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A Novel Inverse-Free Neurodynamic Approach for Solving Absolute Value Equations

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Abstract

We propose a novel inverse-free neurodynamic approach (NIFNA) for solving absolute value equations (AVE). The NIFNA guarantees global convergence and notably improves convergence speed by achieving fixed-time convergence. To validate the theoretical findings, numerical simulations are conducted, demonstrating the effectiveness and efficiency of the proposed NIFNA.

Keywords

Absolute Value Equations, Neurodynamic Approach, Fixed-Time Convergence, Numerical Simulations

1. Introduction

In this paper, we consider the absolute value equations (AVE) given by

$$Ax - |x| = b, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ (the set of $n\times n$ matrices) is a large sparse matrix, $b \in \mathbb{R}^n$ (the set of n-dimensional column vectors) is a vector, and $|x| = \left(|x_1|,|x_2|,\cdots,|x_n|\right)^{\mathrm{T}} \in \mathbb{R}^n$ represents the vector of component-wise absolute values of $x \in \mathbb{R}^n$. The absolute value is denoted by $|\cdot|$, and the transpose of a matrix or vector is denoted by $(\cdot)^{\mathrm{T}}$. Throughout this paper, these notations will retain their defined meanings. The AVE (1) is a special case of the generalized absolute value equations (GAVE), which has the following form:

$$Ax - B|x| = b, \tag{2}$$

where $B \in \mathbb{R}^{n\times n}$ was introduced by Rohn [1] and further studied in [2] [3]. Both the AVE (1) and the GAVE (2) are closely related to various mathematical programming problems, such as the linear complementarity problem (LCP) [3]-[6] and the generalized linear complementarity problem (GLCP) [3]. A notable reformulation of the AVE (1) arises when 1 is not an eigenvalue of A. In such cases, the AVE (1) can be transformed into an LCP [3]. Specifically, we seek a vector $z \in \mathbb{R}^n$ such that

$$z \geq 0,\ (A+I)(A-I)^{-1} z + q \geq 0,\ \left\langle z, (A+I)(A-I)^{-1} z + q \right\rangle = 0 \tag{3}$$

with

$$q = \left((A+I)(A-I)^{-1} - I\right) b,\ z = (A-I)x - b.$$

If $z \in \mathbb{R}^n$ is a solution of the LCP [3], then $x^* = (A-I)^{-1}\left(z^* + b\right)$ is a solution of the AVE (1). Moreover, the AVE (1) can be reformulated as a generalized linear complementarity problem (GLCP) [3]. In this reformulation, we seek a vector $x \in \mathbb{R}^n$ such that

$$Q(x) = Ax + x - b \geq 0,\ F(x) = Ax - x - b \geq 0,\ \left\langle Q(x), F(x) \right\rangle = 0. \tag{4}$$

Over the past decades, the AVE [1] [3] has attracted significant attention from researchers [2] [3] [5] [6]. Some have concentrated on identifying sufficient or necessary and sufficient conditions to ensure the existence and uniqueness of solutions to the AVE. For a deeper exploration of this topic, refer to the works in [3] [7]-[9]. However, solving the AVE (1) remains a challenge due to the presence of absolute values in x. Assuming a unique solution does exist, various efficient iterative methods have been proposed to solve the AVE (1) [9]-[12].

In contrast to these discrete-time iterative methods, neurodynamic approaches [13]-[15] offer a significant advantage as they allow real-time computation of solution vectors. Extensive research on neurodynamic approaches for solving the AVE (1) has emerged in recent decades [16]-[23]. For example, based on the equivalence between AVE and LCP (3), and the fact that LCP (3) and the projection equation share the same solution, a projection neurodynamic model for solving LCP (3) was proposed by Huang *et al.* [16]; another projection neurodynamic model with asymptotic stability for LCP (3) was also proposed by Mansoori and Erfanian [17]; a double-projection neurodynamic model with asymptotic stability was further proposed in [18]; and a novel projection neurodynamic model with fixed-time convergence for solving the LCP (3) was designed by Ju [19]. Notably, all the aforementioned projection neurodynamic approaches [16]-[19] for AVE are inverse-involved, which may incur high computational costs. Similarly, by leveraging the equivalence between AVE (1) and GLCP (4) alongside the nonsmooth projection equation, a residual equation was introduced, and an inverse-free neurodynamic algorithm for solving the AVE (1) was presented by Chen [20]; three accelerated inverse-free neurodynamic models were further developed in [21]. Moreover, a novel inverse-free dynamical model with fixed-time convergence for solving the AVE (1) was established by Li *et al.* [22], and more recently, a unified single-layer inverse-free neurodynamic model, also with fixed-time convergence, was proposed by [23]. Ultimately, the inverse-free neurodynamic approaches [20]-[23] have lower computational costs compared to projection neurodynamic approaches [16]-[19], and they can solve the AVE (1) directly.

The remainder of this paper is organized as follows. In Section 2, we introduce the notations and present fundamental results related to the AVE and autonomous systems. Section 3 introduces the design of the novel inverse-free neurodynamic approach (NIFNA) and presents two convergence theorems for the proposed approach. In Section 4, we provide numerical simulations to demonstrate the feasibility and effectiveness of the proposed method. Finally, Section 5 offers concluding remarks.

2. Preliminaries

In the following, for $x \in \mathbb{R}^n$, the 2-norm of x is written as $\|x\|$, and for $A \in \mathbb{R}^{n\times n}$, the spectral norm of A is denoted by $\|A\|$.

Lemma 2.1. ([3]) *Assume that* $A \in \mathbb{R}^{n\times n}$ *is invertible. If* $\|A^{-1}\| < 1$, *then AVE* (1) *has a unique solution for any* $b \in \mathbb{R}^n$.

Lemma 2.2. ([1]) *Suppose that* $\|A^{-1}\| < 1$. *Then,* $A + I$ *is nonsingular.*

Consider the autonomous system

$$\dot{x}(t) = f(x(t)),\ x(0) = x_0 \in \mathbb{R}^n, \tag{5}$$

where f is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$. The solution of system (5) will be represented as $x(t;x_0)$, which was determined by the initial value condition $x(0) = x_0$.

Definition 2.1. ([24]) *Let* $x^*(t) \in \mathbb{R}^n$, *then it is called an equilibrium point of the system* (5) *if* $f(x^*(t)) = 0$.

Lemma 2.3. ([24]) *Let* $x^* \in \mathbb{R}^n$ *be the equilibrium point of the system* (5). *Let* $V : \mathbb{R}^n \to \mathbb{R}$ *be a continuously differentiable function such that*

$$V(x) = 0 \ \text{ and } \ V(x) > 0,\ \forall x \neq x^*;$$

$$\dot{V}(x) < 0;$$

$$\|x - x^*\| \to \infty \Rightarrow V(x) \to \infty,$$

then x^* *is globally asymptotically stable.*

Lemma 2.4. ([25]) *Let* $x^* \in \mathbb{R}^n$ *be the equilibrium point of the system* (5). *If there exists a radially unbounded continuous function* $V : \mathbb{R}^n \to \mathbb{R}_+$ *such that*

(i) $V(x) = 0 \Leftrightarrow x = x^*$;

(ii) *any solution* $x(t;x_0)$ *of the system* (5) *satisfies*

$$\dot{V}(x(t;x_0)) \le -\alpha V(x(t;x_0))^{k_1} - \beta V(x(t;x_0))^{k_2}$$

for some $\alpha > 0, \beta > 0, 0 < k_1 < 1, k_2 > 1$. *Then the equilibrium point* x^* *of system* (5) *is globally fixed-time stable, and the upper bound of settling time is*

$$T_{max} = \frac{1}{\alpha(1-k_1)} + \frac{1}{\beta(k_2-1)}.$$

Lemma 2.5. *Assume that* $\|A^{-1}\| < 1$ *and* $E(x) = Ax - |x| - b$. *If* x^* *is a solution of the AVE* (1) *then*

$$(x - x^*)^{\mathrm{T}}(A+I)^{\mathrm{T}} E(x) \ge \|E(x)\|^2, \forall x \in \mathbb{R}^n.$$

Proof. As mentioned earlier, x^* is also a solution of the GLCP (4) with the set $\Omega = \{x \in \mathbb{R}^n \mid x \geq 0\}$. Since Ω is a closed convex set and $Q(x^*) \in \Omega$, it follows from the properties of projection mappings that

$$\left[v - P_\Omega(v)\right]^{\mathrm{T}} \left[P_\Omega(v) - u\right] \geq 0, \ \forall v \in \mathbb{R}^n, \ u \in \Omega.$$

By setting $v := Q(x) - F(x)$ and $u := Q(x^*) \geq 0$, we obtain

$$\left[E(x) - F(x)\right]^{\mathrm{T}} \left[P_\Omega\left(Q(x) - F(x)\right) - Q(x^*)\right] \geq 0.$$

Moreover, since $Q(x) \geq 0$, it follows that

$$Q(x)\left[E(x) - F(x)\right]^{\mathrm{T}} \left[P_\Omega\left(Q(x) - F(x)\right) - Q(x^*)\right] \geq 0.$$

Additionally, using the fact that $Q(x)^{\mathrm{T}} F(x) = 0$ and the identity

$$P_\Omega\left(Q(x) - F(x)\right) - Q(x^*) = \left[Q(x) - Q(x^*)\right] - E(x),$$

we obtain

$$\left[Q(x) - Q(x^*)\right]^{\mathrm{T}} E(x) \geq \|E(x)\|^2.$$

Finally, from the definitions of $Q(x)$ and $F(x)$ in GLCP (4), we conclude that

$$\left(x - x^*\right)^{\mathrm{T}} (A + I)^{\mathrm{T}} E(x) \geq \|E(x)\|^2$$

Lemma 2.6. ([20]) *Assume that* $\|A^{-1}\| < 1$, *then the* AVE (1) *has a unique solution, say* x^*, *and*

$$\frac{1}{L_1 + L_2}\|E(x)\| \leq \|x - x^*\| \leq \frac{L_1 + L_2}{\mu}\|E(x)\|, \ \forall x \in \mathbb{R}^n,$$

where $L_1 = \|A + I\|$ *and* $L_2 = \|A - I\|$ *are Lipschitz constants of the functions* $Q(x)$ *and* $F(x)$ *defined as in GLCP* (4), *respectively, and* $0 < \mu = \dfrac{1}{\|A^{-1}\|^2} - 1$.

3. The Novel Inverse-Free Dynamical Approach and Convergence Analysis

In this section, inspired by the works of [20] [22], we introduce a novel inverse-free neurodynamic approach (NIFNA) designed to enhance numerical stability and address the challenges posed by the ill-conditioning of A. The proposed model is governed by the following differential equation:

$$\dot{x}(t) = -\gamma\rho(x)(A + I)^{\mathrm{T}}\left(Ax - |x| - b\right), \tag{6}$$

with

$$\rho(x) = \begin{cases} \dfrac{\rho_1}{\|E(x)\|^{1-\lambda_1}} + \dfrac{\rho_2}{\|E(x)\|^{1-\lambda_2}}, & \text{if } x \notin \mathrm{Fix}(E), \\ 0, & \text{if } x \in \mathrm{Fix}(E), \end{cases}$$

where $\gamma, \rho_1, \rho_2 > 0$, $0 < \lambda_1 < 1$, $\lambda_2 > 1$, $E(x) = Ax - |x| - b$ and

$\text{Fix}(E) := \{x \in \mathbb{R}^n : E(x) = 0\}$. Notably, the proposed NIFNA modifies the approach in ([22], (3.1)) by replacing A^{T} with $(A+I)^{\text{T}}$.

Theorem 3.1. *Let* $A \in \mathbb{R}^{n\times n}$ *satisfy Lemma* 2.2. *Then* x^* *is an equilibrium of the NIFNA* (6) *if and only if it solves the AVE* (1).

Proof. If x^* is an equilibrium of the NIFNA (6), then

$$\rho(x)(A+I)^{\text{T}} E(x) = 0.$$

Since $A+I$ is invertible by Lemma 2.2, the above equation implies that $\rho(x^*) = 0$ or $E(x^*) = 0$. In both cases, we have

$$Ax^* - |x^*| - b = 0,$$

which shows that x^* solves the AVE (1).

Next, we analyze the convergence of the NIFNA (6). Theorem 3.2 establishes global convergence to the equilibrium point x^* from any initial condition. Furthermore, Theorem 3.3 proves fixed-time convergence, ensuring the NIFNA (6) reaches equilibrium within a predetermined time, regardless of the initial state. This guarantees both reliability and efficiency in solving the AVE.

Theorem 3.2. *Let* $A \in \mathbb{R}^{n\times n}$ *satisfy Lemma* 2.1. *Then, the state vector* $x(t)$ *of the NIFNA* (6) *starting from any initial state* $x(0)$ *globally converges to the equilibrium point* $x^*(t)$ *of the system* (5).

Proof. Consider the following Lyapunov function:

$$v(x) = \text{e}^{\|x-x^*\|^2} - 1, x \in \mathbb{R}^n.$$

It is clear that $v(x^*) = 0$ and $v(x) > 0$ for all $x \neq x^*$. Additionally, by Lemma 2.5, we have

$$\begin{aligned}
\dot{v}(x) &= \left(\frac{\text{d}v}{\text{d}x}\right)^{\text{T}} \frac{\text{d}x}{\text{d}t} \\
&= -2\gamma \text{e}^{\|x-x^*\|^2} (x-x^*)^{\text{T}} (A+I)^{\text{T}} E(x) \\
&\leq -2\gamma \text{e}^{\|x-x^*\|^2} \|E(x)\|^2 \\
&< 0, \ \forall x \neq x^*.
\end{aligned}$$

This shows that $\dot{v}(x)$ is negative definite, *i.e.*, $\dot{v}(t) < 0$ for any $x \neq x^*$, and $\dot{v}(t) = 0$ for $x = x^*$, due to matrix $A+I$ being nonsingular and the positive tuning parameter γ.

By Lemma 2.3, it follows that $x \to x^*$ as $t \to \infty$; equivalently, the state $x(t)$ of the NIFNA (6) is globally converges to the equilibrium point x^*.

Theorem 3.3. *Let* $A \in \mathbb{R}^{n\times n}$ *satisfy Lemma* 2.1. *Then, the state vector* $x(t)$ *of the NIFNA* (6) *starting from any initial state* $x(0)$ *converges to the theoretical solution* $x^*(t)$ *within the fixed time*

$$T_{\max}^{I} = \frac{2}{\alpha(1-\lambda_1)} + \frac{2}{\beta(\lambda_2 - 1)}, \tag{7}$$

where

$$\alpha = 2^{\frac{1+\lambda_1}{2}} \gamma\rho_1 \left(\frac{\mu}{L_1+L_2}\right)^{1+\lambda_1}, \ \beta = 2^{\frac{1+\lambda_2}{2}} \gamma\rho_2 \left(\frac{\mu}{L_1+L_2}\right)^{1+\lambda_2}.$$

Proof. Define $\tilde{x}(t) := x(t) - x^*(t)$, where $x^*(t)$ is the equilibrium point of the NIFNA (6). Consider the following Lyapunov function candidate, $v(x) = \frac{1}{2}\|\tilde{x}(t)\|_2^2 \geq 0$. Taking its time derivative, we obtain

$$\begin{aligned}
\dot{v}(t) &= \tilde{x}^{\mathrm{T}}(t)\dot{\tilde{x}}(t) \\
&= -\gamma\left(\frac{\rho_1}{\|E(x)\|^{1-\lambda_1}} + \frac{\rho_2}{\|E(x)\|^{1-\lambda_2}}\right)\tilde{x}^{\mathrm{T}}(t)(A+I)^{\mathrm{T}} E(x) \\
&= -\gamma\rho_1 \frac{\tilde{x}^{\mathrm{T}}(t)(A+I)^{\mathrm{T}} E(x)}{\|E(x)\|^{1-\lambda_1}} - \gamma\rho_2 \frac{\tilde{x}^{\mathrm{T}}(t)(A+I)^{\mathrm{T}} E(x)}{\|E(x)\|^{1-\lambda_2}} \\
&\leq -\gamma\rho_1 \frac{\|E(x)\|^2}{\|E(x)\|^{1-\lambda_1}} - \gamma\rho_2 \frac{\|E(x)\|^2}{\|E(x)\|^{1-\lambda_1}} \\
&= -\gamma\rho_1 \|E(x)\|^{1+\lambda_1} - \gamma\rho_2 \|E(x)\|^{1+\lambda_2} \\
&\leq -\gamma\rho_1 \left(\frac{\mu}{L_1+L_2}\right)^{1+\lambda_1} \|\tilde{x}(t)\|^{1+\lambda_1} - \gamma\rho_2 \left(\frac{\mu}{L_1+L_2}\right)^{1+\lambda_2} \|\tilde{x}(t)\|^{1+\lambda_2} \\
&= \alpha v(x)^{\frac{1+\lambda_1}{2}} - \beta v(x)^{\frac{1+\lambda_2}{2}}.
\end{aligned}$$

where $\alpha = 2^{\frac{1+\lambda_1}{2}} \gamma\rho_1 \left(\frac{\mu}{L_1+L_2}\right)^{1+\lambda_1} > 0$, $\beta = 2^{\frac{1+\lambda_2}{2}} \gamma\rho_2 \left(\frac{\mu}{L_1+L_2}\right)^{1+\lambda_2} > 0$ and $\gamma, \rho_1, \rho_2, E(x)$ are the same as those in the NIFNA (6). The first inequality holds owing to Lemma 2.5, the second inequality holds owing to Lemma 2.6. Using results from Lemma 2.4, the NIFNA (6) is fixed-time convergence and the upper bound of the settling time is

$$T_{\max}^{I} = \frac{2}{\alpha(1-\lambda_1)} + \frac{2}{\beta(\lambda_2 - 1)}.$$

4. Numerical Simulations

In this section, an example is given to validate the convergence performance of NIFNA (6). The simulations were conducted using MATLAB R2018b on a personal computer with a 1.80GHz (Intel(R)Core(TM)i5-8250U) processor, 8GB memory and Windows 10 operating system. NIFNA (6) is solved by using the build-in function "ode45" in MATLAB.

To validate the results in Theorem 3.1 and Theorem 3.2, we set the parameters in Equation (7) for the NIFNA (6) as follows: $\gamma = 2$, $\rho_1 = \rho_2 = 1$, $\lambda_1 = 0.6$ and $\lambda_2 = 1.4$. Additionally, to facilitate comparison, we ensure that the parameter values in ([22], (3.1)) are consistent with those in Equation (7) of the NIFNA (6).

Example 1. *Consider the matrix* $A \in \mathbb{R}^{n\times n}$, *defined as* $A = M + 2I \in \mathbb{R}^{n\times n}$, *where*

$$M = \text{tridiag}(-I, S, -I) = \begin{bmatrix} S & -I & 0 & \cdots & 0 & 0 \\ -I & S & -I & \cdots & 0 & 0 \\ 0 & -I & S & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & S & -I \\ 0 & 0 & \cdots & \cdots & -I & S \end{bmatrix} \in \mathbb{R}^{n \times n}$$

is a block-tridiagonal matrix, and

$$S = \text{tridiag}(-1.5, 8, 0.5) = \begin{bmatrix} 8 & -0.5 & 0 & \cdots & 0 & 0 \\ -1.5 & 8 & -0.5 & \cdots & 0 & 0 \\ 0 & -1.5 & 8 & \cdots & 0 & 0 \\ \vdots & \vdots & & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \cdots & 8 & -0.5 \\ 0 & 0 & \cdots & \cdots & -1.5 & 8 \end{bmatrix} \in \mathbb{R}^{m \times m}$$

is a tridiagonal matrix, with $n = m^2$ *. The right-hand side vector is given by* $b = Ax^* - |x^*| \in \mathbb{R}^n$ *, where* $x^* = (-1, 1, -1, 1, \cdots, -1, 1) \in \mathbb{R}^n$ *.*

In our computational analysis, we set $n = 100$ and utilized five distinct initial vectors: $\mathbf{1}_{100}$ (a 100 × 1 vector with all elements equal to 1), $\mathbf{-0.5}_{100}$ (a 100 × 1 vector with all elements equal to −0.5), $\mathbf{0}_{100}$ (a 100 × 1 vector with all elements equal to 0), $\mathbf{1.5}_{100}$ (a 100 × 1 vector with all elements equal to 1.5), and $\mathbf{-2}_{100}$ (a 100 × 1 vector with all elements equal to −2), which fall within the range of [−2, 2].

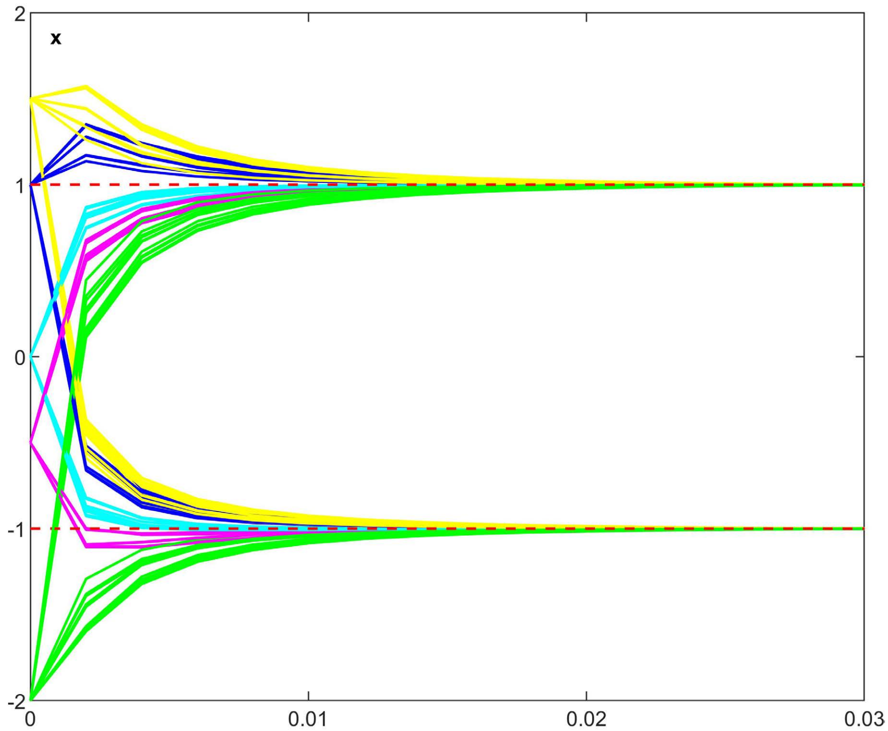

Figure 1. State trajectory of NIFNA (6).

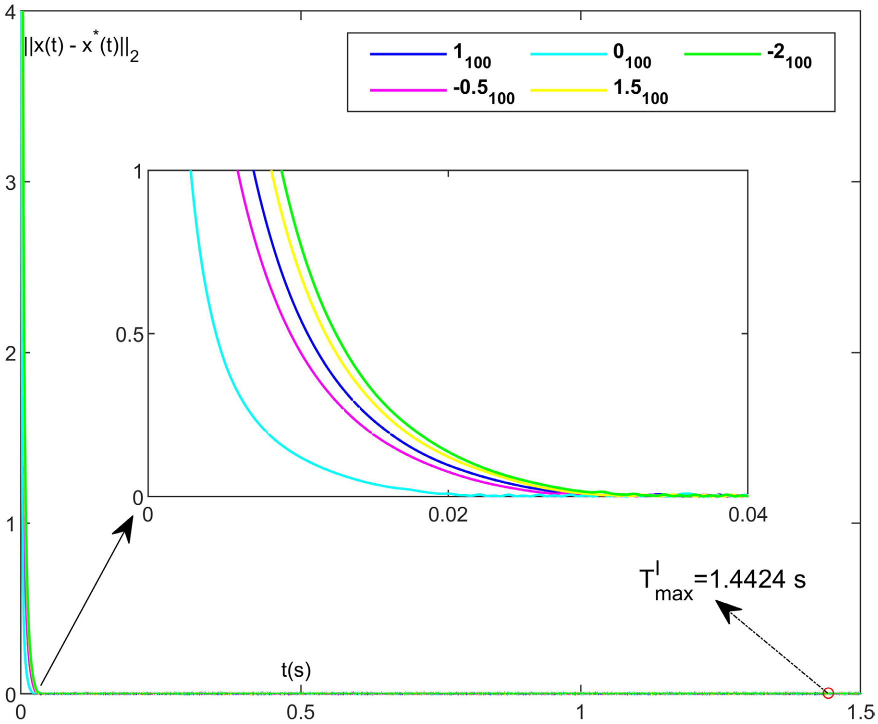

Figure 2. Residual error of NIFNA (6).

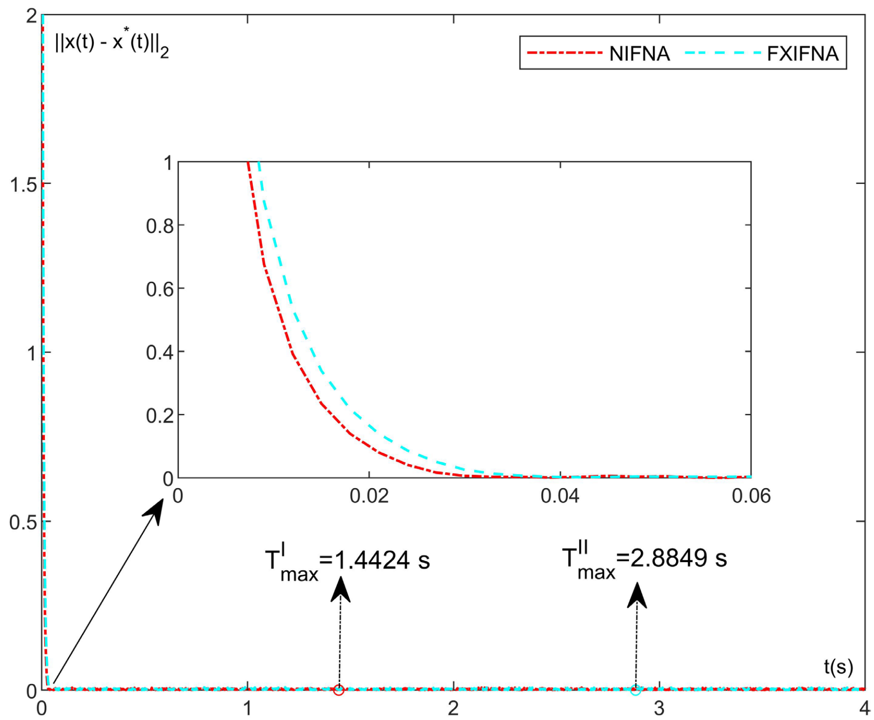

Figure 3. Comparison of residual errors of NIFNA (6) and FXIFNA [22].

The state trajectories $x(t)$ produced by the NIFNA (6) for these five initial vectors are illustrated in **Figure 1**. The figure shows that the online solutions $x(t)$ (solid lines) converge towards the theoretical solution $x^*(t)$ (dotted line) across all initial conditions.

In **Figure 2**, we present the residual error $\|x(t)-x^*(t)\|_2$ for the NIFNA (6) with the different initial vectors. As demonstrated in **Figure 2**, the computational error converges to zero for all five initial vectors considered. Additionally, the upper bound on the fixed convergence time for the NIFNA (6) in Example 1 is $T_{\max}^{I}=1.4424$ seconds.

To further demonstrate the effectiveness of the NIFNA (6), we conducted a performance comparison with the FXIFNA [22], as illustrated in **Figure 3**, using an initial vector of $\mathbf{1}_{100}$. The results indicate that the NIFNA (6) converges more quickly than the FXIFNA [22]. Specifically, the upper bound on the settling time for the NIFNA (6) is $T_{\max}^{I}=1.4424$ seconds, while for the FXIFNA [22], it is $T_{\max}^{II}=2.8849$ seconds.

5. Conclusion

This paper proposes a novel inverse-free neurodynamic approach (NIFNA) to solve the AVE. The NIFNA guarantees global convergence and, importantly, enhances convergence speed by achieving fixed-time convergence. A numerical example demonstrates that the proposed NIFNA can converge to the theoretical solution when solving the online AVE. Comparison results of the numerical simulations show the effectiveness of the proposed model.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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