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Resistor-Capacitor Circuit as a Dynamic System

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Abstract

The paper considers the response to the accumulated energy in the resistor (*R*)-capacitor (*C*) circuit. In the (*RC*) circuit, the capacitor *C* is initially charged with the "capacitive" voltage U_0. At that moment $t = 0$, the P circuit switch turns on. By using Kirchhoff's laws on the elements, a homogeneous differential equation of the first order with constant coefficients is obtained with the initial condition $U_C(0) = U_0$. The solution of the differential equation is presented in exponential form $U_C(t) = U_0 \cdot \mathrm{e}^{-t/\tau}$. Qualitative analysis *RC* of the circuit gives a phase portrait on the line. From the phase portrait on the line, it can be seen that the charge $U_C(t) \to U_C^* = 0$ when $t \to \infty$ stabilizes, regardless of the initial conditions. It is shown that from $U_C(t) = U_0 \cdot \mathrm{e}^{-t/\tau}$ a dynamic system defined by the function $\varphi(t, U_C) = U_C \cdot \mathrm{e}^{-t/\tau}$ can be formed from. It has also been shown that, from the formed dynamic system, an autonomous system (circuit equation *RC*) can be found whose solution describes the formed dynamic system. It is also shown that the dynamic system $\varphi(t, U_C) = U_C \cdot \mathrm{e}^{-t/\tau}$ has one attractive fixed point $U_C = 0$.

Keywords

RC Circuit, Voltage, Dynamic System

1. Introduction and Preliminary Notes

Dynamic systems describe the interdependence of system variables and their change over time. Dynamic systems are described by differential and difference equations. H. Poincaré first considers the properties of solutions of ordinary differential equations of the second order. He introduces topological methods and the concept of trajectories. A more abstract formulation of dynamic systems was given by A.A. Markov, H. Whitney, and G.D. Birkhoff in 1927 [1]. A significant contribution to the study of periodic points of continuous functions on the segment was made by A.N. Sharkovsky in 1964 [2]. Important developments in the theory of dynamic systems can also be observed in the works of T.-Y. Li and J.A. Yorke in 1975 [3], W. Melo and S. Strien in 1993 [4], and J. Milnor and W. Thurston in 1977 [5]. Recent theories of dynamic systems (continuous and discrete) are presented in the works [6]-[8].

The aim of the research in this manuscript is the possibility of applying dynamic systems to the behavior and discharge of capacitors C in the RC circuit.

Before abstractly defining a dynamic system on some set, the concepts of an open set and open neighborhood of a point will be defined.

Definition 1. Set $A \subseteq R^n$ is open if it holds $\forall x \in A$, $\exists \varepsilon > 0$, $K(x,\varepsilon) \subseteq A$.

Definition 2. Open neighborhood of a point $x \in R^n$ is any open set that contains the point *x*.

Definition 3. To set $A \subseteq R$ open in the set *R* if for $\forall x \in A$, $\exists \varepsilon \in A$ such that $(x-\varepsilon, x+\varepsilon) \subset A$.

The entire set *R* is an open set, and so is $\varnothing$.

Let be the given set of initial values $A \subseteq R^n$. Let be the given set of initial values. The dynamic system represents the evolution of this set over time.

Definition 4. Let be $A \subseteq R^n$ open set. A dynamic system is a set *A* of functions $\varphi(t,x): R \times A \to A$ class C1 for which applies [1]-[3]:

$$\varphi(0,x) = x, \quad x \in A \tag{1}$$

$$\varphi(t+p,x) = \varphi(t,\varphi(p,x)), \quad t,p \in R \tag{2}$$

If we introduce the notation $\varphi_t(x) = \varphi(t,x)$ properties (1) and (2) are equivalent to the properties:

$$\varphi_0 = id_A \quad \text{(identity mapping)} \tag{1'}$$

$$\varphi_{t+p} = \varphi_t \circ \varphi_p \tag{2'}$$

A dynamic system can also be defined so that the function φ is not defined on the entire product $R \times A$ but on the product $\{0\} \times A$ (local dynamic system). Similarly, a mapping can be introduced $\varphi(t,x): R_0^+ \times A \to A$ that generates a dynamic system with the time set R_0^+. Dynamic system is called a flow if it is $t \in R$ or a semiflow if it is $t \in R_0^+$.

For a flow, the mapping is $x(t) = \varphi(t,x)$ transformation invertible, as it is $\varphi(-t,x) = \varphi(t,x)^{-1}$. If time is considered $t \in Z$ for a dynamic system, we say that it is discrete.

Definition 5. For $x \in A$, let's define:

1) Positive orbit $O_\varphi^+(x) = \bigcup_{t \ge 0} \varphi(t,x)$

2) Negative orbit $O_\varphi^-(x) = \bigcup_{t \le 0} \varphi(t,x)$

3) Orbit $O_\varphi(x) = O_\varphi^+(t,x) \cup O_\varphi^-(t,x) = \bigcup_t \varphi(t,x)$

Definition 6. If for $\forall t \in R, R_0^+, Z$ or N_0 if the relation $\varphi(t,x) = x$ then *x* fixed point. In dynamical systems, hyperbolic fixed points play a special role. For such points, the following holds $|\varphi'(t,x)| \ne 1$, for $\forall t \in R$, $x \in A$. If it:

- $|\varphi'(t,x)| < 1$ The fixed point is attractive (sink).

- $|\varphi'(t,x)|>1$ The fixed point is repulsive (source).

An attractive fixed point x for the mapping φ has the property that there exists an open set U containing the point x, whose all points converge to x. Repellent fixed point x for the mapping φ, it has the property that there exists an open set U containing the point x, which repels all points from its vicinity U.

Definition 7. For $x \in A$, we say that it is a periodic point with period $n>0$, if it $\varphi^n(t,x)=x$ The smallest non-negative integer n for which the $\varphi^n(t,x)=x$ is called the fundamental period of x.

Theorem 1. Let be $\varphi(t,x): R\times A \to A$ a dynamical system. Suppose there $x(t)=\varphi(t,x_0)$ exists $x_0 \in A$. Such that $x(t)$ is a solution to the autonomous system:

$$\left.\begin{array}{l} x' = f(x) \\ x(0)=x_0 \end{array}\right\} \tag{3}$$

where is function $f(x)$ defined by

$$f(x)=\frac{\partial\varphi(0,x)}{\partial t} \tag{4}$$

Proof: Let $x(t)=\varphi(t,x_0)$. Then, according to property (1) of the dynamic system, $x(0)=\varphi(0,x_0)=x_0$. According to property (2) of the dynamic system, we have:

$$x'(t)=\frac{\partial\varphi(t,x_0)}{\partial t}=\frac{\partial\varphi(0+t,x_0)}{\partial t}=\frac{\partial\varphi(0,\varphi(t,x_0))}{\partial t}=\frac{\partial\varphi(0,x(t))}{\partial t}=f(x(t)) \tag{5}$$

According to, $x(t)$ is a solution of the system $x'=f(x)$.

2. Results

Now, let's consider the autonomous system (RC circuit) depicted in Figure 1. Initially, the capacitor C is charged with a "capacitive" voltage U_0. At time $t=0$, the switch P is turned on, so the capacitor through the resistor begins to discharge. In this case, the condenser acts as a voltage source.

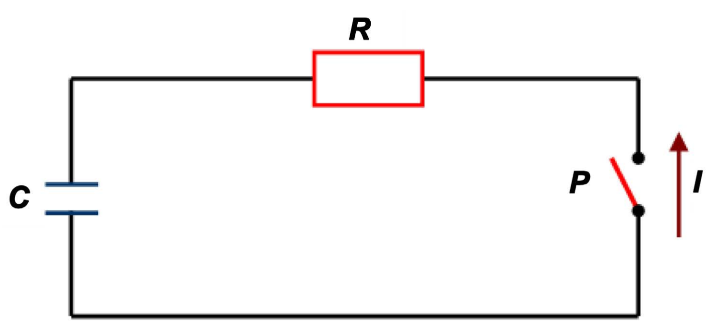

Figure 1. The response to the accumulated energy of the RC circuit.

From Kirchhoff's laws, it follows [9] [10]:

$$U_C + U_R = 0 \tag{6}$$

By using the relationships between the voltage and current of individual circuit components, we have:

$$U_R = R \cdot I, \quad I = C \cdot U_C' \tag{7}$$

By substituting the expression from Equation (7) into Equation (6), we obtain the differential equation of the RC circuit.

$$R \cdot C \cdot U_C'(t) + U_C(t) = 0 \tag{8}$$

The initial condition is the initial voltage across the capacitor $U_C(0) = U_0$.

If we introduce the time constant $\tau = R \cdot C$, Equation (8) takes the form:

$$U_C'(t) + \frac{1}{\tau} \cdot U_C(t) = 0 \tag{9}$$

with the initial condition $U_C(0) = U_0$.

Equation (9) is a homogeneous first-order differential equation with constant coefficients, and its solution is:

$$U_C(t) = K \cdot \mathrm{e}^{-\frac{t}{\tau}} \tag{10}$$

where K is a constant.

The value of the constant K is determined by the condition $U_C(0) = U_0 = K$.

Now, the general solution is represented as:

$$U_C(t) = U_0 \cdot \mathrm{e}^{-\frac{t}{\tau}} \tag{11}$$

which is defined over the entire set R_0^+ and $\tau > 0$.

The graph depicting the dependence of the capacitor voltage $U_C(t)$ on time is shown in Figure 2.

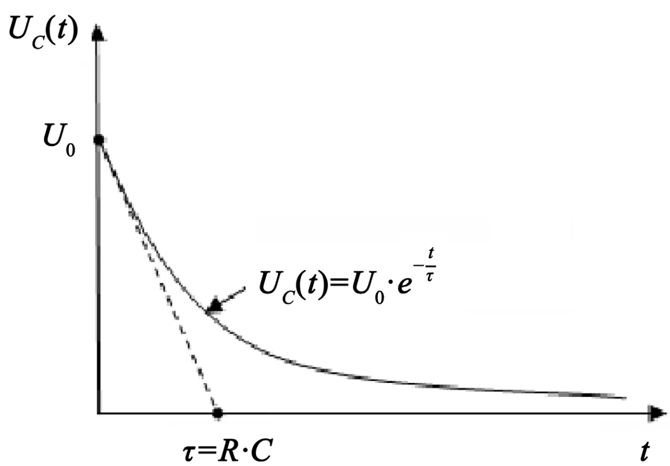

Figure 2. Dependence of capacitor voltage on time in an RC circuit.

It can be observed that the capacitor discharges not instantaneously, but according to an exponential law. The time constant $\tau = RC$ influences the rate of capacitor discharge. The larger the time constant, the slower the discharge. At the moment $t = \tau = RC$ when the "capacitive" voltage on the capacitor reaches 36.79%

of its maximum value. If the transient lasts approx $t = 5\tau$ then in engineering applications the capacitor is considered to be nearly discharged.

The "capacitive" voltage on the capacitor is then equal to 0.67% of its maximum value, and then the stationary state occurs. It can be considered that the transient process is over and that the voltage on the capacitor has reached its final value.

In a similar way, the expression for determining the current I in the RC circuit is obtained:

$$I = I_0 \cdot e^{-t/\tau} \tag{12}$$

where is:

$$I_0 = \frac{U_0}{R} \tag{13}$$

The graph of Equation (12) is similar to the graph shown in Figure 2, which also shows an exponential decline in time $t = \tau$ and the current also drops to 36.79% from its maximum value.

The qualitative analysis of the circuit will provide a phase portrait along the trajectory. Equation (9) is equivalent to the equation $U_C'(t) = -\frac{1}{\tau} U_C(t)$ We define a function $f(U_C) = -\frac{1}{\tau} U_C$, $U_C \geq 0$, $\tau > 0$ whose zero point is $U_C^* = 0$ and the considered function is decreasing. From $U_C' < 0$ for $U_C > U_C^*$, it is concluded that U_C decreases in that part and we get a phase portrait (Figure 3).

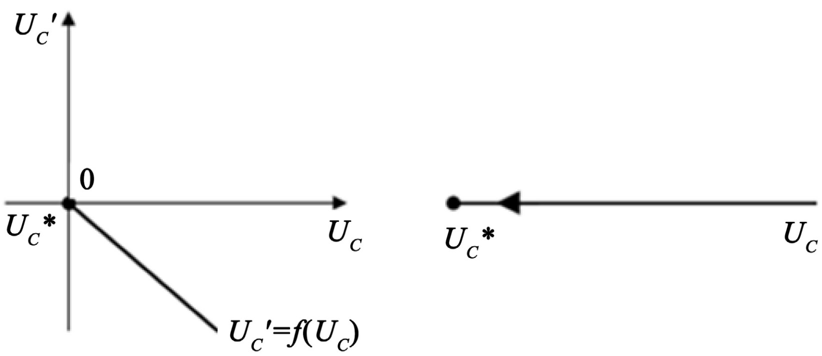

Figure 3. Phase portrait of capacitor discharge in an RC circuit.

From the phase portrait on the line, it can be seen that the charge (capacitive voltage) $U_C(t) \to U_C^* = 0$ when $t \to \infty$ regardless of the initial conditions, it stabilizes. Figure 2 shows the same but also provides information about time t.

Since all Solutions (11) of the autonomous System (9) are defined on the set R_0^+, the corresponding dynamic system can be formed using the function:

$$\varphi(t, U_C) = U_C \cdot e^{-\frac{t}{\tau}} \tag{14}$$

The function $\varphi(t, U_C): R_0^{+2} \to R_0^+$ defined by Expression (14) satisfies both conditions of the dynamic system. It is easy to check that it is valid:

$\varphi(0, U_C) = U_C \cdot e^0 = U_C$, condition (1).

$$\varphi\left(t,\varphi\left(p,U_C\right)\right)=\varphi\left(t,U_C\cdot \mathrm{e}^{-\frac{p}{\tau}}\right)=U_C\cdot \mathrm{e}^{-\frac{p}{\tau}}\cdot \mathrm{e}^{-\frac{t}{\tau}}=U_C\cdot \mathrm{e}^{-\frac{t+p}{\tau}}=\varphi\left(t+p,U_C\right),$$

$t, p\in R_0^+$, condition (2).

According to definition 6, we get $\varphi\left(t,U_C\right)=U_C\cdot \mathrm{e}^{-t/\tau}=U_C$ from where it follows $U_C\cdot\left(1-\mathrm{e}^{-t/\tau}\right)=0$, so it's obvious $U_C=0$ fixed point of the dynamic model. Besides that, it is valid $\left|\varphi'\left(t,U_C\right)\right|=\left|\varphi'\left(t,0\right)\right|=\left|\mathrm{e}^{-t/\tau}\right|<1$ for $\forall t\in R^+$ so the fixed point $U_C=0$ is attractive; that is, it represents an abyss (Figure 3).

The numerical simulation of the dynamic system (Equation (14)) is shown in Figure 4. It can be seen that the diagram shown in Figure 2 is obtained by the intersection of the surface $\varphi\left(t,U_C\right)=U_C\cdot \mathrm{e}^{-\frac{t}{\tau}}$ with the plane $U_C=\text{const}$. Figure 4 also shows the behavior of the RC circuit with variable voltage and time.

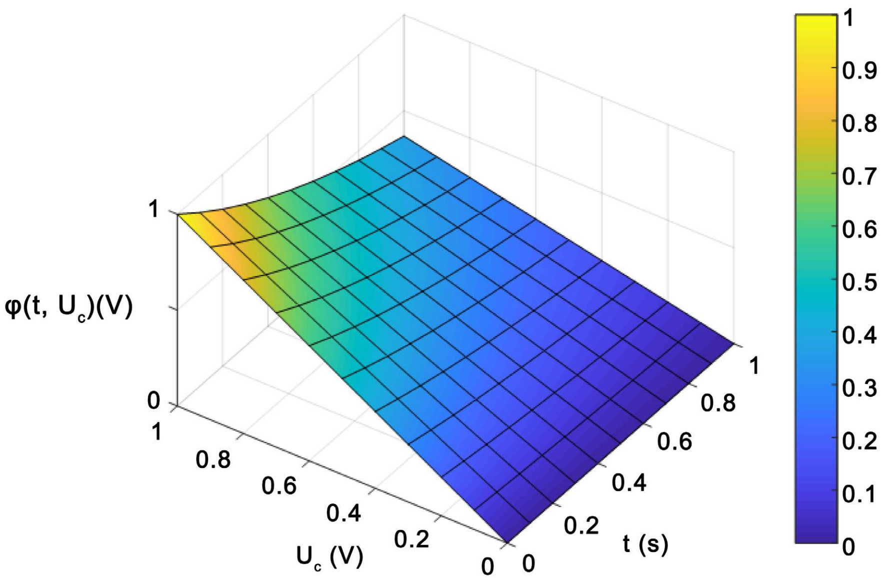

Figure 4. Graphic representation of the dynamic system $\varphi\left(t,U_C\right)=U_C\cdot \mathrm{e}^{-\frac{t}{\tau}}$.

Using theorem 1, it can be shown that, starting from the dynamic system, the corresponding autonomous system (circuit equation) can be found. We have that:

$$f\left(U_C\right)=\frac{\partial\varphi\left(0,U_C\right)}{\partial t}=-\frac{1}{\tau}\cdot U_C$$

and according to (3) it follows $U_C'=-\frac{1}{\tau}\cdot U_C$ or $U_C'+\frac{1}{\tau}\cdot U_C=0$, which is Equation (9), *i.e.*, the equation of the RC circuit.

3. Conclusions

The paper considers an electric circuit that is initially charged with a "capacitive" voltage U_0. At the moment $t=0$, the P circuit switch is turned on, and the discharge of the "capacitive" voltage on the capacitor is monitored over time, and the following conclusions were reached:

- The differential equation that describes the dynamics of the *RC* circuit is a homogeneous differential equation of the first order with constant coefficients (Equation (9)).
- The solution is presented in exponential form (Equation (11)).
- The solutions are defined on the set R_0^+, so then the corresponding dynamic RC circuit system can be formed using the function (14).
- The reverse is also shown, *i.e.*, starting from the dynamic system (14), one can find the corresponding autonomous system (equation of *RC* circuit, Equation (9)).
- Qualitative analysis of the *RC* circuit gives a phase portrait on the line, where it is concluded that U_C decreases on that part of the lead. From the phase portrait on the line, it can be seen that the charge stabilizes when regardless of the initial conditions.
- It is shown that U_C represents an attractive fixed point (sink) of the considered dynamic system, which is in accordance with the qualitative analysis of the *RC* circuit.
- The dynamic system (Equation (14)) represents a deeper insight into the behavior of the *RC* circuit and can be useful in engineering practice.
- It is shown that the discharge of the capacitor *C* in the *RC* circuit can be understood as one dynamic system.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Nomenclatures

A—expensive

C—condenser

$f(x)$—function of x

I—electricity

K—constant

$K(x,\varepsilon)$—a sphere with the center at the point of the radius

N_0—set of natural numbers and zero

O—orbit

P—switch

R—a set of real numbers

R_0^+—the set of non-negative real numbers

R—resistance

t—time

τ—time constant

U—voltage

x—independent variable