t, x-ct, u(t,x)\right)$. It can be understood as the functional response to the time-periodic variation. Periodicity frequently appears in mathematical modelings due to seasonal changes typically related to climate changes. In the case when $r_i(t)$ become $r_i(t, x-ct), i=1,2$ in [19], we can get the following time periodic Lotka-Volterra cooperative system

$$\begin{cases} u_t(t,x) = d_1 u_{xx}(t,x) + u(t,x)\left[r_1(t,x-ct) - u(t,x) + a_1 v(t,x)\right], t \in \mathbb{R}^+, x \in \mathbb{R}, \\ v_t(t,x) = d_2 v_{xx}(t,x) + v(t,x)\left[r_2(t,x-ct) - v(t,x) + a_2 u(t,x)\right], t \in \mathbb{R}^+, x \in \mathbb{R}, \end{cases} \tag{3}$$

where $u(t,x)$ and $v(t,x)$ are the population densities of two species competing for common resource at time *t* and position *x*; d_1 and d_2 are the diffusive coefficients; the parameters a_1 and a_2 reflect the strength of interspecies cooperation and $a_i>0, i=1,2$. Most importantly, the terms $r_1(t,x-ct)$ and $r_2(t,x-ct)$

are dependent on time t and the climate shifting variable $x-ct$. $r_i(t,\cdot), i=1,2$ are assumed to be T-periodic in the first variable t for some positive number T. We have studied the existence, asymptotic behavior, and stability of forced pulsating waves for Equation (3) in our previous work. For the monostable case, Zhao and Ruan [23] showed that system Equation (2) possesses periodic traveling waves only when the wave speed is greater than or equal to a minimal wave speed $c_{\min}$. Liang, Yi, and Zhao [24] investigated spreading speeds and traveling wave solutions for general periodic evolution systems.

Note that the classical reaction-diffusion equation, like Equation (2), is based on the assumption that the internal interaction of species is random and local, *i.e.*, individuals move randomly between adjacent spatial locations. However, this is not always the case in reality. The movements and interactions of many species in ecology and biology can occur between non-adjacent spatial locations, see [25]. Thus, considering the shifting environment and nonlocal predation, Equation (3) was modified to the following form:

$$\begin{cases} u_t(t,x)=d_1u_{xx}(t,x)+u(t,x)\left[r_1(t,x-ct)-u(t,x)+a_1\left(J_1*v(t,x)\right)\right], t\in\mathbb{R}^+, x\in\mathbb{R}, \\ v_t(t,x)=d_2v_{xx}(t,x)+v(t,x)\left[r_2(t,x-ct)-v(t,x)+a_2\left(J_2*u(t,x)\right)\right], t\in\mathbb{R}^+, x\in\mathbb{R}, \end{cases} \tag{4}$$

where

$$(J_1*v)(t,x)=\int_{\mathbb{R}}J_1(x-y)v(t,y)\mathrm{d}y=\int_{\mathbb{R}}J_1(y)v(t,x-y)\mathrm{d}y,$$
$$(J_2*u)(t,x)=\int_{\mathbb{R}}J_2(x-y)u(t,y)\mathrm{d}y=\int_{\mathbb{R}}J_2(y)u(t,x-y)\mathrm{d}y.$$

This paper is devoted to the existence, asymptotic behaviors and stability of forced pulsating waves of the Equation (4).

Throughout the present paper, the following assumptions are valid.

(H_1) Assume that $r_i(t,z), i=1,2$ is continuous, T-periodic in t and increasing in z. Moreover,

$$\lim_{z\to-\infty}r_i(t,z)=\beta_i(t)<0, \lim_{z\to\infty}r_i(t,z)=\theta_i(t)>0, i=1,2, \tag{5}$$

uniformly in t, where $\theta_i(t),\beta_i(t)\in C^\gamma(\mathbb{R},\mathbb{R})$ for some γ *with* $\gamma\in(0,1)$ and they are T-periodic functions, that is $\beta_i(t+T)=\beta(t)$, $\theta_i(t+T)=\theta_i(t)$ for all $t\in\mathbb{R}^+$.

(H_2) There is

$$\left\|\theta_i(t)-r_i(t,z)\right\|\sim A_i\mathrm{e}^{-\alpha_i z}, z\to\infty,$$

for some positive numbers $\alpha_i, A_i(t), i=1,2$. Here, the symbol "~" is the standard sign in asymptotic analysis.

(H_3) $J_i(x)\in C(\mathbb{R},\mathbb{R}^+)$ are symmetric with $\int_{\mathbb{R}}J_i(y)\mathrm{d}y=1$ and there exists some $\lambda_0>0$ such that $\int_{\mathbb{R}}J_i(y)\mathrm{e}^{\lambda y}\mathrm{d}y<\infty$, $\forall\lambda\in(0,\lambda_0]$.

Next, we consider the following system of ordinary differential equations.

$$\begin{cases} u'(t)=u\left(\theta_1(t)-u+a_1v\right), \\ v'(t)=v\left(\theta_2(t)-v+a_2u\right). \end{cases}$$

Let $\overline{r_i} = \frac{1}{T}\int_0^T \theta_i(s)\mathrm{d}s > 0$ for $i = 1,2$. According to Theorem 1 of [26], the above the equation has a unique and globally asymptotically stable periodic positive solution $(p(t), q(t))$ under condition (H$_1$).

By a forced pulsating wave solution of the system Equation (4), we mean a particular solution in the form of

$$(u,v)(t,x) = (\tilde{U},\tilde{V})(t,x-ct) =: (\tilde{U},\tilde{V})(t,z),\ z = x - ct, \tag{6}$$

satisfying

$$(\tilde{U},\tilde{V})(t+T,z) = (\tilde{U},\tilde{V})(t,z).$$

A substitution of Equation (4) leads to the following wave profile system

$$\begin{cases} \tilde{U}_t = d_1\tilde{U}_{zz} + c\tilde{U}_z + \tilde{U}\left[r_1(t,z) - \tilde{U} + a_1\left(J_1 * \tilde{V}\right)\right], t \in \mathbb{R}^+, z \in \mathbb{R}, \\ \tilde{V}_t = d_2\tilde{V}_{zz} + c\tilde{V}_z + \tilde{V}\left[r_2(t,z) - \tilde{V} + a_2\left(J_2 * \tilde{U}\right)\right], t \in \mathbb{R}^+, z \in \mathbb{R}, \end{cases} \tag{7}$$

subjected to

$$\lim_{z\to-\infty}(\tilde{U},\tilde{V})(t,z) = (0,0),\ \lim_{z\to\infty}(\tilde{U},\tilde{V})(t,z) = (p(t),q(t)) \tag{8}$$

uniformly in t.

To our knowledge, the heterogeneity caused by the shifting and periodic coefficients brings nontrivial difficulties. Our contributions in this paper can be summarized in three parts. In Section 2, we establish the existence of the forced pulsating waves by applying the alternatively-coupling upper-lower solution method. In Section 3, we establish the asymptotic behaviors of the forced pulsating waves. In Section 4, with proper initial, the stability of the forced pulsating waves is studied by the squeezing technique based on the comparison principle.

2. Existence of Forced Pulsating Waves for Equation (4)

Throughout this paper, we will use notation $\overline{f(x)}$ to denote the average value of $f(x)$ on the interval $[0,T]$, namely,

$$\overline{f(x)} = \frac{1}{T}\int_0^T f(x)\mathrm{d}x.$$

For functions $R(t,z)$ and $\theta(t)$, we also use $\lim_{z\to\infty} R(t,z) = \theta(t)$ to denote $\lim_{z\to\infty}(R(t,z) - \theta(t)) = 0$ uniformly in t.

In order to demonstrate the existence of forced pulsating waves for Equation (4), we will provide an important lemma and examine the solvability of a spatio-temporal heterogenetic equation that may be thought of as a simplified form of the original system Equation (7).

Lemma 2.1. Assume that $R(t,z)$ is a nondecreasing and continuous function in z, and T-periodic in t, satisfying $\lim_{z\to-\infty} R(t,z) < 0$, $\lim_{z\to\infty} R(t,z) > 0$ uniformly in t. Then, for any $c > -c_0(d)$ with

$$c_0(d) := \min_{\mu>0}\frac{d\mu^2 + \overline{\theta(t)}}{\mu} = 2\sqrt{d\overline{\theta(t)}},$$

there exists a unique positive solution $w(t,z)$, which is nondecreasing in z and T-periodic in t, for the following boundary problem.

$$\begin{cases} w_t = dw_{zz} + cw_z + w\left(R(t,z) - w\right), t \in \mathbb{R}^+, z \in \mathbb{R}, \\ w(t,-\infty) = 0,\, w(t,\infty) = p(t), t \in \mathbb{R}^+. \end{cases} \tag{9}$$

Moreover, if one views the solution w as a functional of R and d, then $I(R,d) := w(R,d)(t,z)$ is nondecreasing in the variable R.

Proof. Lemma 2.1 in [27] can be utilized in a comparable manner to finish the proof, with the exception of uniqueness. Consequently, the details are omitted. For the uniqueness, we refer the reader to Lemma 3.2 in [22].

Theorem 2.1. Assume

$$c > c^*, \tag{10}$$

where $c^* = -\min\{c_0(d_1), c_0(d_2)\}$. Then, there exists a T-periodic solution $(\tilde{U},\tilde{V})(t,z)$ to the system Equations (7)-(8). Moreover, $\tilde{U}(t,z)$ and $\tilde{V}(t,z)$ are nondecreasing in variable z, respectively.

Proof. By Lemma 2.1, we can define two sequences of functions as below.

$$\begin{aligned} &\tilde{V}_0 := 0, \tilde{U}_0 := I(r_1, d_1), \\ &\tilde{V}_1 := I\left(r_2 + a_2\left(J_2 * \tilde{U}_0\right), d_2\right), \tilde{U}_1 := I\left(r_1 + a_1\left(J_1 * \tilde{V}_1\right), d_1\right), \\ &\cdots \\ &\tilde{V}_n := I\left(r_2 + a_2\left(J_2 * \tilde{U}_{n-1}\right), d_2\right), \tilde{U}_n := I\left(r_1 + a_1\left(J_1 * \tilde{V}_n\right), d_1\right). \end{aligned}$$

By taking $R(t,z) = r_1(t,z)$ in Lemma 2.1, we can conclude that $\tilde{U}_0(t,z) := I(r_1,d_1)$ is well-defined. Moreover, $\tilde{U}_0(t,-\infty) = 0$ and $\tilde{U}_0(t,\infty) = p(t)$ are held uniformly in t.

To proceed, we take $R(t,z) = r_2(t,z) + a_2\left(J_2 * \tilde{U}_0\right)(t,z)$ in Lemma 2.1. It is easy to see that $R(t,-\infty) = \beta(t) < 0$, $R(t,\infty) = \theta(t) + a_2 p(t) > 0$. Hence, it follows from Lemma 2.1 that $\tilde{V}_1 := I\left(r_2 + a_2\left(J_2 * \tilde{U}_0\right), d_2\right)$ is well-defined and $\tilde{V}_1(t,-\infty) = 0$ and $\tilde{V}_1(t,\infty) = q(t)$.

By a similar procedure, we can define $\tilde{U}_n(t,z), n \ge 1$ and $\tilde{V}_n(t,z), n \ge 2$ successively. Furthermore, under condition Equation (10), it can be derived directly from Lemmas 2.1 that $\tilde{U}_n(t,z)$ and $\tilde{V}_n(t,z)$ are nondecreasing in variable z for each $n \ge 1$. In the meantime, $\tilde{U}_n \le \tilde{U}_{n+1}$ and $\tilde{V}_n \le \tilde{V}_{n+1}$ for all $n \ge 0$. Further, together with the boundedness of two sequences $\{\tilde{U}_n(t,z)\}$ and $\{\tilde{V}_n(t,z)\}$, *i.e.*,

$$0 \le \tilde{U}_n, \tilde{V}_n \le \max_{t \in [0,T]} \theta(t),$$

for all $(t,z) \in \mathbb{R}^+ \times \mathbb{R}$, there exist $\tilde{U}(t,z)$ and $\tilde{V}(t,z)$ such that $(\tilde{U}_n,\tilde{V}_n)(t,z) \to (\tilde{U},\tilde{V})(t,z)$ pointwise as $n \to \infty$.

Moreover, $\tilde{U}(t,z)$ and $\tilde{V}(t,z)$ are nondecreasing with respect to z and T-periodic with respect to t. We can claim $\tilde{U}(t,z), \tilde{V}(t,z) \in C^1\left(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}\right)$. In fact, it can be justified by applying the standard regularity analysis on the integral forms. Consequently, the existence of $(\tilde{U},\tilde{V})(t,z)$ for $c \in \left(-\min\left(c_0(d_1), c_0(d_2)\right), \infty\right)$

is proved by virtue of Lemma 2.1.

Next, we check that the functions $(\tilde{U},\tilde{V})(t,z)$ satisfies the boundary condition Equation (8). Since $\tilde{U}(t,z)$ and $\tilde{V}(t,z)$ are bounded and monotone functions in *z*, we notice that there exist the limits of $(\tilde{U},\tilde{V})(t,z)$ at $z=\pm\infty$ for each $t\in\mathbb{R}^+$. Denote these limits by $\tilde{U}(t,\pm\infty)$ and $\tilde{V}(t,\pm\infty)$ which are nonnegative, *T*-periodic and bounded, respectively. Let $z\to\infty$ in Equation (7) yields

$$\begin{cases}\dfrac{\mathrm{d}\tilde{U}(t,\infty)}{\mathrm{d}t}=\tilde{U}(t,\infty)\left[\theta(t)-\tilde{U}(t,\infty)+a_1(J_1*\tilde{V})(t,\infty)\right],t\in\mathbb{R}^+,\\ \dfrac{\mathrm{d}\tilde{V}(t,\infty)}{\mathrm{d}t}=\tilde{V}(t,\infty)\left[\theta(t)-\tilde{V}(t,\infty)+a_2(J_2*\tilde{U})(t,\infty)\right],t\in\mathbb{R}^+.\end{cases}$$

Let

$$f_1(t,\tilde{U},\tilde{V})=\tilde{U}(t,\infty)\left[\theta(t)-\tilde{U}(t,\infty)+a_1(J_1*\tilde{V})(t,\infty)\right],$$
$$f_2(t,\tilde{U},\tilde{V})=\tilde{V}(t,\infty)\left[\theta(t)-\tilde{V}(t,\infty)+a_2(J_2*\tilde{U})(t,\infty)\right].$$

Since $\theta(t)\in C^{\gamma}(\mathbb{R},\mathbb{R})$ for some γ with $\gamma\in(0,1)$ and $\theta(t+T)=\theta(t)$ for all $t\in\mathbb{R}^+$, we can assume $0<\theta(t)\le N$. For $0\le\tilde{U}(t,\infty),\tilde{V}(t,\infty)\le M$ and $0\le\tilde{U}_1(t,\infty),\tilde{V}_1(t,\infty)\le M$, we have

$$\begin{aligned}&\left|f_1(t,\tilde{U},\tilde{V})-f_1(t,\tilde{U}_1,\tilde{V}_1)\right|\\&=\left|\left(\theta(t)-(\tilde{U}+\tilde{U}_1)+a_1(J_1*\tilde{V})\right)(\tilde{U}-\tilde{U}_1)+a_1(J_2*\tilde{U}_1)(\tilde{V}-\tilde{V}_1)\right|\\&\le(N+a_1M)\left|\tilde{U}-\tilde{U}_1\right|+a_1M\left|\tilde{V}-\tilde{V}_1\right|\\&\le(N+a_1M)\left(\left|\tilde{U}-\tilde{U}_1\right|+\left|\tilde{V}-\tilde{V}_1\right|\right),\end{aligned}$$

and

$$\begin{aligned}&\left|f_2(t,\tilde{U},\tilde{V})-f_2(t,\tilde{U}_1,\tilde{V}_1)\right|\\&=\left|a_2(J_1*\tilde{V}_1)(\tilde{U}-\tilde{U}_1)+\left(\theta(t)-(\tilde{V}+\tilde{V}_1)+a_2(J_2*\tilde{U})\right)(\tilde{V}-\tilde{V}_1)\right|\\&\le a_2M\left|\tilde{U}-\tilde{U}_1\right|+(N+a_2M)\left|\tilde{V}-\tilde{V}_1\right|\\&\le(N+a_2M)\left(\left|\tilde{U}-\tilde{U}_1\right|+\left|\tilde{V}-\tilde{V}_1\right|\right).\end{aligned}$$

That is to say

$$\begin{aligned}&\left\|f_i(t,\tilde{U},\tilde{V})-f_i(t,\tilde{U}_1,\tilde{V}_1)\right\|\\&\le L\left(\left\|\tilde{U}-\tilde{U}_1\right\|+\left\|\tilde{V}-\tilde{V}_1\right\|\right),i=1,2,\end{aligned}$$

where

$$L=\max\{N+a_1M,N+a_2M\}.$$

It indicates that Equation (7) has a solution at $z\to\infty$, denoted $(p(t),q(t))$, *i.e.*,

$$\lim_{z\to\infty}(\tilde{U},\tilde{V})(t,z)=(p(t),q(t)).$$

For $z\to-\infty$, Equation (7) can be rewritten

$$\begin{cases}\dfrac{\mathrm{d}\tilde{U}(t,-\infty)}{\mathrm{d}t}=\tilde{U}(t,-\infty)\left[\beta(t)-\tilde{U}(t,-\infty)+a_1\left(J_1*\tilde{V}\right)(t,-\infty)\right], t\in\mathbb{R}^+,\\ \dfrac{\mathrm{d}\tilde{V}(t,-\infty)}{\mathrm{d}t}=\tilde{V}(t,-\infty)\left[\beta(t)-\tilde{V}(t,-\infty)+a_2\left(J_2*\tilde{U}\right)(t,-\infty)\right], t\in\mathbb{R}^+.\end{cases}$$

We can see that $(0,0)$ is a solution, *i.e.*, $\lim\limits_{z\to-\infty}\left(\tilde{U},\tilde{V}\right)(t,z)=(0,0)$.

The above analysis leads to

$$\lim_{z\to-\infty}\left(\tilde{U},\tilde{V}\right)(t,z)=(0,0),\ \lim_{z\to\infty}\left(\tilde{U},\tilde{V}\right)(t,z)=\left(p(t),q(t)\right).$$

This completes the proof.

3. Asymptotic Behaviors of Forced Pulsating Waves for Equation (4)

In this section, we investigate the asymptotic behaviors of $(U,V)(t,z)$ for Equations (7)-(8) around (0, 0).

Lemma 3.1. Assume that (H$_1$), (H$_2$) *and* $c>-\min\left(c_0(d_1),c_0(d_2)\right)$ hold. Then the asymptotic behaviors of the forced pulsating wave solution $(U,V)(t,z)$ as $z\to-\infty$ (see Equation (5)) can be described below.

$$\begin{pmatrix}U(t,z)\\ V(t,z)\end{pmatrix}\sim\begin{pmatrix}A_1\breve{\phi}_0(t)\mathrm{e}^{-\mu_1 z}\\ A_2\breve{\psi}_0(t)\mathrm{e}^{-\mu_2 z}\end{pmatrix},\ z\to-\infty, \tag{11}$$

where $A_i, i=1,2$ are positive numbers, and

$$\mu_1=\frac{c+\sqrt{c^2-4d_1\overline{\beta_1(t)}}}{2d_1},\ \mu_2=\frac{c+\sqrt{c^2-4d_2\overline{\beta_2(t)}}}{2d_2}, \tag{12}$$

Proof. By $z\to-\infty$ in Equation (5) and by virtue of the boundary conditions Equation (6) as well as the assumptions on $r_i(t,z), i=1,2$ (see (H$_1$) and (H$_2$)), the limiting system that follows can be deduced

$$\begin{cases}\breve{U}_t=d_1\breve{U}_{zz}+c\breve{U}_z+\breve{U}\beta_1(t), t\in\mathbb{R}^+, x\in\mathbb{R},\\ \breve{V}_t=d_2\breve{V}_{zz}+c\breve{V}_z+\breve{V}\beta_2(t), t\in\mathbb{R}^+, x\in\mathbb{R}.\end{cases} \tag{13}$$

Making an ansatz $\breve{U}(t,z)=A_1\breve{\phi}_0(t)\mathrm{e}^{-\mu_1 z}$ with $\breve{\phi}_0(t)$ being a *T*-periodic function. When it is substituted into the first equation of Equation (13), the corresponding eigenvalue problem arises

$$\frac{\breve{\phi}_0'(t)}{\breve{\phi}_0(t)}=d_1\mu_1^2-c\mu_1+\beta_1(t). \tag{14}$$

We can solve from Equation (14) that

$$\breve{\phi}_0(t)=\breve{\phi}_0(0)\exp\left(\int_0^t\left(d_1\mu_1^2-c\mu_1+\beta_1(s)\right)\mathrm{d}s\right),$$

where

$$\mu_1=\frac{c+\sqrt{c^2-4d_1\overline{\beta_1(t)}}}{2d_1}.$$

Similarly, making an ansatz $\breve{V}(t,z)=A_2\breve{\psi}_0(t)\mathrm{e}^{-\mu_2 z}$ with $\breve{\psi}_0(t)$ being a *T*-periodic function. When it is substituted into the second equation of Equation (14), the corresponding eigenvalue problem arises

$$\frac{\breve{\psi}_0'(t)}{\breve{\psi}_0(t)} = d_2\mu_2^2 - c\mu_2 + \beta_2(t). \tag{15}$$

We can solve from Equation (15) that

$$\breve{\psi}_0(t) = \breve{\psi}_0(0)\exp\left(\int_0^t \left(d_2\mu_2^2 - c\mu_2 + \beta_2(s)\right)\mathrm{d}s\right),$$

where

$$\mu_2 = \frac{c + \sqrt{c^2 - 4d_2\overline{\beta_2(t)}}}{2d_2}.$$

Thus, the proof is completed.

4. Stability of Forced Pulsating Waves for Equation (4)

In this section, we study the stability of the forced pulsating wave of Equation (4). First, we consider the initial value problem.

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = d_1\dfrac{\partial^2 u(t,x)}{\partial x^2} + u(t,x)\left[r_1(t,x-ct) - u(t,x) + a_1(J_1 * v)(t,x)\right], \\ \dfrac{\partial v(t,x)}{\partial t} = d_2\dfrac{\partial^2 v(t,x)}{\partial x^2} + v(t,x)\left[r_2(t,x-ct) - v(t,x) + a_2(J_2 * u)(t,x)\right], \\ \left(u(0,x), v(0,x)\right) = \left(u_0(x), v_0(x)\right), \end{cases} \tag{16}$$

where $\left(u_0(x), v_0(x)\right) \in C\left(\mathbb{R}, \mathbb{R}^2\right)$ satisfy

$$(0,0) \le \left(u_0(x), v_0(x)\right) \le \left(p(0), q(0)\right),\ x \in \mathbb{R}.$$

Define $P(t) = \left(P_1(t), P_2(t)\right): X \to X$ by

$$P_1(t)u_0(x) = \frac{\mathrm{e}^{-\eta_1 t}}{\sqrt{4\pi d_1 t}}\int_{-\infty}^{+\infty} \mathrm{e}^{\frac{-y^2}{4d_1 t}} u_0(x-y)\,\mathrm{d}y$$

$$P_2(t)v_0(x) = \frac{\mathrm{e}^{-\eta_2 t}}{\sqrt{4\pi d_2 t}}\int_{-\infty}^{+\infty} \mathrm{e}^{\frac{-y^2}{4d_2 t}} v_0(x-y)\,\mathrm{d}y$$

where $\eta_1 = 2\max\limits_{t\in[0,T)} p_1(t)$, $\eta_2 = 2\max\limits_{t\in[0,T)} p_2(t)$. It is easy to obtain that $P(t)$ is a strongly continuous real analytic semigroup on *X*.

In the process of studying the stability of the forced pulsating waves, the conditions

$$a_1 q(t) - p(t) < 0,\ a_2 p(t) - q(t) < 0$$

are always true.

Lemma 4.1. For any $x \in \mathbb{R}$, $t \in \mathbb{R}^+$, the mild solution of Equation (16) is satisfied

$$\begin{cases} u\left(t,x,u_0(x),v_0(x)\right) = P_1(t)u_0(x) + \int_0^t P_1(t-s)F_1(u,v)(s,x)\,\mathrm{d}s, \\ v\left(t,x,u_0(x),v_0(x)\right) = P_2(t)v_0(x) + \int_0^t P_2(t-s)F_2(u,v)(s,x)\,\mathrm{d}s, \end{cases}$$

where

$$F_1(u,v)(t,x)=\eta_1 u(t,x)+u(t,x)\big(r_1(t,x-ct)-u(t,x)+a_1(J_1*v)(t,x)\big),$$
$$F_2(u,v)(t,x)=\eta_2 v(t,x)+v(t,x)\big(r_2(t,x-ct)-v(t,x)+a_2(J_2*u)(t,x)\big).$$

Remark 4.1. Assume that $\big(u(t,x,u_0(x)),v(t,x,v_0(x))\big)$, $\big(u(t,x,\varphi_0(x)),v(t,x,\psi_0(x))\big)$ are mild solutions to Equation (16). If

$$(0,0)\le\big(u_0(x),v_0(x)\big)\le\big(\varphi_0(x),\psi_0(x)\big),\ t\in\mathbb{R}^+,\ x\in\mathbb{R},$$

then

$$(0,0)\le\big(u(t,x,u_0(x)),v(t,x,v_0(x))\big)\le\big(u(t,x,\varphi_0(x)),v(t,x,\psi_0(x))\big),\ t\in\mathbb{R}^+,\ x\in\mathbb{R}.$$

Lemma 4.2. Assume that $\big(u(t,x,u_0(x)),v(t,x,v_0(x))\big)$, $\big(u(t,x,\vartheta_0(x)),v(t,x,\varphi_0(x))\big)$ are mild solutions to Equation (16). If $\big(u_0(x),v_0(x)\big),\big(\vartheta_0(x),\varphi_0(x)\big)\in X^+$, $\big(\vartheta_0(x),\varphi_0(x)\big)\le\big(u_0(x),v_0(x)\big)$, then

$$\begin{aligned}&u(t,x,u_0(x))-u(t,x,\vartheta_0(x))\\&\ge\frac{\mathrm{e}^{-\eta_1(t-t_0)}}{\sqrt{4\pi d_1(t-t_0)}}\mathrm{e}^{-\frac{(J+1)^2}{4d_1(t-t_0)}}\int_z^{z+1}\big[u(t_0,y,u_0,v_0),u(t_0,y,\vartheta_0,\varphi_0)\big]\mathrm{d}y\\&\ge 0,\end{aligned}$$

$$\begin{aligned}&v(t,x,v_0(x))-v(t,x,\varphi_0(x))\\&\ge\frac{\mathrm{e}^{-\eta_2(t-t_0)}}{\sqrt{4\pi d_2(t-t_0)}}\mathrm{e}^{-\frac{(J+1)^2}{4d_2(t-t_0)}}\int_z^{z+1}\big[v(t_0,y,u_0,v_0),v(t_0,y,\vartheta_0,\varphi_0)\big]\mathrm{d}y\\&\ge 0,\end{aligned}$$

where $J\ge 0$, $x,z\in\mathbb{R}$ and $|x-z|\le J$, $t\ge t_0\ge 0$.

The proof of the Lemma 4.2 is similar to the Theorem 2.1 of [28], which will not be proved here.

Lemma 4.3. Assume that $\big(u(t,x,u_0(x)),v(t,x,v_0(x))\big)$, $\big(u(t,x,\varphi_0(x)),v(t,x,\psi_0(x))\big)$ are mild solutions to Equation (16). If $\big(\varphi_0(x),\psi_0(x)\big)\le\big(u_0(x),v_0(x)\big)\le\big(p(0),q(0)\big)$, then

$$\begin{aligned}&\big\|u(t,x,u_0(x))-u(t,x,\varphi_0(x))\big\|\\&\le\min\Big\{\mathrm{e}^{\mu t}\big(\|u_0(x)-\varphi_0(x)\|+\|v_0(x)-\psi_0(x)\|\big),p(t)\Big\},\end{aligned}$$

$$\begin{aligned}&\big\|v(t,x,v_0(x))-v(t,x,\psi_0(x))\big\|\\&\le\min\Big\{\mathrm{e}^{\mu t}\big(\|u_0(x)-\varphi_0(x)\|+\|v_0(x)-\psi_0(x)\|\big),q(t)\Big\},\end{aligned}$$

where $\|\cdot\|$ is the maximum value norm of $C(\mathbb{R},\mathbb{R})$, $\mu=2\max\{M_1,M_2\}>0$, and

$$M_1=\max\Big\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}|\partial_u f_1|,\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}|\partial_v f_1|\Big\},$$
$$M_2=\max\Big\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}|\partial_u f_2|,\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}|\partial_v f_2|\Big\}.$$

Proof. Assume $\eta=\min\{\eta_1,\eta_2\}$, we have

$$\begin{aligned}
&\left\|u\left(t,x,u_0\left(x\right)\right)-u\left(t,x,\varphi_0\left(x\right)\right)\right\| \\
&\le \int_0^t P_1\left(t\right)\left\| f_1\left(u\left(s,x,u_0\left(x\right)\right),v\left(s,x,v_0\left(x\right)\right)\right)- f_1\left(u\left(s,x,\varphi_0\left(x\right)\right),v\left(s,x,\psi_0\left(x\right)\right)\right)\right\|\mathrm{d}s \\
&\quad +\left\|P_1\left(t\right)u_0\left(x\right)-P_1\left(t\right)\varphi_0\left(x\right)\right\| \\
&\le \int_0^t \mathrm{e}^{-\eta(t-s)}\Big(\max\left|\partial_u f_1\right|\cdot\left\|u\left(s,x,u_0\left(x\right)\right)-u\left(s,x,\varphi_0\left(x\right)\right)\right\| \\
&\quad +\max\left|\partial_v f_1\right|\cdot\left\|v\left(s,x,v_0\left(x\right)\right)-v\left(s,x,\psi_0\left(x\right)\right)\right\|\Big)\mathrm{d}s+P_1\left(t\right)\left\|u_0\left(x\right)-\varphi_0\left(x\right)\right\| \\
&\le M_1\int_0^t \mathrm{e}^{-\eta(t-s)}\Big(\left\|u\left(s,x,u_0\left(x\right)\right)-u\left(s,x,\varphi_0\left(x\right)\right)\right\| \\
&\quad +\left\|v\left(s,x,v_0\left(x\right)\right)-v\left(s,x,\psi_0\left(x\right)\right)\right\|\Big)\mathrm{d}s+\mathrm{e}^{-\eta t}\left\|u_0\left(x\right)-\varphi_0\left(x\right)\right\|,
\end{aligned}$$

where

$$M_1=\max\left\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}\left|\partial_u f_1\right|,\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}\left|\partial_v f_1\right|\right\}.$$

Similarly, we can see

$$\begin{aligned}
&\left\|v\left(t,x,v_0\left(x\right)\right)-v\left(t,x,\psi_0\left(x\right)\right)\right\| \\
&\le M_2\int_0^t \mathrm{e}^{-\eta(t-s)}\Big(\left\|u\left(s,x,u_0\left(x\right)\right)-u\left(s,x,\varphi_0\left(x\right)\right)\right\| \\
&\quad +\left\|v\left(s,x,v_0\left(x\right)\right)-v\left(s,x,\psi_0\left(x\right)\right)\right\|\Big)\mathrm{d}s+\mathrm{e}^{-\eta t}\left\|v_0\left(x\right)-\psi_0\left(x\right)\right\|,
\end{aligned}$$

where

$$M_2=\max\left\{\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}\left|\partial_u f_2\right|,\max_{(t,u,v)\in[0,T]\times[0,p(t)]\times[0,q(t)]}\left|\partial_v f_2\right|\right\}.$$

Further, we can obtain

$$\begin{aligned}
&\left\|u\left(t,x,u_0\left(x\right)\right)-u\left(t,x,\varphi_0\left(x\right)\right)\right\|+\left\|v\left(t,x,v_0\left(x\right)\right)-v\left(t,x,\psi_0\left(x\right)\right)\right\| \\
&\le \mu\int_0^t \mathrm{e}^{-\eta(t-s)}\Big(\left\|u\left(s,x,u_0\left(x\right)\right)-u\left(s,x,\varphi_0\left(x\right)\right)\right\|+\left\|v\left(s,x,v_0\left(x\right)\right)-v\left(s,x,\psi_0\left(x\right)\right)\right\|\Big)\mathrm{d}s \\
&\quad +\mathrm{e}^{-\eta t}\Big(\left\|u_0\left(x\right)-\varphi_0\left(x\right)\right\|+\left\|v_0\left(x\right)-\psi_0\left(x\right)\right\|\Big),
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\mathrm{e}^{\eta t}\Big(\left\|u\left(t,x,u_0\left(x\right)\right)-u\left(t,x,\varphi_0\left(x\right)\right)\right\|+\left\|v\left(t,x,v_0\left(x\right)\right)-v\left(t,x,\psi_0\left(x\right)\right)\right\|\Big) \\
&\le \mu\int_0^t \mathrm{e}^{\eta s}\Big(\left\|u\left(s,x,u_0\left(x\right)\right)-u\left(s,x,\varphi_0\left(x\right)\right)\right\|+\left\|v\left(s,x,v_0\left(x\right)\right)-v\left(s,x,\psi_0\left(x\right)\right)\right\|\Big)\mathrm{d}s \\
&\quad +\left\|u_0\left(x\right)-\varphi_0\left(x\right)\right\|+\left\|v_0\left(x\right)-\psi_0\left(x\right)\right\|,
\end{aligned}$$

where $\mu=2\max\left\{M_1,M_2\right\}$. By Gronwall's inequality, we can establish

$$\begin{aligned}
&\left\|u\left(t,x,u_0\left(x\right)\right)-u\left(t,x,\varphi_0\left(x\right)\right)\right\|+\left\|v\left(t,x,v_0\left(x\right)\right)-v\left(t,x,\psi_0\left(x\right)\right)\right\| \\
&\le \mathrm{e}^{\mu t}\Big(\left\|u_0\left(x\right)-\varphi_0\left(x\right)\right\|+\left\|v_0\left(x\right)-\psi_0\left(x\right)\right\|\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left\|u\left(t,x,u_0\left(x\right)\right)-u\left(t,x,\varphi_0\left(x\right)\right)\right\|\le \mathrm{e}^{\mu t}\Big(\left\|u_0\left(x\right)-\varphi_0\left(x\right)\right\|+\left\|v_0\left(x\right)-\psi_0\left(x\right)\right\|\Big), \\
&\left\|v\left(t,x,v_0\left(x\right)\right)-v\left(t,x,\psi_0\left(x\right)\right)\right\|\le \mathrm{e}^{\mu t}\Big(\left\|u_0\left(x\right)-\varphi_0\left(x\right)\right\|+\left\|v_0\left(x\right)-\psi_0\left(x\right)\right\|\Big).
\end{aligned}$$

Thus, the proof is completed.

Definition 4.1. For any $t\in[0,T)$, $x\in\mathbb{R}$, if the continuous function $(\bar{u}(t,x),\bar{v}(t,x))$, $(\underline{u}(t,x),\underline{v}(t,x))$ satisfy

$$\frac{\partial\bar{u}(t,x)}{\partial t}\geq d_1\frac{\partial^2\bar{u}(t,x)}{\partial x^2}+\bar{u}(t,x)\left[r_1(t,x-ct)-\bar{u}(t,x)+a_1(J_1*\bar{v})(t,x)\right],\quad(17)$$

$$\frac{\partial\bar{v}(t,x)}{\partial t}\geq d_2\frac{\partial^2\bar{v}(t,x)}{\partial x^2}+\bar{v}(t,x)\left[r_2(t,x-ct)-\bar{v}(t,x)+a_2(J_2*\bar{u})(t,x)\right],\quad(18)$$

$$\frac{\partial\underline{u}(t,x)}{\partial t}\leq d_1\frac{\partial^2\underline{u}(t,x)}{\partial x^2}+\underline{u}(t,x)\left[r_1(t,x-ct)-\underline{u}(t,x)+a_1(J_1*\underline{v})(t,x)\right],\quad(19)$$

$$\frac{\partial\underline{v}(t,x)}{\partial t}\leq d_2\frac{\partial^2\underline{v}(t,x)}{\partial x^2}+\underline{v}(t,x)\left[r_2(t,x-ct)-\underline{v}(t,x)+a_2(J_2*\underline{u})(t,x)\right],\quad(20)$$

then $(\bar{u}(t,x),\bar{v}(t,x))$, $(\underline{u}(t,x),\underline{v}(t,x))$ are a pair of upper and lower solutions of the system Equation (16).

Lemma 4.4. Assume that $(\bar{u}(t,x),\bar{v}(t,x))$, $(\underline{u}(t,x),\underline{v}(t,x))$ are a pair of upper and lower solutions of the system Equation (16). If $(\underline{u}(0,x),\underline{v}(0,x))\leq(\bar{u}(0,x),\bar{v}(0,x))$, then $(\bar{u}(t,x),\bar{v}(t,x))$ and $(\underline{u}(t,x),\underline{v}(t,x))$ satisfies $(\underline{u}(t,x),\underline{v}(t,x))\leq(\bar{u}(t,x),\bar{v}(t,x))$ for any $t\in[0,T)$, $x\in\mathbb{R}$. Therefore, Equation (16) has a unique classical solution $(u(t,x),v(t,x))$ satisfies $(\underline{u}(t,x),\underline{v}(t,x))\leq(u(t,x),v(t,x))\leq(\bar{u}(t,x),\bar{v}(t,x))$.

Lemma 4.5. Assume that $(\bar{\mu}(t,x),\bar{\nu}(t,x))$, $(\bar{w}(t,x),\bar{\varpi}(t,x))$ are the upper solutions of Equation (16) and $(\underline{u}(t,x),\underline{v}(t,x))$ is a lower solution of the system Equation (16). If

$$(\underline{u}(0,x),\underline{v}(0,x))\leq\min\left\{(\bar{\mu}(0,x),\bar{\nu}(0,x)),(\bar{w}(0,x),\bar{\varpi}(0,x))\right\},$$

then $(\bar{\mu}(t,x),\bar{\nu}(t,x))$, $(\bar{w}(t,x),\bar{\varpi}(t,x))$ and $(\underline{u}(t,x),\underline{v}(t,x))$ satisfy $(\underline{u}(t,x),\underline{v}(t,x))\leq\min\left\{(\bar{\mu}(t,x),\bar{\nu}(t,x)),(\bar{w}(t,x),\bar{\varpi}(t,x))\right\}$ for any $t\in[0,T)$, $x\in\mathbb{R}$. So Equation (16) has a unique classical solution $(u(t,x),v(t,x))$ satisfies $(\underline{u}(t,x),\underline{v}(t,x))\leq(u(t,x),v(t,x))\leq\min\left\{(\bar{\mu}(t,x),\bar{\nu}(t,x)),(\bar{w}(t,x),\bar{\varpi}(t,x))\right\}$.

Lemma 4.4 and Lemma 4.5 can be derived from the classical theory of parabolic equation mixed quasi-monotonic systems in Smoller [29] and Ye *et al.* [30], so the proof is omitted here.

Remark 4.2. By Lemma 4.5, $\min\left\{(\bar{\mu}(t,x),\bar{\nu}(t,x)),(\bar{w}(t,x),\bar{\varpi}(t,x))\right\}$ is still the upper solution of the Equation (16).

Theorem 4.1. Assume that the initial function $(u_0(x),v_0(x))$ is satisfied

(i) $(0,0)\leq(u_0(x),v_0(x))\leq(p(0),q(0))$;

(ii) $(\underline{u},\underline{v})\leq(u_0(x),v_0(x))\leq(\bar{u},\bar{v})$, where $(\underline{u},\underline{v}),(\bar{u},\bar{v})$ are a set of lower and upper solutions defined by Definition 4.1;

(iii) $\liminf\limits_{x\to\infty}u_0(x)>0,\liminf\limits_{x\to\infty}v_0(x)>0$;

(iv) $\lim\limits_{x\to-\infty}\dfrac{u_0(x)}{K_1\mathrm{e}^{\lambda_1x}}=1,\ \lim\limits_{x\to-\infty}\dfrac{v_0(x)}{K_2\mathrm{e}^{\lambda_2x}}=1$.

Let (Φ^c,Ψ^c) be a solution defined by Theorem 2.1, then we have

$$\limsup_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{u(t,x,u_0)}{\Phi^c(t,x-ct)}-1\right|=0,\ \limsup_{t\to\infty}\sup_{x\in\mathbb{R}}\left|\frac{v(t,x,v_0)}{\Psi^c(t,x-ct)}-1\right|=0.$$

Next we use the following lemmas to prove Theorem 4.1.

Lemma 4.6. $\left(\Phi^c(t,z),\Psi^c(t,z)\right)$ is strictly monotonically increasing with respect to *z*, *i.e.*,

$$\Phi_z^c(t,z)>0,\ \Psi_z^c(t,z)>0.$$

Proof. The proof of Lemma 4.6 is similar to Lemma 2.4 of [23], which will not be proved here.

Lemma 4.7. Assume that $\xi^+\in\mathbb{R}$ and $\varepsilon\in(0,\bar{\varepsilon}]$, where $\bar{\varepsilon}\in(0,1)$. If $\gamma>0$ is sufficiently small, $\sigma>0$ and $\sigma\gamma$ is sufficiently large, then $(\bar{u},\bar{v})(t,x)$ is an upper solution of Equation (16), where

$$\bar{u}(t,x)=\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c\left(t,x-ct-\xi^+-\varepsilon\sigma e^{-\gamma t}\right),$$
$$\bar{v}(t,x)=\left(1+\varepsilon e^{-\gamma t}\right)\Psi^c\left(t,x-ct-\xi^+-\varepsilon\sigma e^{-\gamma t}\right).$$

Proof. We only prove that $\bar{u}(t,x)$ satisfies inequality Equation (17), since $\bar{v}(t,x)$ satisfies inequality Equation (18) that can be handled similarly.

Let $\tau=x-ct-\xi^+-\varepsilon\sigma e^{-\gamma t}$, $\bar{u}(t,x)=\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)$, we can get

$$\begin{aligned}\bar{u}_t(t,x)=&-\varepsilon\gamma e^{-\gamma t}\Phi^c(t,\tau)+\left(1+\varepsilon e^{-\gamma t}\right)\Phi_t^c(t,\tau)\\&-c\left(1+\varepsilon e^{-\gamma t}\right)\Phi_\tau^c(t,\tau)+\varepsilon\sigma\gamma e^{-\gamma t}\left(1+\varepsilon e^{-\gamma t}\right)\Phi_\tau^c(t,\tau),\end{aligned}$$

$$\bar{u}_{xx}=\left(1+\varepsilon e^{-\gamma t}\right)\Phi_{\tau\tau}^c(t,\tau),$$

and

$$\begin{aligned}&\bar{u}\left(r_1(t,x-ct)-\bar{u}+a_1\left(J_1*\bar{v}\right)\right)\\&=\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)\Big[r_1(t,x-ct)-\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)\\&\quad+a_1\left(1+\varepsilon e^{-\gamma t}\right)\left(J_1*\Psi^c\right)(t,\tau)\Big]\\&=\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)\left(r_1(t,x-ct)-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right)\\&\quad+\varepsilon e^{-\gamma t}\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)\left(-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right).\end{aligned}$$

Therefore, we can get

$$\begin{aligned}&d_1\bar{u}_{xx}+\bar{u}\left(r_1(t,x-ct)-\bar{u}+a_1\left(J_1*\bar{v}\right)\right)-\bar{u}_t\\&=d_1\left(1+\varepsilon e^{-\gamma t}\right)\Phi_{\tau\tau}^c(t,\tau)+\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)\left(r_1(t,x-ct)-\Phi^c(t,\tau)\right.\\&\quad\left.+a_1\left(J_1*\Psi^c\right)(t,\tau)\right)+\varepsilon e^{-\gamma t}\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)\left(-\Phi^c(t,\tau)\right.\\&\quad\left.+a_1\left(J_1*\Psi^c\right)(t,\tau)\right)+\varepsilon\gamma e^{-\gamma t}\Phi^c(t,\tau)-\left(1+\varepsilon e^{-\gamma t}\right)\Phi_t^c(t,\tau)\\&\quad+c\left(1+\varepsilon e^{-\gamma t}\right)\Phi_\tau^c(t,\tau)-\varepsilon\sigma\gamma e^{-\gamma t}\left(1+\varepsilon e^{-\gamma t}\right)\Phi_\tau^c(t,\tau)\\&=\left(1+\varepsilon e^{-\gamma t}\right)\Big[d_1\Phi_{\tau\tau}^c(t,\tau)+c\Phi_\tau^c(t,\tau)-\Phi_t^c(t,\tau)\\&\quad+\Phi^c(t,\tau)\left(r_1(t,x-ct)-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right)\Big]\\&\quad+\varepsilon e^{-\gamma t}\left(1+\varepsilon e^{-\gamma t}\right)\Phi^c(t,\tau)\left(-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right)\\&\quad+\varepsilon\gamma e^{-\gamma t}\Phi^c(t,\tau)-\varepsilon\sigma\gamma e^{-\gamma t}\left(1+\varepsilon e^{-\gamma t}\right)\Phi_\tau^c(t,\tau).\end{aligned}$$

From the definition of the forced pulsating wave solution, we have

$$\begin{aligned}&d_1\Phi^c_{\tau\tau}(t,\tau)+c\Phi^c_\tau(t,\tau)-\Phi^c_t(t,\tau)\\&+\Phi^c(t,\tau)\left[r_1(t,x-ct)-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right]=0.\end{aligned}$$

In order to get Equation (17), we need to prove

$$\begin{aligned}&\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\tau)\left[-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right]\\&+\gamma\Phi^c(t,\tau)-\sigma\gamma\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\tau(t,\tau)\le 0,\end{aligned}$$

in other words,

$$\begin{aligned}&\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\tau)\left[-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right]\\&\le-\gamma\Phi^c(t,\tau)+\sigma\gamma\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\tau(t,\tau).\end{aligned}\tag{21}$$

(I) Assume $|\tau|\ge H$, when $\tau\to\infty$,

$$-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\to-p(t)+a_1q(t).$$

For $-p(t)+a_1q(t)<0$, so we have

$$\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\tau)\left[-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right]<0.$$

Since $\gamma>0$ is sufficiently small, $\sigma\gamma>0$ is sufficiently large and $\varepsilon\in(0,1)$, $\Phi^c_\tau(t,\tau)>0$, then $-\gamma\Phi^c(t,\tau)\to0$ and $\sigma\gamma\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\tau(t,\tau)\to\infty$. *i.e.*,

$$-\gamma\Phi^c(t,\tau)+\sigma\gamma\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\tau(t,\tau)\to\infty.$$

Thus, Equation (20) is true.

(II) Choose $|\tau|\le H$, we can get $-\gamma\Phi^c(t,\tau)+\sigma\gamma\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\tau(t,\tau)\to+\infty$ by the same proof as (I). Due to $\Phi^c(t,\tau)$, $\Psi^c(t,\tau)$ are bounded, then $\left(1+\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\tau)\left[-\Phi^c(t,\tau)+a_1\left(J_1*\Psi^c\right)(t,\tau)\right]$ are bounded. Therefore, Equation (20) is true, *i.e.*, $\overline{u}(t,x)$ satisfies inequality Equation (17).

The similar method shows that if $a_2p(t)-q(t)<0$, then $\overline{v}(t,x)$ satisfies inequality Equation (18). Thus, $(\overline{u},\overline{v})(t,x)$ is an upper solution of Equation (16).

Lemma 4.8. Assume $\xi^-\in\mathbb{R}$ and $\varepsilon\in(0,\overline{\varepsilon}]$ for $\overline{\varepsilon}\in(0,1)$. If $\gamma>0$ is sufficiently small, $\sigma>0$ and $\sigma\gamma$ is sufficiently large, then $(\underline{u},\underline{v})(t,x)$ is a lower solution of Equation (16), where

$$\begin{aligned}&\underline{u}(t,x)=\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c\left(t,x-ct+\xi^-+\varepsilon\sigma\mathrm{e}^{-\gamma t}\right),\\&\underline{v}(t,x)=\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Psi^c\left(t,x-ct+\xi^-+\varepsilon\sigma\mathrm{e}^{-\gamma t}\right).\end{aligned}$$

Proof. We only prove that $\underline{u}(t,x)$ satisfies inequality Equation (19) since $\underline{v}(t,x)$ satisfies inequality Equation (20) that can be handled similarly.

Let $\varsigma=x-ct+\xi^-+\varepsilon\sigma\mathrm{e}^{-\gamma t}$. When $\underline{u}(t,x)=\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)$, we can obtain

$$\begin{aligned}\frac{\partial\underline{u}(t,x)}{\partial t}&=\varepsilon\gamma\mathrm{e}^{-\gamma t}\Phi^c(t,\varsigma)+\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_t(t,\varsigma)\\&\quad-c\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma)-\varepsilon\sigma\gamma\mathrm{e}^{-\gamma t}\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma),\end{aligned}$$

$$\underline{u}_{xx}=\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_{\varsigma\varsigma}(t,\varsigma),$$

and

$$\begin{aligned}
&\underline{u}\left(r_1(t,x-ct)-\underline{u}+a_1\left(J_1*\underline{v}\right)\right)\\
&=\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\Big[r_1(t,x-ct)-\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\\
&\quad+a_1\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\left(J_1*\Psi^c\right)(t,\varsigma)\Big]\\
&=\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\left(r_1(t,x-ct)-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right)\\
&\quad-\varepsilon\mathrm{e}^{-\gamma t}\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\left(-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right).
\end{aligned}$$

Therefore, we can get

$$\begin{aligned}
&d_1\underline{u}_{xx}+\underline{u}\left(r_1(t,x-ct)-\underline{u}+a_1\left(J_1*\underline{v}\right)\right)-\underline{u}_t\\
&=d_1\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_{\varsigma\varsigma}(t,\varsigma)+\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\Big(r_1(t,x-ct)-\Phi^c(t,\varsigma)\\
&\quad+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\Big)-\varepsilon\mathrm{e}^{-\gamma t}\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\Big(-\Phi^c(t,\varsigma)\\
&\quad+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\Big)-\varepsilon\gamma\mathrm{e}^{-\gamma t}\Phi^c(t,\varsigma)-\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_t(t,\varsigma)\\
&\quad+c\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma)+\varepsilon\sigma\gamma\mathrm{e}^{-\gamma t}(1-\varepsilon)\Phi^c_\varsigma(t,\varsigma)\\
&=\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Big[d_1\Phi^c_{\varsigma\varsigma}(t,\varsigma)+c\Phi^c_\varsigma(t,\varsigma)-\Phi^c_t(t,\varsigma)\\
&\quad+\Phi^c(t,\varsigma)\left(r_1(t,x-ct)-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right)\Big]\\
&\quad-\varepsilon\mathrm{e}^{-\gamma t}\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\left(-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right)\\
&\quad-\varepsilon\gamma\mathrm{e}^{-\gamma t}\Phi^c(t,\varsigma)+\varepsilon\sigma\gamma\mathrm{e}^{-\gamma t}\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma).
\end{aligned}$$

From the definition of the forced pulsating wave solution, we have

$$\begin{aligned}
&d_1\Phi^c_{\varsigma\varsigma}(t,\varsigma)+c\Phi^c_\varsigma(t,\varsigma)-\Phi^c_t(t,\varsigma)\\
&\quad+\Phi^c(t,\varsigma)\left[r_1(t,x-ct)-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right]=0.
\end{aligned}$$

In order to get (4.4), we need to prove

$$\begin{aligned}
&-\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\left[-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right]\\
&-\gamma\Phi^c(t,\varsigma)+\sigma\gamma\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma)\geq 0,
\end{aligned}$$

in other words,

$$\begin{aligned}
&\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\left[-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right]\\
&\leq-\gamma\Phi^c(t,\varsigma)+\sigma\gamma\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma).
\end{aligned}\tag{22}$$

(I) Assume $|\varsigma|\geq N$, when $\varsigma\to\infty$,

$$-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\to-p(t)+a_1q(t).$$

For $-p(t)+a_1q(t)<0$, so we have

$$\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c(t,\varsigma)\left[-\Phi^c(t,\varsigma)+a_1\left(J_1*\Psi^c\right)(t,\varsigma)\right]<0.$$

Since $\gamma>0$ is sufficiently small, $\sigma\gamma>0$ is sufficiently large and $\varepsilon\in(0,1)$, $\Phi^c_\varsigma(t,\varsigma)>0$, then $-\gamma\Phi^c(t,\varsigma)\to 0$ and $\sigma\gamma\left(1-\varepsilon\mathrm{e}^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma)\to\infty$, *i.e.*,

$$-\gamma\Phi^c(t,\varsigma)+\sigma\gamma\left(1-\varepsilon e^{-\gamma t}\right)\Phi^c_\varsigma(t,\varsigma)\to\infty.$$

Thus, Equation (22) is true.

(II) Choose $|\varsigma| \le N$, we can get $-\gamma\Phi^c(t,\varsigma) + \sigma\gamma(1-\varepsilon e^{-\gamma t})\Phi^c_\varsigma(t,\varsigma) \to +\infty$ by the same proof as (I). Since $\Phi^c(t,\varsigma)$, $\Psi^c(t,\varsigma)$ are bounded, then $(1-\varepsilon e^{-\gamma t})\Phi^c(t,\varsigma)\left[-\Phi^c(t,\varsigma) + a_1(J_1 * \Psi^c)(t,\varsigma)\right]$ are bounded. Therefore, Equation (22) is true, *i.e.*, $\underline{u}(t,x)$ satisfies inequality Equation (19).

The similar method shows that for $a_2 p(t) - q(t) < 0$, $\underline{v}(t,x)$ satisfies inequality Equation (20). Thus, $(\underline{u},\underline{v})(t,x)$ is a lower solution of Equation (16).

Lemma 4.9. For $\varepsilon > 0$, there is $\tau_1 = \tau_1(\varepsilon)$, for any $\tau \le \tau_1$, such that

$$\begin{aligned} &\inf_{t\ge0} u(t,\tau - ct - 2\varepsilon, u_0) \le \Phi^c(t,\tau) \le \sup_{t\ge0} u(t,\tau - ct - 2\varepsilon, u_0), \\ &\inf_{t\ge0} v(t,\tau - ct - 2\varepsilon, v_0) \le \Psi^c(t,\tau) \le \sup_{t\ge0} v(t,\tau - ct - 2\varepsilon, v_0). \end{aligned} \tag{23}$$

Proof. We know that

$$\inf_{t\ge0} \Phi^c(t,\tau) \le \Phi^c(t,\tau) \le \sup_{t\ge0} \Phi^c(t,\tau)$$

for any $\tau \in \mathbb{R}$. Since (Φ^c, Ψ^c) is a solution of Equation (16), there is $\tau_1 = \tau_1(\varepsilon)$ for any $\tau \le \tau_1$ such that

$$\begin{aligned} &\inf_{t\ge0} u(t,\tau - ct - 2\varepsilon, u_0) \le \inf_{t\ge0} \Phi^c(t,\tau), \\ &\sup_{t\ge0} u(t,\tau - ct - 2\varepsilon, u_0) \ge \sup_{t\ge0} \Phi^c(t,\tau). \end{aligned}$$

Thus, the first equation of Equation (23) is true. The second inequality of Equation (23) can be proved similarly.

Lemma 4.10. There exist positive constants $\varepsilon \in (0,1)$, γ, σ, z_0, such that

$$\begin{aligned} &(1-\varepsilon e^{-\gamma t})\Phi^c(t,\xi - z_0 + \varepsilon\sigma e^{-\gamma t}) \le u(t,x,u_0) \le (1+\varepsilon e^{-\gamma t})\Phi^c(t,\xi + z_0 - \varepsilon\sigma e^{-\gamma t}), \\ &(1-\varepsilon e^{-\gamma t})\Psi^c(t,\xi - z_0 + \varepsilon\sigma e^{-\gamma t}) \le v(t,x,v_0) \le (1+\varepsilon e^{-\gamma t})\Psi^c(t,\xi + z_0 - \varepsilon\sigma e^{-\gamma t}) \end{aligned} \tag{24}$$

for all $t \ge 1$, $x \in \mathbb{R}$. Then for all $t > 1$, we have

$$\begin{aligned} &1 - \varepsilon e^{-\gamma t} \le \inf_{\mathbb{R}} \frac{u(t,\cdot - ct, u_0)}{\Phi^c(t,\cdot + z_0)} \le \sup_{\mathbb{R}} \frac{u(t,\cdot - ct, u_0)}{\Phi^c(t,\cdot - z_0)} \le 1 + \varepsilon e^{-\gamma t}, \\ &1 - \varepsilon e^{-\gamma t} \le \inf_{\mathbb{R}} \frac{v(t,\cdot - ct, v_0)}{\Psi^c(t,\cdot + z_0)} \le \sup_{\mathbb{R}} \frac{v(t,\cdot - ct, v_0)}{\Psi^c(t,\cdot - z_0)} \le 1 + \varepsilon e^{-\gamma t}. \end{aligned}$$

Proof. According to Lemma 4.2 and 4.7 - 4.9, there exist constants $\varepsilon \in (0,1)$, $\gamma > 0$, $\sigma > 0$, $z_0 \ge 0$, such that

$$\begin{aligned} &(1-\varepsilon e^{-\gamma t})\Phi^c(t,\xi + z_0 + \varepsilon\sigma e^{-\gamma t}) \le u(t,x,u_0) \le (1+\varepsilon e^{-\gamma t})\Phi^c(t,\xi - z_0 - \varepsilon\sigma e^{-\gamma t}), \\ &(1-\varepsilon e^{-\gamma t})\Psi^c(t,\xi + z_0 + \varepsilon\sigma e^{-\gamma t}) \le v(t,x,v_0) \le (1+\varepsilon e^{-\gamma t})\Psi^c(t,\xi - z_0 - \varepsilon\sigma e^{-\gamma t}) \end{aligned}$$

for all $\xi \in \mathbb{R}$. At the same time, these constants also satisfy the conditions of Lemma 4.7 - 4.8 when z_0 are sufficiently large. Therefore, the conclusion can be obtained from Lemma 4.4.

Lemma 4.11. *For all* $\varepsilon \in (0,1)$, there exists a positive integer H_0, such that

$$\begin{aligned} &(1-\varepsilon)\Phi^c(t,\xi + 3\varepsilon\sigma) \le \Phi^c(t,\xi) \le (1+\varepsilon)\Phi^c(t,\xi - 3\varepsilon\sigma), \xi \ge H_0, \\ &(1-\varepsilon)\Psi^c(t,\xi + 3\varepsilon\sigma) \le \Psi^c(t,\xi) \le (1+\varepsilon)\Psi^c(t,\xi - 3\varepsilon\sigma), \xi \ge H_0. \end{aligned} \tag{25}$$

Proof. Considering the function $(1+\eta)\Phi^c(t,\xi-3\eta\sigma)$, we can obtain

$$\frac{\mathrm{d}}{\mathrm{d}\eta}\left\{(1+\eta)\Phi^c(t,\xi-3\eta\sigma)\right\}=\Phi^c(t,\xi-3\eta\sigma)-3\sigma(1+\eta)\Phi^c_\eta(t,\xi-3\eta\sigma).$$

From the asymptotic behavior of the forced pulsating wave solution, there exists a constant $H_0>0$ such that

$$\Phi^c(t,\xi-3\eta\sigma)-3\sigma(1+\eta)\Phi^c_\eta(t,\xi-3\eta\sigma)\geq 0$$

for any $\xi\geq H_0$. Therefore, we have

$$(1-\varepsilon)\Phi^c(t,\xi+3\varepsilon\sigma)\leq\Phi^c(t,\xi)\leq(1+\varepsilon)\Phi^c(t,\xi-3\varepsilon\sigma).$$

The second inequality of Equation (25) can be proved similarly.

Lemma 4.12. Let z,H be the positive constants and $\left(u^+(t,x),v^+(t,x)\right)$, $\left(u^-(t,x),v^-(t,x)\right)$ be solutions to the initial value problem of Equation (16). Define $\chi(y)=\min\{\max\{0,-y\},1\}$ for any $y\in\mathbb{R}$, and assume that the initial values satisfy

$$\begin{aligned}&\left(u^\pm(0,x-c),v^\pm(0,x-c)\right)\\&=\Big(\Phi^c(0,x\pm z)\chi(x+H)+\Phi^c(0,x\pm 2z)\left[1-\chi(x+H)\right],\\&\quad\Psi^c(0,x\pm z)\chi(x+H)+\Psi^c(0,x\pm 2z)\left[1-\chi(x+H)\right]\Big).\end{aligned}$$

Then, there is a constant $\varepsilon\in\left(0,\min\left\{\frac{1}{2},\frac{z}{3\sigma}\right\}\right)$ such that

$$\begin{aligned}&\left(u^+(1,x-c),v^+(1,x-c)\right)\\&\leq\left((1+\varepsilon)\Phi^c(t,x+2z-3\varepsilon\sigma),(1+\varepsilon)\Psi^c(t,x+2z-3\varepsilon\sigma)\right),\\&\left(u^-(1,x-c),v^-(1,x-c)\right)\\&\geq\left((1-\varepsilon)\Phi^c(t,x-2z+3\varepsilon\sigma),(1-\varepsilon)\Psi^c(t,x-2z+3\varepsilon\sigma)\right)\end{aligned}\tag{26}$$

for any $x\in[-H,\infty)$.

Proof. According to the definition of $\chi(y)$, we can see $\left(u^+(0,x-c),v^+(0,x-c)\right)\leq\left(\Phi^c(0,x+2z),\Psi^c(0,x+2z)\right)$. On the nonempty subset of $\mathbb{R}$, we can obtain $\left(u^+(1,x-c),v^+(1,x-c)\right)\leq\left(\Phi^c(1,x+2z),\Psi^c(1,x+2z)\right)$ from the regularity of $T(t)$ and the comparison principle. Let H_0 satisfy the condition of Lemma 4.12. Since u^+,v^+,Φ^c,Ψ^c are continuous functions, they are uniformly continuous on a bounded set. Then, there exists a constant $\varepsilon\in\left(0,\min\left\{\frac{1}{2},\frac{z}{3\sigma}\right\}\right)$ such that

$$\begin{aligned}&\left(u^+(1,x-c),v^+(1,x-c)\right)\\&\leq\left((1+\varepsilon)\Phi^c(t,x+2z-3\varepsilon\sigma),(1+\varepsilon)\Psi^c(t,x+2z-3\varepsilon\sigma)\right)\end{aligned}$$

for $x\in[-H,H_0-2z]$.

From Lemma 4.11, we have that

$$\begin{aligned}&\left(u^{+}(t,x-c),v^{+}(t,x-c)\right)\\&<\left(\Phi^{c}(t,x+2z),\Psi^{c}(t,x+2z)\right)\\&\le\left((1+\varepsilon)\Phi^{c}(t,x+2z-3\varepsilon\sigma),(1+\varepsilon)\Psi^{c}(t,x+2z-3\varepsilon\sigma)\right)\end{aligned}$$

for $x\in[H_0-2z,\infty)$.

The similar method can be used to prove the second inequality of Equation (26). Thus, the proof is completed.

Now let us prove Theorem 4.1, we only prove $\limsup\limits_{t\to\infty}\sup\limits_{x\in\mathbb{R}}\left|\dfrac{u(t,x,u_0)}{\Phi^{c}(t,x-ct)}-1\right|=0$. The rest can be proved similarly.

Proof. Define $z^{+}:=\inf\{z\mid z\in D^{+}\}$, $z^{-}:=\inf\{z\mid z\in D^{-}\}$, where

$$D^{+}=\left\{z\ge 0\mid \limsup_{t\to\infty}\sup_{\xi\in\mathbb{R}}\frac{u(t,\xi-ct,u_0)}{\Phi^{c}(t,\xi+2z)}\le 1\right\},$$

$$D^{-}=\left\{z\ge 0\mid \liminf_{t\to\infty}\inf_{\xi\in\mathbb{R}}\frac{u(t,\xi-ct,u_0)}{\Phi^{c}(t,\xi-2z)}\ge 1\right\}.$$

According to Lemma 4.10, we can obtain $\left[\frac{1}{2}z_0,\infty\right)\subset D^{\pm}$, $z^{\pm}\in\left[0,\frac{1}{2}z_0\right]$. If $z^{\pm}=0$, the proof is completed.

Assume $z^{+}>0$, let $z=z^{+}$, $H=z^{+}\left(1-\frac{\xi_1}{2}\right)$, $\varepsilon\in\left(0,\min\left\{\frac{1}{2},\frac{z}{3\sigma}\right\}\right)$. Since $z^{+}\in D^{+}$, there exists $T\ge 0$ such that

$$\sup_{\mathbb{R}}\frac{u(T,\xi-cT,u_0)}{\Phi^{c}(T,\xi+2z^{+})}\le 1+\frac{\bar{\varepsilon}}{\max\limits_{t\in[0,T)}p(t)},$$

where $4\bar{\varepsilon}=\varepsilon e^{-\mu}\min\left\{\min\limits_{t\in[0,T)}\Phi^{c}(t,-H-3\varepsilon\sigma),\min\limits_{t\in[0,T)}\Psi^{c}(t,-H-3\varepsilon\sigma)\right\}$, $\mu=2\max\{M_1,M_2\}>0$.

From Lemma 4.12, for $\xi\in[-H,\infty)$, we can obtain

$$u(T,\xi-cT,u_0)\le\Phi^{c}(T,\xi+2z^{+})+\bar{\varepsilon}=u^{+}(0,\xi-c)+\bar{\varepsilon}.$$

For $\xi\in(-\infty,-H]$, we can see

$$u(T,\xi-cT,u_0)\le\Phi^{c}(T,\xi+z^{+})\le u^{+}(0,\xi-c).$$

Thus,

$$\begin{aligned}&u(T+1,\xi-c(T+1),u_0)\\&\le u^{+}(1,\xi-c)+4\bar{\varepsilon}e^{\mu}\le u^{+}(1,\xi-c)+\varepsilon\Phi^{c}(T,-H-3\varepsilon\sigma).\end{aligned}$$

By Lemma 4.12, we have that

$$\begin{aligned}&u(T+1,\xi-c(T+1),u_0)\\&\le u^{+}(1,\xi-c)+\varepsilon\Phi^{c}(T,-H-3\varepsilon\sigma)\\&\le(1+\varepsilon)\Phi^{c}(T,\xi+2z^{+}-3\varepsilon\sigma)+\varepsilon\Phi^{c}(T,-H-3\varepsilon\sigma)\\&\le(1+2\varepsilon)\Phi^{c}(T,\xi+2z^{+}-3\varepsilon\sigma)\end{aligned}$$

for $e^{-\gamma t} \geq 1$, $\xi \in [-H, \infty)$. Since $3\varepsilon\sigma \leq z^+$, we can see that

$$u(T+1, \xi - c(T+1), u_0) \leq \Phi^c(T, \xi + z^+) \leq \Phi^c(T, \xi + 2z^+ - 3\varepsilon\sigma)$$

for $\xi \in (-\infty, -H]$. Thus,

$$u(T+1, \xi - c(T+1)) \leq \min\left\{(1+2\varepsilon)\Phi^c(T, \xi + 2z^+ - 3\varepsilon\sigma), p(t)\right\}.$$

By the comparison principle, we can obtain

$$\begin{aligned} & u(T+1+t, \xi - c(T+1+t), u_0) \\ & \leq \min\left\{(1+2\varepsilon e^{-\gamma t})\Phi^c(T, \xi + 2z^+ - \varepsilon\sigma - 2\varepsilon\sigma e^{-\gamma t}), p(t)\right\}. \end{aligned}$$

If $t \geq 0$, $\xi \in \mathbb{R}$, we have

$$\limsup_{t\to\infty} \sup_{\xi\in\mathbb{R}} \frac{u(t, \xi - ct, u_0)}{\Phi(t, \xi + 2z^+ - \varepsilon\sigma)} \leq 1.$$

So, we can see $z^+ - \dfrac{\varepsilon\sigma}{2} \in D^+$ from the inequality. It is a contradiction. Therefore, $z^+ = 0$. For the case $z^- = 0$, we can prove it similarly.
Thus, the proof is completed.

5. Conclusion

We are concerned with the existence, asymptotic behaviors and stability of forced pulsating waves for a Lotka-Volterra cooperative system with nonlocal effects under shifting habitats. Firstly, we establish the existence of the forced pulsating waves by using the alternatively-coupling upper-lower solution method, as long as the shifting speed falls in a finite interval where the endpoints are obtained from KPP-Fisher speeds. Secondly, the asymptotic behaviors of the forced pulsating waves are derived respectively by way of asymptotic analysis. Finally, with proper initial, the stability of the forced pulsating waves is studied by the squeezing technique based on the comparison principle. The methods adopted in the present paper can be used to investigate the aforementioned properties of forced pulsating waves for a more general time periodic Lotka-Volterra cooperation system with nonlocal effects.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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