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Exact Solutions of Forced Schrödinger Equation and How to Choose the External Forces

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Abstract

Schrödinger equations are very common equations in physics and mathematics for nonlinear physics to model the dynamics of wave propagation in waveguides such as power lines, atomic chains, optical fibers, and even in quantum mechanics. But all these equations are most often studied without worrying about what would happen if this equation were maintained, that is to say, had a second member synonymous with an external force. It is true that on a physical level, such equations can be considered as describing the generation of waves on a waveguide using an external force. However, the in-depth analysis of this aspect is not at the center of our reflection in this article, but for us, it is a question of proposing exact solutions to this type of equation and above all proposing the general form of the external force so that the obtaining exact solutions is possible.

Keywords

Schrödinger Equation, Solitary Wave, Exact Solutions, External Forces, iB-Functions

1. Introduction

The development of Mathematics for nonlinear Physics experienced great growth thanks to the discovery of soliton waves by the Scottish engineer John Scott Russell. Thus, the very first analytical beginnings were made by setting up the KdV equations that model the dynamics of sea waves [1]-[5]. Since then, the concept has been transported on transmission media such as optical fibers, electrical lines, atomic chains, plasmas, Bose-Einstein condensate, etc. [6]-[11]. The observation that we make is that the nonlinear partial differential equations that model the propagation dynamics in the majority of these transmission media are of the Schrödinger type. They generally describe the dynamics of the propagation of solitary waves and numerous works have focused on proposing solutions [12]-[18]. Our curiosity from a purely mathematical angle pushed us to know what would happen if the cubic Schrödinger equation were subjected to external action. However, it should already be noted that this article is our second article, which solves a nonlinear partial differential equation with an external force, because we have successfully treated the case of KdV with an external force [19].

In the case where we accept this possibility, what must be the external force for the solutions obtained to be exact? Faced with these questions, we resolved to construct the general solution of the cubic Schrödinger equation with an external force and, at the same time, propose a model of general external forces so that exact solutions are possible. However, in front of this commitment to proposing a general solution, there is difficulty in the method or technique used to achieve our objective. For this purpose, we used an ansatz solution based on iB-functions [20]-[24] to facilitate the approach. Indeed, this choice is not random; it is just that it offers wide possibilities for obtaining solutions. So, this article will be organized as follows. In Section 2, we present the model of the forced Schrödinger equation, which will be at the center of our analyses. In Section 3, we present the method to be used to obtain the results; we will use the main iB-functions to construct the first series of solutions and the external forces in Section 4. Section 5 will use the secondary forms of the iB-functions to construct the second series of solutions and the external forces. Section 6 shows the profiles of the solutions obtained in the case where the main forms of the iB-functions are used. Section 7 shows the solution profiles in the case where secondary forms of iB-functions are used. We end the work with a conclusion and some remarks.

2. Forced Schrödinger Equation

The Schrödinger equation, as we want to propose solutions in this article, is given by

$$i\frac{\partial A}{\partial t}+P\frac{\partial^2 A}{\partial x^2}+Q|A|^2 A=F(x,t), \tag{1}$$

where P represents the coefficient of dispersion, Q the coefficient of nonlinearity, A represents the wave envelope and $F(x,t)$ the external force. The equation without a second member associated with Equation (1) is given by

$$i\frac{\partial A}{\partial t}+P\frac{\partial^2 A}{\partial x^2}+Q|A|^2 A=0. \tag{2}$$

We will then construct the general solution of Equation (1) and propose some particular solutions. But before moving on to the solution itself, it is appropriate to dwell a little on the technique that must be used to obtain these solutions.

3. Method Used

The ansatz solutions used come from the two forms of iB-functions, notably the main form and the secondary form. The main and secondary forms are defined respectively in dimension one by [18]-[22]

$$J_{n,m}(\xi)=\frac{\sinh^m(\xi)}{\cosh^n(\xi)}, \tag{3}$$

and

$$T_{n,m}(\xi)=\frac{\sin^m(\xi)}{\cos^n(\xi)}. \tag{4}$$

The two previous functions are linked by relationships

$$J_{n,m}(i\xi)=(i)^m T_{n,m}(\xi),\ i^2=-1, \tag{5}$$

and

$$T_{n,m}(i\xi)=(i)^m J_{n,m}(\xi),\ i^2=-1. \tag{6}$$

In the following, we use these two forms of iB-functions to constitute our ansatz solutions.

4. First Series of Solutions: Use of iB-Main Functions

In this section, we propose to construct the solutions of Equation (1). To do this, we first start by solving this equation without a second member Equation (2) to get an idea of the general form of solution to Equation (1) to construct. Thus, by seeking the solution of Equation (2) in the form:

$$A(x,t)=aJ_{n,0}(\alpha x)\exp(i\alpha t),\ i^2=-1, \tag{7}$$

where a and n are real numbers to be determined and α a parameter which can be real or complex. Thus, inserting Equation (7) into Equation (2) leads to the equation

$$\left(an^2\alpha^2P-a\alpha^2\right)J_{n,0}-an(n+1)\alpha^2PJ_{n+2,0}+Q|a|^2aJ_{3n,0}=0. \tag{8}$$

The search for values of n for which certain terms of Equation (8) come together gives $n=0$ and $n=1$. For $n=0$, we have a trivial solution. In the case where $n=1$, Equation (8) becomes

$$a\alpha^2(P-1)J_{1,0}+a\left(Q|a|^2-2\alpha^2P\right)aJ_{3,0}=0. \tag{9}$$

Equation (9) is verified if for $a\neq 0$, we have $P=1$ and $a=\alpha\sqrt{2/Q}\exp i\theta$, $\alpha\succ 0$, $Q\succ a$, $\theta\in R$. The solution to Equation (2) under these conditions is given by

$$A(x,t)=\alpha\sqrt{2/Q}J_{1,0}(\alpha x)\exp i\left(\alpha^2t+\theta\right)\equiv\alpha\sqrt{2/Q}\operatorname{sech}(\alpha x)\exp i\left(\alpha^2t+\theta\right) \tag{10}$$

Solution Equation (10) serves as a guide for us to choose the general form of solution of Equation (1) to construct. Thus, we propose to subsequently construct the solution to Equation (1) in the form

$$\begin{aligned}A(x,t)&=\alpha\sqrt{2/Q}J_{1,0}(f(x))\exp i\left(\alpha^2t+\theta\right)\\&\equiv\alpha\sqrt{2/Q}\operatorname{sech}(f(x))\exp i\left(\alpha^2t+\theta\right),\end{aligned} \tag{11}$$

where $f(x)$ is an arbitrary function continuously differentiable in its domain of definition. The insertion of Equation (11) in Equation (1) supposes the evaluation of its different terms. Thus, we have

$$\frac{\partial A}{\partial t}=i\alpha^{3}\sqrt{2/Q}\,J_{1,0}\left(f\left(x\right)\right)\exp i\left(\alpha^{2}t+\theta\right), \tag{12}$$

$$\frac{\partial A}{\partial x}=-\alpha\sqrt{2/Q}\,f'\left(x\right)J_{2,1}\left(f\left(x\right)\right)\exp i\left(\alpha^{2}t+\theta\right), \tag{13}$$

$$\frac{\partial^{2}A}{\partial x^{2}}=-\alpha\sqrt{2/Q}\left[f''J_{2,1}\left(f\right)-f'^{2}J_{1,0}\left(f\right)+2f'^{2}J_{3,0}\left(f\right)\right]\exp i\left(\alpha^{2}t+\theta\right), \tag{14}$$

and

$$\left|A\right|^{2}A=\left(2\alpha^{3}/Q\right)\sqrt{2/Q}\,J_{3,0}\exp i\left(\alpha^{2}t+\theta\right). \tag{15}$$

In expressions Equation (12) to Equation (15), f', f'' denote respectively the first derivative of $f\left(x\right)$ with respect to x and the second derivative of $f\left(x\right)$ with respect to x. Insertion of terms Equation (12) to Equation (15) in Equation (1) allows to have external strength as

$$F\left(x,t\right)=\left(F_{1}\left(x\right)J_{1,0}\left(f\right)+F_{2}\left(x\right)J_{3,0}\left(f\right)+F_{3}\left(x\right)J_{2,1}\left(f\right)\right)\exp i\left(\alpha^{2}t+\theta\right), \tag{16}$$

where the functions $F_{i}\left(x\right), i=1,2,3$ are given

$$F_{1}\left(x\right)=\alpha\sqrt{2/Q}\left(f'^{2}-\alpha^{2}\right), \tag{17}$$

$$F_{2}\left(x\right)=2\alpha\sqrt{2/Q}\left(\alpha^{2}-f'^{2}\right), \tag{18}$$

and

$$F_{3}\left(x\right)=-\alpha\sqrt{2/Q}\,f''. \tag{19}$$

The expression for the external force having been found, the cubic Schrödinger equation with external force must be defined in the following way so that obtaining the exact solution is possible.

$$\begin{aligned}&i\frac{\partial A}{\partial t}+\frac{\partial^{2}A}{\partial x^{2}}+Q\left|A\right|^{2}A\\&=\left(F_{1}\left(x\right)J_{1,0}\left(f\right)+F_{2}\left(x\right)J_{3,0}\left(f\right)+F_{3}\left(x\right)J_{2,1}\left(f\right)\right)\exp i\left(\alpha^{2}t+\theta\right).\end{aligned} \tag{20}$$

The choice of external force is very important, it must be adequate for the reaction to produce the expected effects. Thus, any solitary wave solution of the form $A\left(x,t\right)=\alpha\sqrt{2/Q}\,J_{1,0}\left(f\left(x\right)\right)\exp i\left(\alpha^{2}t+\theta\right)\equiv\alpha\sqrt{2/Q}\,\operatorname{sech}\left(f\left(x\right)\right)\exp i\left(\alpha^{2}t+\theta\right)$, where $f\left(x\right)$ is a continuously differentiable function in its domain is always an exact solution to Equation (20). Thus, we can generalize the exact solution of the forced Schrödinger Equation (20) as follows

$$\begin{aligned}A\left(x,t\right)&=\alpha\sqrt{2/Q}\,J_{1,0}\left(f\left(x\right)+C\right)\exp i\left(\alpha^{2}t+\theta\right)\\&\equiv\alpha\sqrt{2/Q}\,\operatorname{sech}\left(f\left(x\right)+C\right)\exp i\left(\alpha^{2}t+\theta\right),\end{aligned} \tag{21}$$

where α can be real or complex, Q the nonlinearity coefficient and C an arbitrary constant.

Some Exact Solutions and External Forces

In this subsection, we propose some exact solutions as well as the forced

Schrödinger equation that they verify.

- For $f(x)=\cos x$ the corresponding forced Schrödinger equation is given by Equation (20), as we have

$$F_1(x)=\alpha\sqrt{2/Q}\left(\sin^2 x-\alpha^2\right), \tag{22}$$

$$F_2(x)=2\alpha\sqrt{2/Q}\left(\alpha^2-\sin^2 x\right), \tag{23}$$

and

$$F_3(x)=\alpha\sqrt{2/Q}\cos x. \tag{24}$$

The exact general solution in this case is given by

$$\begin{aligned}A(x,t)&=\alpha\sqrt{2/Q}\,J_{1,0}\left(\cos x+C\right)\exp i\left(\alpha^2 t+\theta\right)\\&\equiv\alpha\sqrt{2/Q}\,\mathrm{sech}\left(\cos x+C\right)\exp i\left(\alpha^2 t+\theta\right).\end{aligned} \tag{25}$$

- For $f(x)=x^2$ the corresponding forced Schrödinger equation is given by Equation (20), as we have

$$F_1(x)=\alpha\sqrt{2/Q}\left(4x^2-\alpha^2\right), \tag{26}$$

$$F_2(x)=2\alpha\sqrt{2/Q}\left(\alpha^2-4x^2\right), \tag{27}$$

and

$$F_3(x)=-2\alpha\sqrt{2/Q}. \tag{28}$$

The exact general solution in this case is given by

$$\begin{aligned}A(x,t)&=\alpha\sqrt{2/Q}\,J_{1,0}\left(x^2+C\right)\exp i\left(\alpha^2 t+\theta\right)\\&\equiv\alpha\sqrt{2/Q}\,\mathrm{sech}\left(x^2+C\right)\exp i\left(\alpha^2 t+\theta\right).\end{aligned} \tag{29}$$

- For $f(x)=\mathrm{sech}\,x$ the corresponding forced Schrödinger equation is given by Equation (20), as we have

$$F_1(x)=\alpha\sqrt{2/Q}\left(\mathrm{sech}^4 x\sinh^2 x-\alpha^2\right), \tag{30}$$

$$F_2(x)=2\alpha\sqrt{2/Q}\left(\alpha^2-\mathrm{sech}^4 x\sinh^2 x\right), \tag{31}$$

and

$$F_3(x)=\alpha\sqrt{2/Q}\,\mathrm{sech}\,x\left(1+2\,\mathrm{sech}^2 x\sinh^2 x\right). \tag{32}$$

The exact general solution in this case is given by

$$\begin{aligned}A(x,t)&=\alpha\sqrt{2/Q}\,J_{1,0}\left(\mathrm{sech}\,x+C\right)\exp i\left(\alpha^2 t+\theta\right)\\&\equiv\alpha\sqrt{2/Q}\,\mathrm{sech}\left(\mathrm{sech}\,x+C\right)\exp i\left(\alpha^2 t+\theta\right).\end{aligned} \tag{33}$$

- For $f(x)=\tanh x$ the corresponding forced Schrödinger equation is given by Equation (20), as we have

$$F_1(x)=\alpha\sqrt{2/Q}\left(\mathrm{sech}^4 x-\alpha^2\right), \tag{34}$$

$$F_2(x)=2\alpha\sqrt{2/Q}\left(\alpha^2-\mathrm{sech}^4 x\right), \tag{35}$$

and

$$F_3(x) = 2\alpha\sqrt{2/Q}\,\mathrm{sech}^3 x \sinh x. \tag{36}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}\,J_{1,0}(\tanh x + C)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\,\mathrm{sech}(\tanh x + C)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{37}$$

- For $f(x) = \tan x$ the corresponding forced Schrödinger equation is given by Equation (20), as we have

$$F_1(x) = \alpha\sqrt{2/Q}\left(\sec^4 x - \alpha^2\right), \tag{38}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 - \sec^4 x\right), \tag{39}$$

and

$$F_3(x) = -2\alpha\sqrt{2/Q}\sec^3 x \sin x. \tag{40}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}\,J_{1,0}(\tan x + C)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\,\mathrm{sech}(\tan x + C)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{41}$$

- For $f(x) = \mathrm{Arctan}\, x$ the corresponding forced Schrödinger equation is given by Equation (20), as we have

$$F_1(x) = \alpha\sqrt{2/Q}\left(\left(\frac{1}{1+x^2}\right)^2 - \alpha^2\right), \tag{42}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 - \left(\frac{1}{1+x^2}\right)^2\right), \tag{43}$$

and

$$F_3(x) = \frac{2\alpha\sqrt{2/Q}\,x}{\left(1+x^2\right)^2}. \tag{44}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}\,J_{1,0}(\mathrm{Arctan}\, x + C)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\,\mathrm{sech}(\mathrm{Arctan}\, x + C)\exp i\left(\alpha^2 t + \theta\right). \end{aligned}$$

- For $f(x) = \exp x$ the corresponding forced Schrödinger equation is given by Equation (20), as we have

$$F_1(x) = \alpha\sqrt{2/Q}\left(\exp 2x - \alpha^2\right), \tag{45}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 - \exp 2x\right), \tag{46}$$

and

$$F_3(x) = -\alpha\sqrt{2/Q}\exp x. \tag{47}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}\,J_{1,0}(\exp x + C)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\,\mathrm{sech}(\exp x + C)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{48}$$

5. Second Series of Solutions: Use of Secondary iB-Functions

In this section, we propose to construct the solutions of Equation (1) using the second form of iB-functions. To do this, we first start by solving this equation without a second member to get an idea of the general solution to Equation (1), which is given by (2). Thus, by looking for the solution to Equation (2) in the form

$$\eta(\xi) = aT_{n,0}(\alpha x)\exp i\alpha^2 t, \tag{49}$$

where a and n are real numbers to be determined. Thus, the insertion of Equation (49) into Equation (2) leads to the equation

$$-\alpha^2(1+P)aT_{n,0} + \alpha^2 Pan(n+1)T_{n+2,0} + Q|a|^2 aT_{3n,0} = 0. \tag{50}$$

Looking for the values of n for which certain terms in Equation (50) group together gives $n=0$ and $n=1$. For $n=0$, we have a trivial solution. In the case where $n=1$, Equation (50) becomes

$$-\alpha^2(1+P)aT_{1,0} + \left(2\alpha^2 P + Q|a|^2\right)aT_{3,0} = 0 \tag{51}$$

Equation (51) is verified if for $a \neq 0$, we have $P=-1$ and $|a| = \alpha\sqrt{\frac{2}{Q}}$ $\Rightarrow \exists\theta \in R / a = \alpha\sqrt{\frac{2}{Q}}\exp i\theta$, $i^2=-1$. The solution to Equation (2) under these conditions is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{\frac{2}{Q}}T_{1,0}(\alpha x)\exp i\left(\alpha^2 t+\theta\right) \\ &= \alpha\sqrt{\frac{2}{Q}}\sec(\alpha x)\exp i\left(\alpha^2 t+\theta\right). \end{aligned} \tag{52}$$

Solution Equation (52) serves as a guide to choose the general form of solution of Equation (1) to construct. Thus, we propose to subsequently construct the solution to Equation (1) in the form

$$\begin{aligned} A(x,t) &= \alpha\sqrt{\frac{2}{Q}}T_{1,0}(f(x))\exp i\left(\alpha^2 t+\theta\right) \\ &= \alpha\sqrt{\frac{2}{Q}}\sec(f(x))\exp i\left(\alpha^2 t+\theta\right). \end{aligned} \tag{53}$$

where $f(x)$ is an arbitrary function continuously differentiable in its domain of definition. The insertion of Equation (53) in Equation (1) obliges to evaluate its different terms. Thus, we have

$$\frac{\partial A}{\partial t} = i\alpha^3\sqrt{\frac{2}{Q}}T_{1,0}(f(x))\exp i\left(\alpha^2 t+\theta\right), \tag{54}$$

$$|A|^2 A = \frac{2\alpha^3}{Q}\sqrt{\frac{2}{Q}}T_{3,0}(f)\exp i\left(\alpha^2 t+\theta\right), \tag{55}$$

and

$$\frac{\partial^2 A}{\partial x^2} = \alpha\sqrt{\frac{2}{Q}}\left[-f'^2 T_{1,0}(f) + f'' T_{2,1}(f) + 2f'^2 T_{3,0}(f)\right]\exp i\left(\alpha^2 t+\theta\right), \tag{56}$$

In Equation (56) f' and f'' denote respectively the first derivative of $f(x)$ with respect to x and the second derivative of $f(x)$ with respect to x.The insertion of the terms Equation (54) to Equation (56) in the Equation (1) allows to have the external force under

$$F(x,t)=\left[F_1(x)T_{1,0}(f)+F_2(x)T_{3,0}(f)+F_3(x)T_{2,1}(f)\right]\exp i\left(\alpha^2 t+\theta\right), \quad (57)$$

where the functions $F_i(x), i=1,2,3$ are given by

$$F_1(x)=-\alpha\sqrt{\frac{2}{Q}}\left(\alpha^2+f'^2\right), \quad (58)$$

$$F_2(x)=2\alpha\sqrt{\frac{2}{Q}}\left(\alpha^2+f'^2\right), \quad (59)$$

and

$$F_3(x)=\alpha\sqrt{\frac{2}{Q}}f'', \quad (60)$$

The forced Schrödinger equation in this case is corrected as follows

$$\begin{aligned} &i\frac{\partial A}{\partial t}-\frac{\partial^2 A}{\partial x^2}+Q|A|^2 A \\ &=\left(F_1(x)T_{1,0}(f)+F_2(x)T_{3,0}(f)+F_3(x)T_{2,1}(f)\right)\exp i\left(\alpha^2 t+\theta\right). \end{aligned} \quad (61)$$

The choice of external force is very important; it must be adequate for the reaction to produce the expected effects. Thus any solitary wave solution of the form $A(x,t)=\alpha\sqrt{\frac{2}{Q}}T_{1,0}(f(x))\exp i\left(\alpha^2 t+\theta\right)$ where $f(x)$ is a continuously differentiable function in its domain is always an exact solution to Equation (1). Thus, we can generalize the exact solution of the forced Schrödinger Equation (1) as follows

$$\begin{aligned} A(x,t)&=\alpha\sqrt{\frac{2}{Q}}T_{1,0}\left(f(x)+K\right)\exp i\left(\alpha^2 t+\theta\right) \\ &\equiv\alpha\sqrt{\frac{2}{Q}}\sec\left(f(x)+K\right)\exp i\left(\alpha^2 t+\theta\right), \end{aligned} \quad (62)$$

where Q is the nonlinear coefficient of order 3, α^2 the angular frequency and K an arbitrary constant. It should be noted that we can go from solution Equation (21) to solution Equation (62) and vice versa by making the following matches $\lambda \leftarrow i\lambda$, $f(\xi)\leftarrow if(\xi)$ et $K \leftarrow iK$.

Some Exact Solutions and External Forces

In this subsection, we propose some exact solutions as well as the forced Schrödinger equation that they verify.

- For $f(x)=\cos x$ the corresponding forced Schrödinger equation is given by Equation (61), as we have

$$F_1(x)=-\alpha\sqrt{2/Q}\left(\sin^2 x+\alpha^2\right), \quad (63)$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 + \sin^2 x\right), \tag{64}$$

and

$$F_3(x) = -\alpha\sqrt{2/Q}\cos x. \tag{65}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}T_{1,0}\left(\cos x + K\right)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\sec\left(\cos x + K\right)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{66}$$

For $f(x) = x^2$ the corresponding forced Schrödinger equation is given by Equation (61), as we have

$$F_1(x) = -\alpha\sqrt{2/Q}\left(4x^2 + \alpha^2\right), \tag{67}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 + 4x^2\right), \tag{68}$$

and

$$F_3(x) = 2\alpha\sqrt{2/Q}. \tag{69}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}T_{1,0}\left(x^2 + K\right)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\sec\left(x^2 + K\right)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{70}$$

- For $f(x) = \operatorname{sech} x$ the corresponding forced Schrödinger equation is given by Equation (61), as we have

$$F_1(x) = -\alpha\sqrt{2/Q}\left(\operatorname{sch}^4 x \sinh^2 x + \alpha^2\right), \tag{71}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 + \operatorname{sech}^4 x \sinh^2 x\right), \tag{72}$$

and

$$F_3(x) = -\alpha\sqrt{2/Q}\operatorname{sech} x\left(1 + 2\operatorname{sech}^2 x \sinh^2 x\right). \tag{73}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}T_{1,0}\left(\operatorname{sech} x + K\right)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\sec\left(\operatorname{sech} x + K\right)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{74}$$

- For $f(x) = \tanh x$ the corresponding forced Schrödinger equation is given by Equation (61), as we have

$$F_1(x) = -\alpha\sqrt{2/Q}\left(\operatorname{sech}^4 x + \alpha^2\right), \tag{75}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 + \operatorname{sech}^4 x\right), \tag{76}$$

and

$$F_3(x) = -2\alpha\sqrt{2/Q}\operatorname{sech}^3 x \sinh x. \tag{77}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}T_{1,0}\left(\tanh x + K\right)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\sec\left(\tanh x + K\right)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{78}$$

- For $f(x) = \tan x$ the corresponding forced Schrödinger equation is given by

Equation (61), as we have

$$F_1(x) = -\alpha\sqrt{2/Q}\left(\sec^4 x + \alpha^2\right), \tag{79}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 + \sec^4 x\right), \tag{80}$$

and

$$F_3(x) = 2\alpha\sqrt{2/Q}\sec^3 x \sin x. \tag{81}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}T_{1,0}\left(\tan x + K\right)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\,\mathrm{sech}\left(\tan x + K\right)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{82}$$

- For $f(x) = \mathrm{Arctan}\,x$ the corresponding forced Schrödinger equation is given by Equation (61), as we have

$$F_1(x) = -\alpha\sqrt{2/Q}\left(\left(\frac{1}{1+x^2}\right)^2 + \alpha^2\right), \tag{83}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 + \left(\frac{1}{1+x^2}\right)^2\right), \tag{84}$$

and

$$F_3(x) = \frac{-2x\alpha\sqrt{2/Q}}{\left(1+x^2\right)^2}. \tag{85}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}T_{1,0}\left(Arc\tan x + K\right)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\sec\left(Arc\tan x + K\right)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{86}$$

- For $f(x) = \exp x$ the corresponding forced Schrödinger equation is given by Equation (61), as we have

$$F_1(x) = -\alpha\sqrt{2/Q}\left(\exp 2x + \alpha^2\right), \tag{87}$$

$$F_2(x) = 2\alpha\sqrt{2/Q}\left(\alpha^2 + \exp 2x\right), \tag{88}$$

and

$$F_3(x) = \alpha\sqrt{2/Q}\exp x. \tag{89}$$

The exact general solution in this case is given by

$$\begin{aligned} A(x,t) &= \alpha\sqrt{2/Q}T_{1,0}\left(\exp x + K\right)\exp i\left(\alpha^2 t + \theta\right) \\ &\equiv \alpha\sqrt{2/Q}\sec\left(\exp x + K\right)\exp i\left(\alpha^2 t + \theta\right). \end{aligned} \tag{90}$$

Figures 1-5 represent some profiles of the solutions obtained in Section 4 and **Figures 6-10** represent some profiles of solutions obtained in Section 5. It is important to point out that for any defined function that is continuous and differentiable in its domain of definition, we can obtain a corresponding solution profile. But except that, not all the curves obtained can properly describe the solitary waves as expected.

6. Solution Profiles: Case of the First Series of Solutions

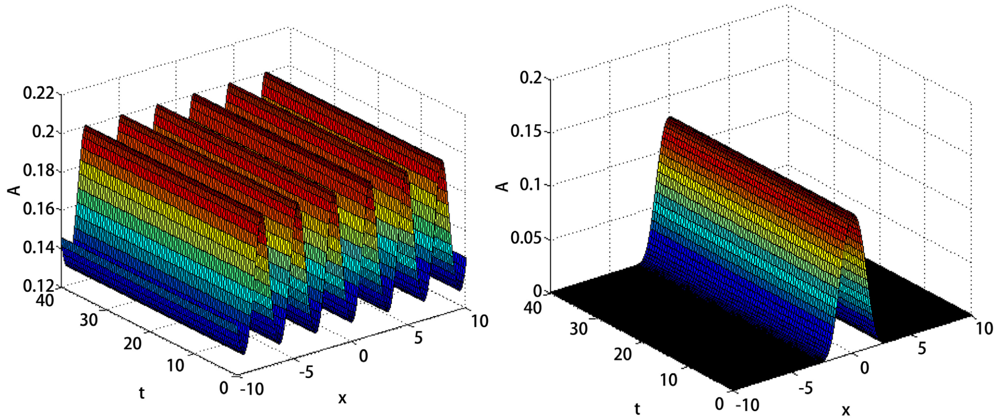

Figure 1. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,J_{1,0}\left(f(x)+C\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $C = 0$: the left curve is obtained for $f(x) = \cos(x)$ and the right curve for $f(x) = \cosh(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

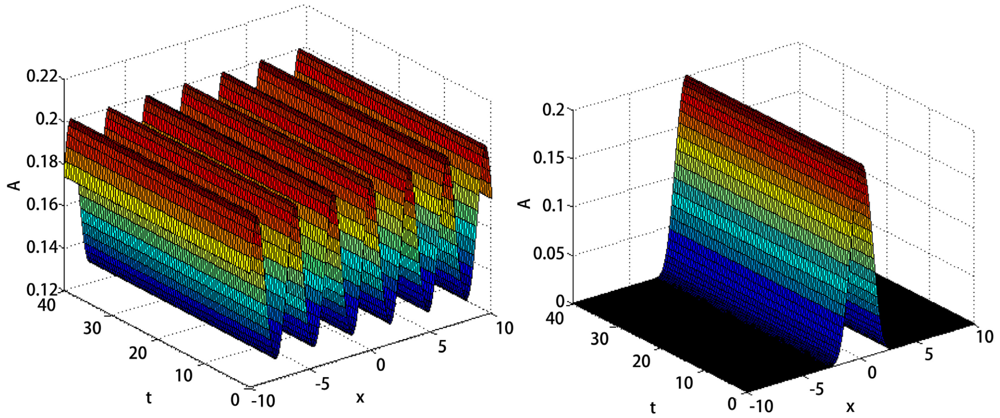

Figure 2. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,J_{1,0}\left(f(x)+C\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $C = 0$: the left curve is obtained for $f(x) = \sin(x)$ and the right curve for $f(x) = \sinh(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

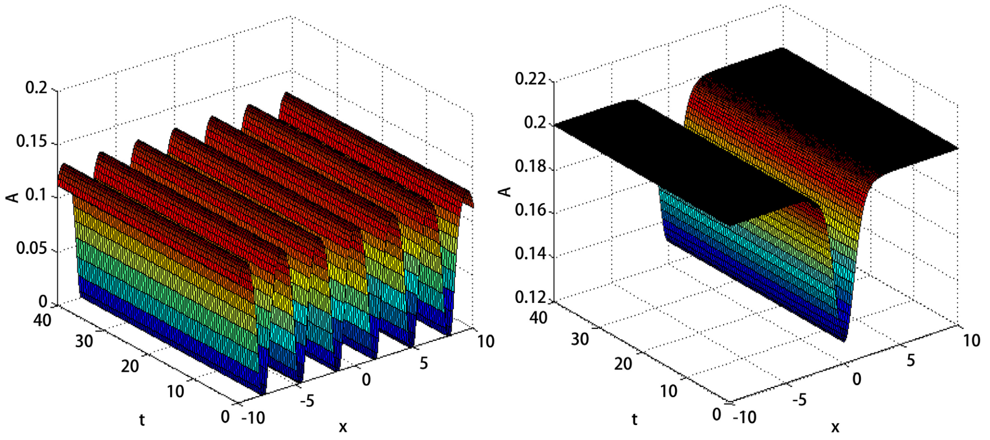

Figure 3. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,J_{1,0}\left(f(x)+C\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $C = 0$: the left curve is obtained for $f(x) = \sec(x)$ and the right curve for $f(x) = \mathrm{sech}(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

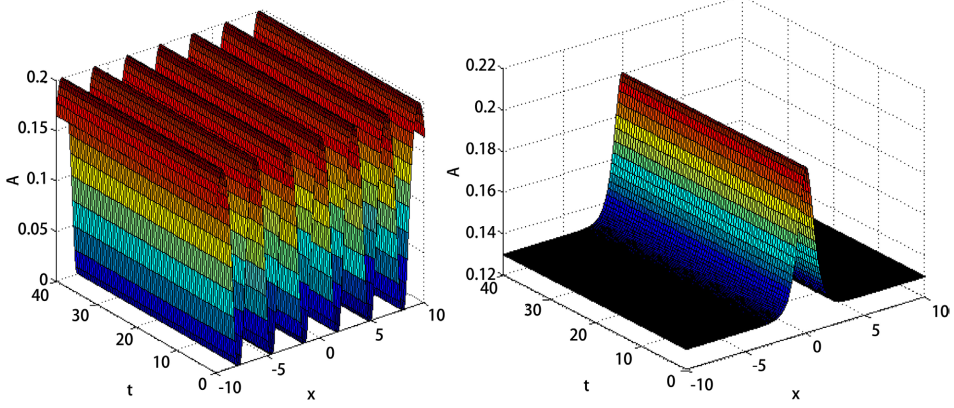

Figure 4. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,J_{1,0}\left(f(x)+C\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $C = 0$: the left curve is obtained for $f(x) = \tan(x)$ and the right curve for $f(x) = \tanh(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

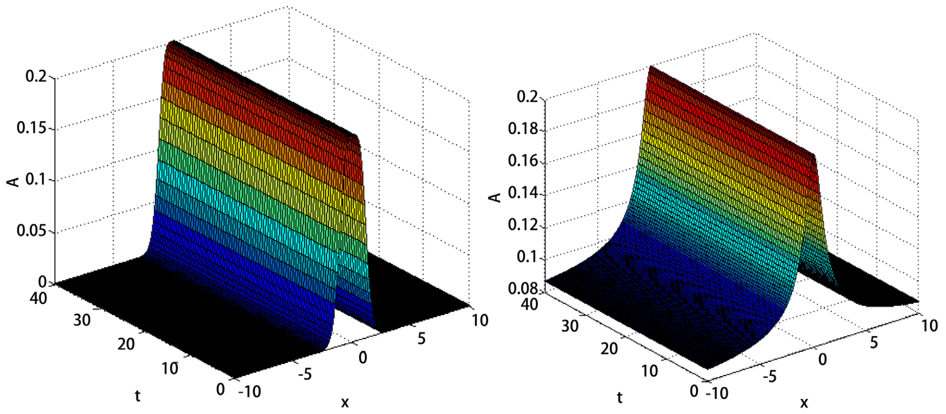

Figure 5. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,J_{1,0}\left(f(x)+C\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $C = 0$: the left curve is obtained for $f(x) = x^2$ and the right curve for $f(x) = \arctan(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

7. Solution Profiles: Case of the Second Series of Solutions

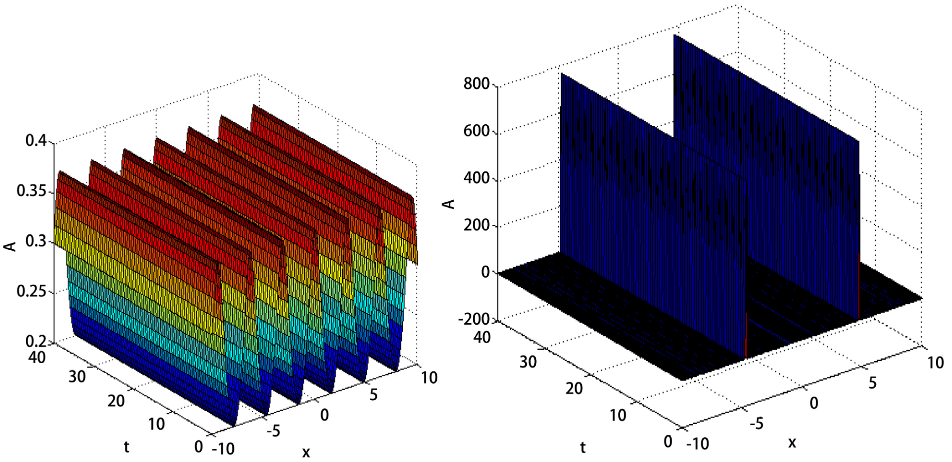

Figure 6. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,T_{1,0}\left(f(x)+K\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $K = 0$: the left curve is obtained for $f(x) = \cos x$ and the right curve for $f(x) = \cosh(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

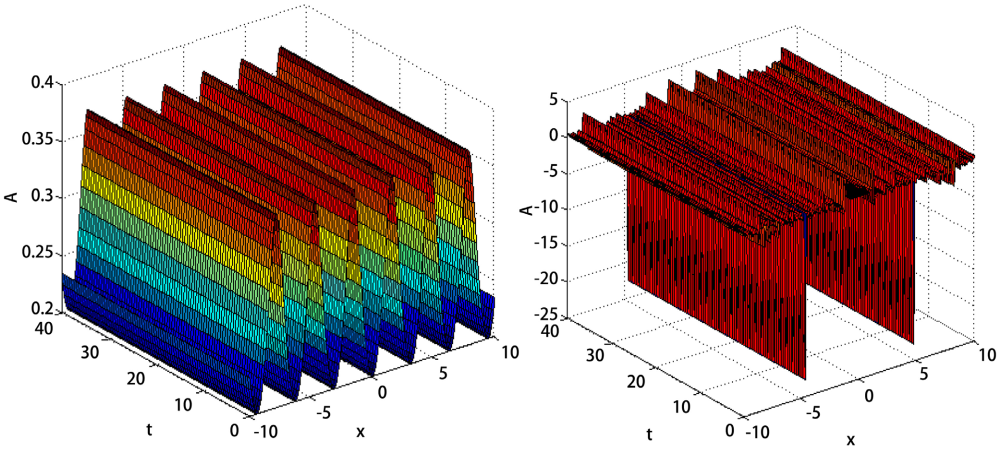

Figure 7. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,T_{1,0}\left(f(x)+K\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $K = 0$: the left curve is obtained for $f(x) = \sin(x)$ and the right curve for $f(x) = \sinh(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

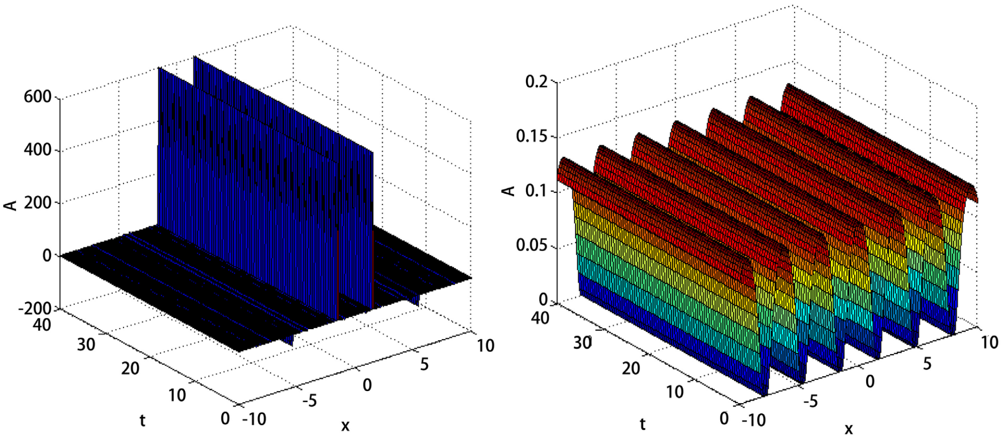

Figure 8. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,T_{1,0}\left(f(x)+K\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $K = 0$: the left curve is obtained for $f(x) = \sec(x)$ and the right curve for $f(x) = \operatorname{sech}(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

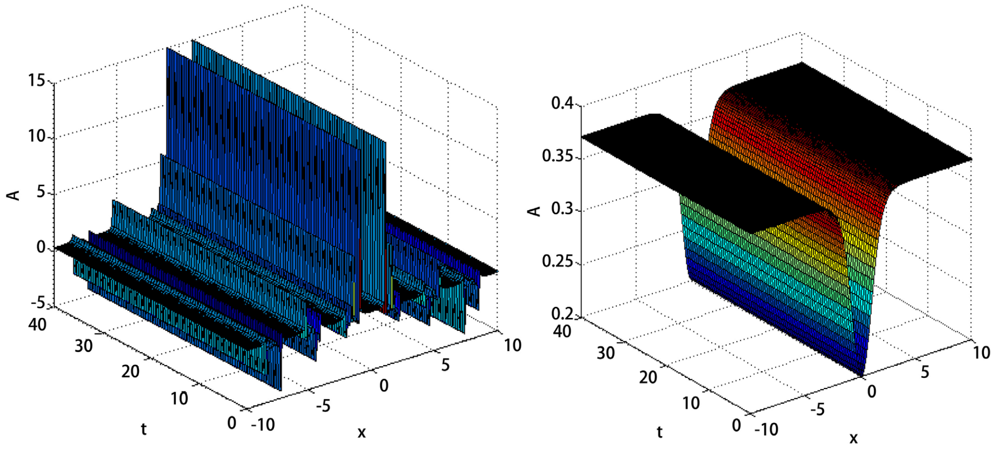

Figure 9. Curves represent the solution $\left|A(x,t)\right| = \alpha\sqrt{2/Q}\,T_{1,0}\left(f(x)+K\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $K = 0$: the left curve is obtained for $f(x) = \tan(x)$ and the right curve for $f(x) = \tanh(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

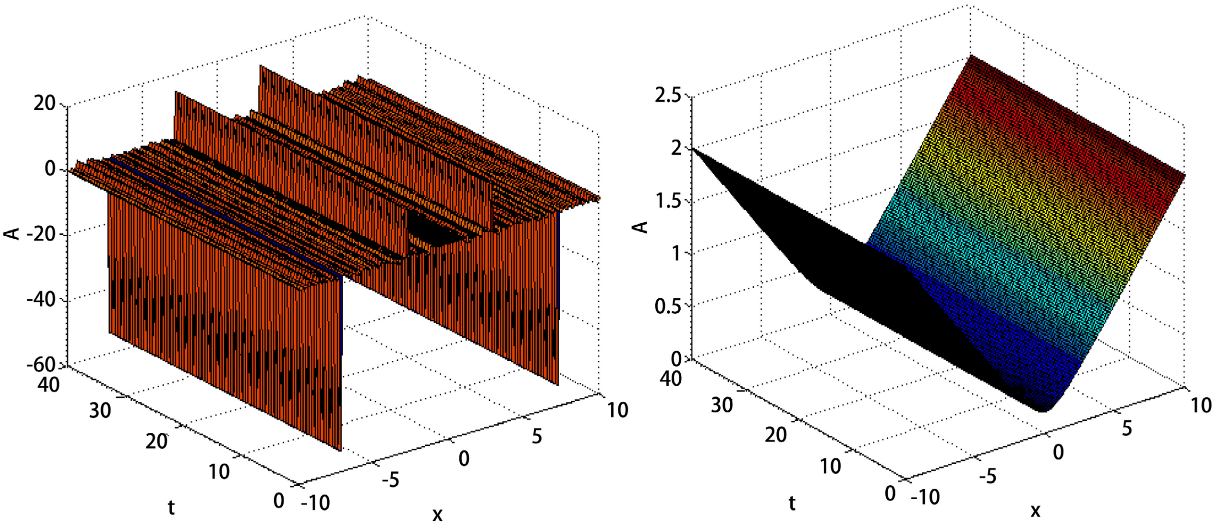

Figure 10. Curves represent the solution $|A(x,t)| = \alpha\sqrt{2/Q}\,T_{1,0}\left(f(x)+K\right)\exp i\left(\alpha^2 t+\theta\right)$ for $\alpha = 0.1$, $Q = 0.5$ and $K = 0$: the left curve is obtained for $f(x) = x^2$ and the right curve for $f(x) = \text{Arctan}(x)$ such that we have $x \in [-10,10]$, $t \in [0,40]$.

The curves of Section 6 are the projections of the curves of Section 7 in R and the curves of Section 7 are the projections of the curves of Section 6 in an interval of length 2π or in a circular domain. We can notice that the curves obtained in the case where the solutions are main iB-functions, have similar appearances to the curves obtained in the case where the solutions sought are secondary forms of the iB-functions.

8. Conclusions

This article is a mathematical contribution within the framework of the resolution of nonlinear partial differential equations where it is assumed that it has an external action or even an external force. Beyond classic physical considerations, we have further thought to see how to resolve them in the case where there is an external force that generates this dynamic. Having admitted this possibility, the main concern was how to choose the external force so that this equation admits exact solutions. The equation that we considered in this work is the cubic Schrödinger equation with an external force. We have noted for this purpose that when we construct solution (1) in the form $A(x,t) = \alpha\sqrt{\frac{2}{Q}}J_{1,0}\left(f(x)\right)\exp i\left(\alpha^2 t+\theta\right)$, the major constraint for this to be an exact solution is $P = 1$ and such that the external force is given in Equation (20). When the solution is of the form $A(x,t) = \alpha\sqrt{\frac{2}{Q}}T_{1,0}\left(f(x)\right)\exp i\left(\alpha^2 t+\theta\right)$, the constraint is $P = -1$ such that the external force is given by Equation (61). Thus, we proposed forms of exact solutions based on the choice of the function $f(x)$ which must first be continuous and differentiable several times in its domain of definition.

We have considered two sets of solutions in this manuscript, but it should be noted that in practice, it is advisable to consider a single case to search for solutions and subsequently move on to other forms of solutions through transformations.

$$J_{n,m}\left(i\xi\right)=\left(i\right)^{m}T_{n,m}\left(\xi\right),\ i^{2}=-1\quad\text{or}\quad T_{n,m}\left(i\xi\right)=\left(i\right)^{m}J_{n,m}\left(\xi\right),\ i^{2}=-1\,.$$

Figures 1-10 show some profiles of wave solutions resulting from the results obtained. We humbly believe that this approach to solving and analyzing the forced Schrödinger equation can be a significant contribution to the resolution of a forced nonlinear partial differential equation. The resolution of nonlinear partial differential equations is very difficult and so worrying that researchers generally do not have the time to know what is happening when these equations present a right side, that is to say when there is an external action. This work is then included in the short list of works that propose an approach to finding solutions to nonlinear partial differential equations with external action. Although the aim of this article was not to explore all facets of the physical applications of the solutions obtained, these solutions can help generate waves or excitations in the transmission media considered. You just need to choose the right external forces.

Data Availability Statement

All data that support the findings of this study are included in the article.

Credit Authorship Contribution Statement

M.N. ZamboAbou'o: Investigation, Writing, Printing, Investigation, Computation, Verification of results.

J.R. Bogning: Conception of the project, execution of the project, writing, Printing, Investigation, Computation and Numerical study, Interpretation of results.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work in this paper.

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