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Solution of the Matrix Second Semi-Tensor Product Equation $A \circ_l X \circ_l B = C$

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Abstract

In this paper, the solution of the matrix second semi-tensor product equation $A \circ_l X \circ_l B = C$ is studied. Firstly, the solvability of the matrix-vector second semi-tensor product equation is investigated. At the same time, the compatibility conditions, the sufficient and necessary conditions and the specific solution methods for the matrix solution are given. Secondly, we further consider the solvability of the second semi-tensor product equation of the matrix. For each part, several examples are given to illustrate the validity of the results.

Keywords

Matrix Equation, The Second Semi-Tensor Product, Compatibility Condition, Sufficient and Necessary Conditions, Vectorization

1. Introduction

The second semi-tensor product of the matrix is a new matrix multiplication constructed by Professor Cheng by replacing I_k in the matrix semi-tensor product with J_k in 2019. [1] In the same year, Professor Cheng systematically introduced the second semi-tensor product of the matrix. [2] It provides a new way to solve the problem of the control system. For example, in control theory, the cross-dimensional system is a very important dimension-free system. [3] There are many mathematical models that can describe this cross-dimensional system, such as generators, spacecraft, and biological systems. And we know that switching is the classic way to solve the problem of variable dimensional systems. But, the disadvantage is that it neglects the dynamic characteristics of the system in the process of changing dimension. However, the second matrix semi-tensor product can provide a new way to establish a unified morphological model for such switching systems, so as to better discuss the inter-dimensional systems.

The research of matrix equation $AXB = C$ not only has important theoretical significance, but also has wide application in parameter identification, dynamic science, biology, dynamic analysis, nonlinear programming and so on. Mitra studied the solutions of a pair of linear matrix equations $A_1XB_1 = C_1$ and $A_2XB_2 = C_2$. [4] Zhang *et al.* [5] proposed an iterative method to solve the equation $AXB = C$ when X is a symmetric matrix. Ji *et al.* [6] investigated the solution of the matrix equation $AXB = C$ under the semi-tensor product. Liu *et al.* [7] studied cyclic solution and optimal approximation of the quaternion stein equation. L. Chen [8] found a networked evolutionary model of a snow-drift game based on a semi-tensor product.

The second semi-tensor product of the matrix is a new kind of multiplication in recent years, so few people have studied the solvability of the matrix equation $AXB = C$ for the second semi-tensor product. In 2021, Wang Jin [9] studied the solution of the matrix equation $AX = B$ under the second semi-tensor product, and this paper will study the solution of the matrix equation $AXB = C$ combined with the conclusion of the matrix equation $AX = B$.

In this paper, $\mathbb{R}^n$ represents the n-dimensional vector space over the real number field, and $\mathbb{R}^{m\times n}$ represents the linear space of all $m\times n$ matrices over the real number field. $\mathbb{C}^{m\times n}$ represents the linear space of all $m\times n$ matrices over a complex number field. $A=\left(a_{ij}\right)_{m\times n}$ represents the $m\times n$ dimensional matrix with elements a_{ij}. $lcm(m,n)$ is the smallest common multiple of integer m,n, $gcd(m,n)$ is the greatest common divisor of integer m,n. $\text{vec}(A)$ represents the column expansion of matrix A. $\mathbf{1}_k$ represents $k\times k$ dimensional matrix, whose elements are all 1. $[x]$ is the rounding down function. I_n represents the identity matrix of dimensions $n\times n$.

2. Preliminaries

Definition 2.1. [6] Let $A=\left(a_{ij}\right)\in\mathbb{C}^{m\times n}$, $B=\left(b_{ij}\right)\in\mathbb{C}^{n\times r}$, the definition of the Kronecker product of A and B is:

$$A\otimes B=\begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}.$$

Definition 2.2. [6] Let $A=\left(a_{ij}\right)\in\mathbb{C}^{m\times n}$, then

$$\text{vec}(A)=\begin{bmatrix} a_{11} & \cdots & a_{m1} & a_{12} & \cdots & a_{m2} & \cdots & a_{1n} & \cdots & a_{mn} \end{bmatrix}^{\mathrm{T}}.$$

Lemma 2.1. [10] Let $A=\left(a_{ij}\right)\in\mathbb{C}^{m\times n}$, $B=\left(b_{ij}\right)\in\mathbb{C}^{n\times r}$, $C=\left(c_{ij}\right)\in\mathbb{C}^{r\times l}$, there is

$$\text{vec}(ABC)=\left(C^{\mathrm{T}}\otimes A\right)\text{vec}(B).$$

In particular, when A takes I_n, there is

$$\text{vec}(BC)=\left(C^{\mathrm{T}}\otimes I_n\right)\text{vec}(B),$$

when C takes I_r, there is

$$\text{vec}(AB)=\left(I_r\otimes A\right)\text{vec}(B).$$

Definition 2.3. [6] The left (or right) second semi-tensor product of matrix, denoted as $A\circ_l B\left(A\circ_r B\right)$, $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{r\times l}$, is defined as:

$$A\circ_l B=\left(A\otimes J_{t/n}\right)\left(B\otimes J_{t/r}\right)\in\mathbb{C}^{(mt/n)\times(lt/r)}$$

$$\left(A\circ_r B=\left(J_{t/n}\otimes A\right)\left(J_{t/r}\otimes B\right)\in\mathbb{C}^{(mt/n)\times(lt/r)}\right),$$

where $t=lcm\{n,p\}$, $J_k=\frac{1}{k}\mathbf{1}_{k\times k}$ is a $k\times k$ -dimensional matrix.

3. The Solution of Matrix-Vector Second Semi-Tensor Product Equation

In this section, we explore the problem of solving the matrix-vector second semi-tensor product equation of the following form:

$$A\circ_l X\circ_l B=C, \tag{1}$$

where $A\in\mathbb{C}^{m\times n},B\in\mathbb{C}^{r\times l},C\in\mathbb{C}^{h\times k}$ is given, and $X\in\mathbb{R}^p$ is an unknown vector. Here we first study the case of $m=h$, and then consider the general case.

3.1. The Case $m=h$

In this part, the solvability of Equation (1) under $m=h$ is studied. Similar to the proof of the necessary conditions for the dimensionality of matrix semi-tensor product equations, we can obtain the dimension necessary condition for the solution of the matrix-vector second semi-tensor product Equation (1):

Lemma 3.1. [6] If $m=h$, and the matrix-vector second semi-tensor product Equation (1) has a p-dimensional solution vector, then $\frac{k}{l},\frac{n}{r}$ must be positive integers and the Equation (1) satisfy $\frac{k}{l}\Big|\frac{n}{r}, p=\frac{ln}{rk}$.

Remark 3.1. The condition in the lemma is necessary for the matrix-vector second semi-tensor product Equation (1) to have a solution, which is called the compatibility condition for the matrix-vector second semi-tensor product Equation (1). Matrices A, B and C are said to be compatible if they satisfy the conditions.

According to Lemma 3.1, if the matrix-vector second semi-tensor product Equation (1) with $m=h$ has a solution, then $X\in\mathbb{R}^p$, $p=\frac{n}{k}$. Let $t_1=lcm(n,p)$ and by definition we can get:

$$A\circ_l X\circ_l B=\frac{1}{\frac{t_1}{n}}\left(A\otimes\mathbf{1}_{\frac{t_1}{n}}\right)\cdot\frac{1}{\frac{t_1}{p}}\left(X\otimes\mathbf{1}_{\frac{t_1}{p}}\right)\circ_l B$$

$$=A\cdot\frac{p}{n}\left(X\otimes\mathbf{1}_{\frac{n}{p}}\right)\circ_l B=\frac{p}{n}A\cdot\begin{bmatrix}x_1\mathbf{1}_{\frac{n}{p}}\\ x_2\mathbf{1}_{\frac{n}{p}}\\ \vdots\\ x_p\mathbf{1}_{\frac{n}{p}}\end{bmatrix}\circ_l B$$

$$= \frac{p}{n}\begin{bmatrix} a_{1,1} & \cdots & a_{1,\frac{n}{p}} & \vdots & a_{1,\frac{n}{p}+1} & \cdots & a_{1,2\frac{n}{p}} & \vdots & \cdots & \cdots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \cdots \\ a_{m,1} & \cdots & a_{m,\frac{n}{p}} & \vdots & a_{m,\frac{n}{p}+1} & \cdots & a_{m,2\frac{n}{p}} & \vdots & \cdots & \cdots \end{bmatrix.$$

$$\left.\begin{matrix} \cdots & \vdots & a_{1,(p-1)\frac{n}{p}+1} & \cdots & a_{s,n} \\ \cdots & \vdots & \vdots & \ddots & \vdots \\ \cdots & \vdots & a_{m,(p-1)\frac{n}{p}+1} & \cdots & a_{m,n} \end{matrix}\right] \cdot \begin{bmatrix} x_1 \mathbf{1}_{\frac{n}{p}} \\ x_2 \mathbf{1}_{\frac{n}{p}} \\ \vdots \\ x_p \mathbf{1}_{\frac{n}{p}} \end{bmatrix} \circ_l B$$

$$= \frac{p}{n}\left[\dot{A}_1 \vdots \dot{A}_2 \vdots \cdots \vdots \dot{A}_p\right] \cdot \begin{bmatrix} x_1 \mathbf{1}_{\frac{n}{p}} \\ x_2 \mathbf{1}_{\frac{n}{p}} \\ \vdots \\ x_p \mathbf{1}_{\frac{n}{p}} \end{bmatrix} \circ_l B$$

$$= \frac{p}{n}\left(x_1 \dot{A}_1 \cdot \mathbf{1}_{\frac{n}{p}} + x_2 \dot{A}_2 \cdot \mathbf{1}_{\frac{n}{p}} + \cdots + x_p \dot{A}_p \cdot \mathbf{1}_{\frac{n}{p}} \right) \circ_l B$$

$$= \frac{pl}{nk}\left(x_1 \dot{A}_1 \cdot \mathbf{1}_{\frac{n}{p}} \cdot \left(B \otimes \mathbf{1}_{\frac{k}{l}} \right) + x_2 \dot{A}_2 \cdot \mathbf{1}_{\frac{n}{p}} \cdot \left(B \otimes \mathbf{1}_{\frac{k}{l}} \right) + \cdots + x_p \dot{A}_p \cdot \mathbf{1}_{\frac{n}{p}} \cdot \left(B \otimes \mathbf{1}_{\frac{k}{l}} \right) \right),$$

where $\dot{A}_j \in \mathbb{C}^{m \times \frac{n}{p}}, j = 1, 2, \cdots, p$ is p equal partition of matrix A.

Let $\dot{B}_j = \dot{A}_j \cdot \mathbf{1}_{\frac{n}{p}} \cdot \left(B \otimes \mathbf{1}_{\frac{k}{l}} \right)$, we can get:

Lemma 3.2. When $m = h$, the matrix-vector second semi-tensor product Equation (1) can be written as:

$$\frac{pl}{nk}\left(x_l \dot{B}_1 + x_2 \dot{B}_2 + \cdots + x_p \dot{B}_p \right) = C.$$

Theorem 3.1. The matrix-vector second semi-tensor product Equation (1) with $m = h$ has a solution, if and only if $\dot{B}_1, \dot{B}_2, \cdots, \dot{B}_p$ and C are linearly correlated in matrix space $\mathbb{C}^{m \times \frac{n}{p}}$, and when $\dot{B}_1, \dot{B}_2, \cdots, \dot{B}_p$ is linearly independent, the solution is unique.

Similarly, the following corollary can be drawn.

Corollary 3.1. If the matrix-vector second semi-tensor product Equation (1) has a solution when $m = h$, it must satisfy:

$$\operatorname{rank}\left[\dot{B}_1 \vdots \dot{B}_2 \vdots \cdots \vdots \dot{B}_p\right] = \operatorname{rank}\left[\dot{B}_1 \vdots \dot{B}_2 \vdots \cdots \vdots \dot{B}_p \vdots C\right].$$

Take $\ddot{A}_j = \dot{A}_j \mathbf{1}_{\frac{n}{p}}, j = 1, 2, \cdots, p$, and let w_{ij} be the sum of all the elements of row i of $\dot{A}_j$, then

$$\ddot{A}_j=\begin{bmatrix} w_{1j} & w_{1j} & \cdots & w_{1j} \\ w_{2j} & w_{2j} & \cdots & w_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{mj} & \cdots & w_{mj} \end{bmatrix}, j=1,2,\cdots,p.$$

Let v_i be the sum of all the elements in column i of B and

$$B_{ij}=\begin{bmatrix} b_{ij} & \cdots & b_{ij} \\ \vdots & \ddots & \vdots \\ b_{ij} & \cdots & b_{ij} \end{bmatrix},$$

we have

$$\dot{B}_j=\ddot{A}_j\cdot\begin{bmatrix} B_{11} & \cdots & B_{1l} \\ \vdots & \ddots & \vdots \\ B_{r1} & \cdots & B_{rl} \end{bmatrix}$$

$$=\begin{bmatrix} w_{1j}\frac{k}{l}v_1 & w_{1j}\frac{k}{l}v_1 & \cdots & w_{1j}\frac{k}{l}v_1 & w_{1j}\frac{k}{l}v_2 & \cdots & w_{1j}\frac{k}{l}v_l & \cdots & w_{1j}\frac{k}{l}v_l \\ w_{2j}\frac{k}{l}v_1 & w_{2j}\frac{k}{l}v_1 & \cdots & w_{2j}\frac{k}{l}v_1 & w_{2j}\frac{k}{l}v_2 & \cdots & w_{2j}\frac{k}{l}v_l & \cdots & w_{2j}\frac{k}{l}v_l \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ w_{mj}\frac{k}{l}v_1 & w_{mj}\frac{k}{l}v_1 & \cdots & w_{mj}\frac{k}{l}v_1 & w_{mj}\frac{k}{l}v_2 & \cdots & w_{mj}\frac{k}{l}v_l & \cdots & w_{mj}\frac{k}{l}v_l \end{bmatrix}$$

$$=\frac{k}{l}\begin{bmatrix} w_{1j}v_1 & w_{1j}v_1 & \cdots & w_{1j}v_1 & w_{1j}v_2 & \cdots & w_{1j}v_l & \cdots & w_{1j}v_l \\ w_{2j}v_1 & w_{2j}v_1 & \cdots & w_{2j}v_1 & w_{2j}v_2 & \cdots & w_{2j}v_l & \cdots & w_{2j}v_l \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots \\ w_{mj}v_1 & w_{mj}v_1 & \cdots & w_{mj}v_1 & w_{mj}v_2 & \cdots & w_{mj}v_l & \cdots & w_{mj}v_l \end{bmatrix}$$

$$=\frac{k}{l}\begin{bmatrix} w_{1j} \\ \vdots \\ w_{mj} \end{bmatrix}\otimes\begin{bmatrix} v_1 & \cdots & v_1 & v_2 & \cdots & v_2 & \cdots & v_l & \cdots & v_l \end{bmatrix}\in\mathbb{C}^{m\times k},$$

Combining Lemma 3.2, it can be concluded that

$$C=\left[Block_1(C) \quad \cdots \quad Block_l(C)\right],$$

where, for $i=1,2,\cdots,m$, $s=1,2,\cdots,l$, there is

$$Block_s(C)=\begin{bmatrix} c'_{1s} & \cdots & c'_{1s} \\ \vdots & \ddots & \vdots \\ c'_{ms} & \cdots & c'_{ms} \end{bmatrix}, c'_{is}=c_{i,(s-1)\frac{k}{l}+1}.$$

Further

$$\frac{p}{n}\sum_{j=1}^{p}X_j\cdot\begin{bmatrix} w_{1j}v_i \\ w_{2j}v_i \\ \vdots \\ w_{mj}v_i \end{bmatrix}=\begin{bmatrix} c'_{1i} \\ c'_{2i} \\ \vdots \\ c'_{mi} \end{bmatrix}.$$

Let

$$W=\begin{bmatrix} w_{11} & \cdots & w_{1p} \\ \vdots & \ddots & \vdots \\ w_{m1} & \cdots & w_{mp} \end{bmatrix},$$

we have $v_i \frac{p}{n} WX = C_i(C)$, where $C_i(C)=\begin{bmatrix} c'_{1i} \\ c'_{2i} \\ \vdots \\ c'_{mi} \end{bmatrix}$.

Theorem 3.2. If the matrix-vector second half tensor product Equation (1) has a solution at $m=h$, then the matrix $C=(c_{ij}), i=1,2,\cdots,m; j=1,2,\cdots,k$ must have the following form:

$$C=\left[Block_1(C) \quad \cdots \quad Block_l(C)\right],$$

where

$$Block_s(C)=\begin{bmatrix} c'_{1s} & \cdots & c'_{1s} \\ \vdots & \ddots & \vdots \\ c'_{ms} & \cdots & c'_{ms} \end{bmatrix} \in \mathbb{C}^{m\times\frac{k}{l}},$$

and $c'_{is}=c_{i,(s-1)\frac{k}{l}+1}, i=1,2,\cdots,m; s=1,2,\cdots,l$, $c'_{i+1,s}/c'_{is}=v_{i+1}/v_i$.

In addition, we can also get the following theorem and corollary:

Theorem 3.3. The matrix-vector second semi-tensor product Equation (1) for $m=h$ is equivalent to the matrix-vector equation of ordinary multiplication as follows:

$$v_i \frac{p}{n} WX = C_i(C), i=1,2,\cdots,l.$$

Corollary 3.2. If the matrix-vector second semi-tensor product Equation (1) has a solution when $m=h$, the rank should satisfy the following conditions:

$$\text{rank} W = \text{rank}\left[W \vdots C_i(C)\right], i=1,2,\cdots,l.$$

3.2. The General Case

Lemma 3.3. [6] If $m \neq h$ and the matrix-vector second semi-tensor product Equation (1) has a p-dimensional solution vector, then $\frac{h}{m},\frac{k}{l},\frac{rk}{nl}$ must be positive integers and the matrix-vector second semi-tensor product Equation (1) satisfy

$$\beta=\gcd\left(\frac{h}{m},r\right), \gcd\left(\frac{k}{l},\beta\right)=1, \gcd\left(\frac{h}{m},\frac{k}{l}\right)=1, p=\frac{nhl}{mrk}.$$

Next, we assume that the matrix-vector second semi-tensor product Equation (1) always satisfies the compatibility condition. We will find the solution of the matrix-vector second semi-tensor product Equation (1) on $\mathbb{R}^{\frac{nhl}{mrk}}$. By the definition of the second semi-tensor product of a matrix,

$$A \circ_l X \circ_l B = \left(\frac{m}{h}\left(A \otimes \mathbf{1}_{\frac{h}{m}}\right)\right) \circ_l X \circ_l B,$$

If $A' = \frac{m}{h} A \otimes \mathbf{1}_{\frac{h}{m}}$, then the matrix-vector second semi-tensor product Equation (1) is the case of $m = h$, and then the solution of the matrix-vector second semi-tensor product Equation (1) is easily obtained according to the previous conclusion.

Next, we study the conditions for the existence of matrix solutions for $m \neq h$.

Theorem 3.4. If the matrix-vector second semi-tensor product Equation (1) has a solution at $m \neq h$, then the matrix $C = \left(c_{ij}\right), i = 1,2,\cdots,m; j = 1,2,\cdots,k$ must have the following form

$$C = \begin{bmatrix} Block_{11}(C) & \cdots & Block_{1l}(C) \\ \vdots & \ddots & \vdots \\ Block_{m1}(C) & \cdots & Block_{ml}(C) \end{bmatrix},$$

where,

$$Block_{ij}(C) = \begin{bmatrix} c'_{ij} & \cdots & c'_{ij} \\ \vdots & \ddots & \vdots \\ c'_{ij} & \cdots & c'_{ij} \end{bmatrix} \in \mathbb{C}^{\frac{h}{m} \times \frac{k}{l}},$$

and $c'_{ij} = c_{(i-1)\frac{h}{m}+1,(j-1)\frac{k}{l}+1}, i = 1,2,\cdots,m; j = 1,2,\cdots,l$, $c'_{i,j+1} / c'_{ij} = v_{j+1} / v_j$, v_i is the sum of all the elements in column *i* of *B*.

Proof. From Lemma 3.3 we know that the matrix-vector second semi-tensor product Equation (1) has a solution $X \in \mathbb{R}^p$, where $p = \frac{nhl}{mrk}$. Let $n = l_1 \cdot p + l_2$, l_1, l_2 be integers.

For line *i* of *A*, we have:

$$\begin{aligned}
Row_i(A) \circ_l X \circ_l B &= \frac{m}{h}\left(Row_i(A) \otimes \mathbf{1}_{\frac{h}{m}}\right) \cdot \frac{mp}{nh}\left(X \otimes \mathbf{1}_{\frac{nh}{mp}}\right) \circ_l B \\
&= \frac{m^2 p}{nh^2}\begin{bmatrix} a_{i1}\mathbf{1}_{\frac{h}{m}} & a_{i2}\mathbf{1}_{\frac{h}{m}} & \cdots & a_{in}\mathbf{1}_{\frac{h}{m}} \end{bmatrix} \cdot \begin{bmatrix} x_1 \mathbf{1}_{\frac{nh}{mp}} \\ x_2 \mathbf{1}_{\frac{nh}{mp}} \\ \vdots \\ x_p \mathbf{1}_{\frac{nh}{mp}} \end{bmatrix} \circ_l B \\
&= \frac{m^2 p}{nh^2}\left(x_1 \begin{bmatrix} a_{i,1}\mathbf{1}_{\frac{h}{m}} & a_{i,2}\mathbf{1}_{\frac{h}{m}} & \cdots & a_{i,l_1+1}\mathbf{1}_{\frac{h}{m}\times l_2} \end{bmatrix} \cdot \mathbf{1}_{\frac{nh}{mp}} \right. \\
&\quad + x_2 \begin{bmatrix} a_{i,l_1+1}\mathbf{1}_{\frac{h}{m}\times\frac{h}{m}-l_2} & a_{i,l_1+2}\mathbf{1}_{\frac{h}{m}} & \cdots & a_{i,\left[\frac{2n}{p}\right]+1}\mathbf{1}_{\frac{h}{m}\times \mathrm{mod}\left(\frac{2n}{p}\right)} \end{bmatrix} \cdot \mathbf{1}_{\frac{nh}{mp}} \\
&\quad + \cdots + x_{\frac{h}{m}} \begin{bmatrix} a_{i,\frac{nh}{mp}-l_1}\mathbf{1}_{\frac{h}{m}\times l_2} & \cdots & a_{i,\frac{nh}{mp}}\mathbf{1}_{\frac{h}{m}} \end{bmatrix} \cdot \mathbf{1}_{\frac{nh}{mp}} \\
&\quad + x_{\frac{h}{m}+1} \begin{bmatrix} a_{i,\frac{nh}{mp}+1}\mathbf{1}_{\frac{h}{m}} & \cdots & a_{i,\frac{nh}{mp}+l_1}\mathbf{1}_{\frac{h}{m}} & a_{i,\frac{nh}{mp}+l_1+1}\mathbf{1}_{\frac{h}{m}\times l_2} \end{bmatrix} \cdot \mathbf{1}_{\frac{nh}{mp}}
\end{aligned}$$

$$
+\cdots+x_p\begin{bmatrix} a_{i,n-l_1}\mathbf{1}_{\frac{h}{m}\times l_2} & \cdots & a_{i,n}\mathbf{1}_{\frac{h}{m}} \end{bmatrix}\cdot\mathbf{1}_{\frac{nh}{mp}}\Bigg)\circ_l B
$$

$$
=\frac{m^2p}{nh^2}\left(x_1\begin{bmatrix} \frac{h}{m}a_{i,1}+\frac{h}{m}a_{i,2}+\cdots+\frac{h}{m}a_{i,l_1}+l_2a_{i,l_1+1} & \cdots & \frac{h}{m}a_{i,1}+\frac{h}{m}a_{i,2}+\cdots+\frac{h}{m}a_{i,l_1}+l_2a_{i,l_1+1} \\ \vdots & \ddots & \vdots \\ \frac{h}{m}a_{i,1}+\frac{h}{m}a_{i,2}+\cdots+\frac{h}{m}a_{i,l_1}+l_2a_{i,l_1+1} & \cdots & \frac{h}{m}a_{i,1}+\frac{h}{m}a_{i,2}+\cdots+\frac{h}{m}a_{i,l_1}+l_2a_{i,l_1+1} \end{bmatrix}\right.
$$

$$
+x_2\begin{bmatrix} \left(\frac{h}{m}-l_2\right)a_{i,l_1+1}+\frac{h}{m}a_{i,l_1+2}+\cdots+\mathrm{mod}\left(\frac{2n}{p}\right)a_{i,\left[\frac{2n}{p}\right]+1} & \cdots & \left(\frac{h}{m}-l_2\right)a_{i,l_1+1}+\frac{h}{m}a_{i,l_1+2}+\cdots+\mathrm{mod}\left(\frac{2n}{p}\right)a_{i,\left[\frac{2n}{p}\right]+1} \\ \vdots & \ddots & \vdots \\ \left(\frac{h}{m}-l_2\right)a_{i,l_1+1}+\frac{h}{m}a_{i,l_1+2}+\cdots+\mathrm{mod}\left(\frac{2n}{p}\right)a_{i,\left[\frac{2n}{p}\right]+1} & \cdots & \left(\frac{h}{m}-l_2\right)a_{i,l_1+1}+\frac{h}{m}a_{i,l_1+2}+\cdots+\mathrm{mod}\left(\frac{2n}{p}\right)a_{i,\left[\frac{2n}{p}\right]+1} \end{bmatrix}
$$

$$
+\cdots+x_{\frac{h}{m}}\begin{bmatrix} l_2a_{i,\frac{nh}{mp}-l_1}+\frac{h}{m}a_{i,\frac{nh}{mp}-l_1+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}} & \cdots & l_2a_{i,\frac{nh}{mp}-l_1}+\frac{h}{m}a_{i,\frac{nh}{mp}-l_1+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}} \\ \vdots & \ddots & \vdots \\ l_2a_{i,\frac{nh}{mp}-l_1}+\frac{h}{m}a_{i,\frac{nh}{mp}-l_1+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}} & \cdots & l_2a_{i,\frac{nh}{mp}-l_1}+\frac{h}{m}a_{i,\frac{nh}{mp}-l_1+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}} \end{bmatrix}
$$

$$
+x_{\frac{h}{m}+1}\begin{bmatrix} \frac{h}{m}a_{i,\frac{nh}{mp}+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}+l_1}+l_2a_{i,\frac{nh}{mp}+l_1+1} & \cdots & \frac{h}{m}a_{i,\frac{nh}{mp}+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}+l_1}+l_2a_{i,\frac{nh}{mp}+l_1+1} \\ \vdots & \ddots & \vdots \\ \frac{h}{m}a_{i,\frac{nh}{mp}+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}+l_1}+l_2a_{i,\frac{nh}{mp}+l_1+1} & \cdots & \frac{h}{m}a_{i,\frac{nh}{mp}+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}+l_1}+l_2a_{i,\frac{nh}{mp}+l_1+1} \end{bmatrix}
$$

$$+\cdots+x_p\left[\begin{matrix} l_2a_{i,n-l_1}+\frac{h}{m}a_{i,n-l_1+1}+\cdots+\frac{h}{m}a_{i,n} & \cdots & l_2a_{i,n-l_1}+\frac{h}{m}a_{i,n-l_1+1}+\cdots+\frac{h}{m}a_{i,n} \\ \vdots & \ddots & \vdots \\ l_2a_{i,n-l_1}+\frac{h}{m}a_{i,n-l_1+1}+\cdots+\frac{h}{m}a_{i,n} & \cdots & l_2a_{i,n-l_1}+\frac{h}{m}a_{i,n-l_1+1}+\cdots+\frac{h}{m}a_{i,n}\end{matrix}\right]\Bigg)\circ_l B$$

$$\begin{aligned}
&=\frac{m^2p}{nh^2}\Bigg[x_1\left(\frac{h}{m}a_{i,1}+\frac{h}{m}a_{i,2}+\cdots+\frac{h}{m}a_{i,l_1}+l_2a_{i,l_1+1}\right)\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\\
&\quad+x_2\left(\left(\frac{h}{m}-l_2\right)a_{i,l_1+1}+\frac{h}{m}a_{i,l_1+2}+\cdots+\operatorname{mod}\left(\frac{2n}{p}\right)a_{i,\left[\frac{2n}{p}\right]+1}\right)\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\\
&\quad+\cdots+x_{\frac{h}{m}}\left(l_2a_{i,\frac{nh}{mp}-l_1}+\frac{h}{m}a_{i,\frac{nh}{mp}-l_1+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}}\right)\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\\
&\quad+x_{\frac{h}{m}+1}\left(\frac{h}{m}a_{i,\frac{nh}{mp}+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}+l_1}+l_2a_{i,\frac{nh}{mp}+l_1+1}\right)\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\\
&\quad+\cdots+x_p\left(l_2a_{i,n-l_1}+\frac{h}{m}a_{i,n-l_1+1}+\cdots+\frac{h}{m}a_{i,n}\right)\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\Bigg]\circ_l B\\
&=\frac{m^2p}{nh^2}\Bigg[x_1\left(\frac{h}{m}a_{i,1}+\frac{h}{m}a_{i,2}+\cdots+\frac{h}{m}a_{i,l_1}+l_2a_{i,l_1+1}\right)\\
&\quad+x_2\left(\left(\frac{h}{m}-l_2\right)a_{i,l_1+1}+\frac{h}{m}a_{i,l_1+2}+\cdots+\operatorname{mod}\left(\frac{2n}{p}\right)a_{i,\left[\frac{2n}{p}\right]+1}\right)\\
&\quad+\cdots+x_{\frac{h}{m}}\left(l_2a_{i,\frac{nh}{mp}-l_1}+\frac{h}{m}a_{i,\frac{nh}{mp}-l_1+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}}\right)\\
&\quad+x_{\frac{h}{m}+1}\left(\frac{h}{m}a_{i,\frac{nh}{mp}+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}+l_1}+l_2a_{i,\frac{nh}{mp}+l_1+1}\right)\\
&\quad+\cdots+x_p\left(l_2a_{i,n-l_1}+\frac{h}{m}a_{i,n-l_1+1}+\cdots+\frac{h}{m}a_{i,n}\right)\Bigg]\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\circ_l B.
\end{aligned}$$

Let

$$\begin{aligned}
w=\frac{m^2p}{nh^2}\Bigg[&x_1\left(\frac{h}{m}a_{i,1}+\frac{h}{m}a_{i,2}+\cdots+\frac{h}{m}a_{i,l_1}+l_2a_{i,l_1+1}\right)\\
&+x_2\left(\left(\frac{h}{m}-l_2\right)a_{i,l_1+1}+\frac{h}{m}a_{i,l_1+2}+\cdots+\operatorname{mod}\left(\frac{2n}{p}\right)a_{i,\left[\frac{2n}{p}\right]+1}\right)\\
&+\cdots+x_{\frac{h}{m}}\left(l_2a_{i,\frac{nh}{mp}-l_1}+\frac{h}{m}a_{i,\frac{nh}{mp}-l_1+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}}\right)
\end{aligned}$$

$$+x_{\frac{h}{m}+1}\left(\frac{h}{m}a_{i,\frac{nh}{mp}+1}+\cdots+\frac{h}{m}a_{i,\frac{nh}{mp}+l_1}+l_2a_{i,\frac{nh}{mp}+l_1+1}\right)$$
$$+\cdots+x_p\left(l_2a_{i,n-l_1}+\frac{h}{m}a_{i,n-l_1+1}+\cdots+\frac{h}{m}a_{i,n}\right)\Bigg],$$

take v_i to be the sum of the *i*-th column elements of *B*, we have

$$Row_i(A)\circ_l X\circ_l B=w\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\circ_l B=w\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\cdot\left(B\otimes\mathbf{1}_{\frac{k}{l}}\right)$$

$$=\begin{bmatrix} w & \cdots & w \\ \vdots & \ddots & \vdots \\ w & \cdots & w \end{bmatrix}\cdot\begin{bmatrix} b_{11} & \cdots & b_{11} & b_{12} & \cdots & b_{12} & \cdots & b_{1l} & \cdots & b_{1l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{11} & \cdots & b_{11} & b_{12} & \cdots & b_{12} & \cdots & b_{1l} & \cdots & b_{1l} \\ b_{21} & \cdots & b_{21} & b_{22} & \cdots & b_{22} & \cdots & b_{2l} & \cdots & b_{2l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{21} & \cdots & b_{21} & b_{22} & \cdots & b_{22} & \cdots & b_{2l} & \cdots & b_{2l} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{r1} & \cdots & b_{r1} & b_{r2} & \cdots & b_{r2} & \cdots & b_{rl} & \cdots & b_{rl} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{r1} & \cdots & b_{r1} & b_{r2} & \cdots & b_{r2} & \cdots & b_{rl} & \cdots & b_{rl} \end{bmatrix}$$

$$=\begin{bmatrix} w\frac{k}{l}v_1 & \cdots & w\frac{k}{l}v_1 & \cdots & w\frac{k}{l}v_l & \cdots & w\frac{k}{l}v_l \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ w\frac{k}{l}v_1 & \cdots & w\frac{k}{l}v_1 & \cdots & w\frac{k}{l}v_l & \cdots & w\frac{k}{l}v_l \end{bmatrix}$$

$$=\left[Block_{i1}(C)\quad Block_{i2}(C)\quad\cdots\quad Block_{il}(C)\right],$$

where

$$Block_{ij}(C)=\begin{bmatrix} c'_{ij} & \cdots & c'_{ij} \\ \vdots & \ddots & \vdots \\ c'_{ij} & \cdots & c'_{ij} \end{bmatrix}\in\mathbb{C}^{\frac{h}{m}\times\frac{k}{l}}.$$

The theorem is proved.

Several examples are given to illustrate the effectiveness of this method.

Example 3.1. Consider the matrix-vector second semi-tensor product equation $A\circ_l X\circ_l B=C$, where A,B,C is as follows: (For convenience, let's say $A\in\mathbb{C}^{m\times n}$, $B\in\mathbb{C}^{r\times l}$, $C\in\mathbb{C}^{h\times k}$, $X\in\mathbb{R}^p$.)

(1)

$$A=\begin{bmatrix} 8 & 3 & 7 & 8 \\ 5 & 8 & 8 & 2 \end{bmatrix}, B=\begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix}, C=\begin{bmatrix} 8 & 2 & 3 \\ 6 & 2 & 3 \end{bmatrix}.$$

Note that $m=h$, and $\frac{n}{k}=\frac{4}{3}$, so the given matrix is incompatible, and according to lemma 3.1, there is no solution to this equation.

(2)

$$A=\begin{bmatrix}1&0&0&1&1&0&0&1\\0&1&1&1&1&1&1&1\\1&1&1&1&0&1&1&1\end{bmatrix},B=\begin{bmatrix}1&3\\1&2\end{bmatrix},C=\begin{bmatrix}1&2&3&2\\1&2&3&1\\1&2&3&3\end{bmatrix}.$$

It is calculated that $m=h,k/l=2,n/r=4,k/l\,|\,n/r$, $p=2$, so the given matrix is compatible, but C does not have a suitable form, and according to Lemma 3.1, there is no solution to this equation.

(3)

$$A=\begin{bmatrix}2&0&0&2\\0&2&2&2\end{bmatrix},B=\begin{bmatrix}1&3\\2&4\end{bmatrix},C=\begin{bmatrix}15&35\\24&56\end{bmatrix}.$$

It is calculated that $m=h,k/l=1,n/r=2,k/l\,|\,n/r$, $p=2$, and the matrix C has a suitable form, so according to Lemma 3.1, this equation may have a solution $X\in\mathbb{R}^2$. Make

$$\dot{A}_1=\begin{bmatrix}2&0\\2&2\end{bmatrix},\dot{A}_2=\begin{bmatrix}0&2\\2&2\end{bmatrix},$$

we have

$$\ddot{A}_1=\begin{bmatrix}2&2\\2&2\end{bmatrix},\ddot{A}_2=\begin{bmatrix}2&2\\4&4\end{bmatrix},W=\begin{bmatrix}2&2\\2&4\end{bmatrix},v_1=3,v_2=7.$$

Method 1: Know by definition

$$A\circ_l X\circ_l B=\frac{1}{2}A\left(X\otimes\mathbf{1}_{2\times 2}\right)\circ_l B.$$

From

$$\frac{1}{2}\begin{bmatrix}2&0&0&2\\0&2&2&2\end{bmatrix}\begin{bmatrix}x_1&x_1\\x_1&x_1\\x_2&x_2\\x_2&x_2\end{bmatrix}$$

$$=\begin{bmatrix}1&0&0&1\\0&1&1&1\end{bmatrix}\begin{bmatrix}x_1&x_1\\x_1&x_1\\x_2&x_2\\x_2&x_2\end{bmatrix}=\begin{bmatrix}x_1+x_2&x_1+x_2\\x_1+2x_2&x_1+2x_2\end{bmatrix}$$

we have

$$A\circ_l X\circ_l B=\begin{bmatrix}x_1+x_2&x_1+x_2\\x_1+2x_2&x_1+2x_2\end{bmatrix}B=C,$$

solving the equation yields

$$X=\begin{bmatrix}2\\3\end{bmatrix}.$$

Method 2: From $v_1\dfrac{p}{n}WX=C_1$ we have

$$\frac{3}{2}\begin{bmatrix}2&2\\2&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}15\\24\end{bmatrix},$$

solving the equation yields

$$X=\begin{bmatrix}2\\3\end{bmatrix}.$$

(4)

$$A=\begin{bmatrix}4&0&0&4&4&0&0&4\\0&4&4&4&4&4&4&4\\4&4&4&4&0&4&4&4\end{bmatrix},B=\begin{bmatrix}1&3\\1&2\end{bmatrix},C=\begin{bmatrix}12&12&30&30\\22&22&55&55\\20&20&50&50\end{bmatrix}.$$

It is calculated that $m=h,k/l=2,n/r=4,\frac{k}{l}\parallel\frac{n}{r},p=2$, and the matrix C has a suitable form, so according to lemma 3.1, this equation may have a solution $X\in\mathbb{R}^2$. Let

$$\dot{A}_1=\begin{bmatrix}4&0&0&4\\0&4&4&4\\4&4&4&4\end{bmatrix},\dot{A}_2=\begin{bmatrix}4&0&0&4\\4&4&4&4\\0&4&4&4\end{bmatrix},$$

we have

$$\ddot{A}_1=\begin{bmatrix}8&8&8&8\\12&12&12&12\\16&16&16&16\end{bmatrix},\ddot{A}_2=\begin{bmatrix}8&8&8&8\\16&16&16&16\\12&12&12&12\end{bmatrix},W=\begin{bmatrix}8&8\\12&16\\16&12\end{bmatrix},$$

$$v_1=2,v_2=5.$$

From $v_1\frac{p}{n}WX=C_1$ we have

$$\frac{1}{2}\begin{bmatrix}8&8\\12&16\\16&12\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}12\\44\\40\end{bmatrix},$$

solving the equation yields

$$X=\begin{bmatrix}1\\2\end{bmatrix}.$$

(5)

$$A=\begin{bmatrix}1&2&1&1\\2&2&3&1\end{bmatrix},B=\begin{bmatrix}2&2&1&1\\2&2&1&1\\2&2&1&1\\2&2&1&1\end{bmatrix},C=\begin{bmatrix}38&38&19&19\\38&38&19&19\\64&64&32&32\\64&64&32&32\end{bmatrix}.$$

where $m\neq h$, According to lemma 3.3, $\frac{h}{m}=2,\frac{k}{l}=1,\frac{rk}{nl}=1,\beta=2$, and C has a suitable form, this equation may have a solution $X\in\mathbb{R}^2$. First, by the definition of the second semi-tensor product of the matrix, we know that $A\circ_l X\circ_l B=\frac{1}{2}(A\otimes I_2)\circ_l X\circ_l B$, so take

$$A' = A \otimes I_2 = \begin{bmatrix} 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 2 & 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 & 3 & 3 & 1 & 1 \\ 2 & 2 & 2 & 2 & 3 & 3 & 1 & 1 \end{bmatrix},$$

let

$$\dot{A}_1 = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 2 & 2 \\ 2 & 2 & 2 & 2 \end{bmatrix}, \dot{A}_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 3 & 3 & 1 & 1 \\ 3 & 3 & 1 & 1 \end{bmatrix},$$

we have

$$\ddot{A}_1 = \begin{bmatrix} 6 & 6 & 6 & 6 \\ 6 & 6 & 6 & 6 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix}, \ddot{A}_2 = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 4 & 4 & 4 & 4 \\ 8 & 8 & 8 & 8 \\ 8 & 8 & 8 & 8 \end{bmatrix}, W = \begin{bmatrix} 6 & 4 \\ 6 & 4 \\ 8 & 8 \\ 8 & 8 \end{bmatrix},$$

$$v_1 = 8, v_2 = 8, v_3 = 4, v_4 = 4.$$

From $v_1 \frac{p}{2n} WX = C_1$ we have

$$\begin{bmatrix} 6 & 4 \\ 6 & 4 \\ 8 & 8 \\ 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 38 \\ 38 \\ 64 \\ 64 \end{bmatrix},$$

further we have

$$X = \begin{bmatrix} 3 \\ 5 \end{bmatrix}.$$

4. The Solution of Matrix Equation

Now, we will study the solvability of the second semi-tensor product equation of the matrix

$$A \circ_l X \circ_l B = C, \tag{2}$$

where $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{r \times l}, C \in \mathbb{C}^{h \times k}$ is known, $X \in \mathbb{C}^{p \times q}$ is an unknown matrix.

4.1. The Situation of $m = h$

Lemma 4.1. [6] If $m = h$, and the matrix second semi-tensor product Equation (2) has a solution vector of $p \times q$ dimension, then $\frac{k}{l}$ must be a positive integer and satisfy

$$p = \frac{n}{\alpha}, q = \frac{rk}{l\alpha},$$

where α is any common divisor of n and $\frac{rk}{l}$.

Theorem 4.1. When $m = h$, then the matrix second semi-tensor product Equation (2) is equivalent to

$$\frac{l}{\alpha k}\left(\left(B^{\mathrm{T}} \otimes \mathbf{1}_{\frac{k}{l}}\right) \otimes I_m\right)\left(I_q \otimes A'\right)\operatorname{vec}(X) = \operatorname{vec}(C), \tag{3}$$

where

$$A' = \left[\operatorname{vec}\left(\ddot{A}_1\right), \operatorname{vec}\left(\ddot{A}_2\right), \cdots, \operatorname{vec}\left(\ddot{A}_p\right)\right] = \begin{bmatrix} W_1 & W_2 & \cdots & W_p \\ W_1 & W_2 & \cdots & W_p \\ \vdots & \vdots & \ddots & \vdots \\ W_1 & W_2 & \cdots & W_p \end{bmatrix}.$$

Let w_{ij} be the sum of all the elements of row i of $\dot{A}_j$, then

$$\ddot{A}_j = \dot{A}_j \mathbf{1}_\alpha = \begin{bmatrix} w_{1j} & w_{1j} & \cdots & w_{1j} \\ w_{2j} & w_{2j} & \cdots & w_{2j} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m1} & w_{mj} & \cdots & w_{mj} \end{bmatrix}, \dot{A}_j \in \mathbb{C}^{m \times \alpha}, j = 1, 2, \cdots, p$$

is a column p of A divided into equal chunks.

Proof. From Lemma 4.1, we know that $p = \frac{n}{\alpha}$, $q = \frac{rk}{l\alpha}$, α is the common factor of n and $\frac{rk}{l}$, and from the definition of the product of the second semi-tensor, we have

$$\begin{aligned} A \circ_l X \circ_l B &= \frac{1}{\alpha}\left[\dot{A}_1 \vdots \dot{A}_2 \vdots \cdots \vdots \dot{A}_p\right] \cdot \begin{bmatrix} x_{11}\mathbf{1}_\alpha & x_{12}\mathbf{1}_\alpha & \cdots & x_{1q}\mathbf{1}_\alpha \\ x_{21}\mathbf{1}_\alpha & x_{22}\mathbf{1}_\alpha & \cdots & x_{2q}\mathbf{1}_\alpha \\ \vdots & \vdots & \ddots & \vdots \\ x_{p1}\mathbf{1}_\alpha & x_{p2}\mathbf{1}_\alpha & \cdots & x_{pq}\mathbf{1}_\alpha \end{bmatrix} \circ_l B \\ &= \frac{1}{\alpha}\left[\dot{A}_1\mathbf{1}_\alpha \vdots \dot{A}_2\mathbf{1}_\alpha \vdots \cdots \vdots \dot{A}_p\mathbf{1}_\alpha\right]\left[X_1, X_2, \cdots, X_p\right] \circ_l B \\ &= \frac{1}{\alpha}\left[\ddot{A}_1, \ddot{A}_2, \cdots, \ddot{A}_p\right]\left[X_1, X_2, \cdots, X_q\right] \circ_l B \\ &= \frac{l}{\alpha k} A \cdot \left(X \otimes \mathbf{1}_\alpha\right)\left(B \otimes \mathbf{1}_{\frac{k}{l}}\right), \end{aligned} \tag{4}$$

where $\dot{A}_i \in \mathbb{C}^{m \times \alpha}, i = 1, 2, \cdots, p$ is the column p equal block of the matrix A, taking $D = A \cdot \left(X \otimes \mathbf{1}_\alpha\right)$. Using the operator $\operatorname{vec}(\cdot)$ we have

$$\begin{aligned} &\left[\operatorname{vec}\left(\ddot{A}_1\right), \operatorname{vec}\left(\ddot{A}_2\right), \cdots, \operatorname{vec}\left(\ddot{A}_p\right)\right]\left[X_1, X_2, \cdots, X_p\right] \\ &= \left[\operatorname{vec}\left(\dot{D}_1\right), \operatorname{vec}\left(\dot{D}_2\right), \cdots, \operatorname{vec}\left(\dot{D}_q\right)\right] \in \mathbb{C}^{m\alpha \times q}, \end{aligned}$$

$\dot{D}_i \in \mathbb{C}^{m \times \alpha}, i = 1, 2, \cdots, q$ is the columnar fourth block of the matrix D. Further

$$\left(I_q \otimes A'\right)\operatorname{vec}(X) = \operatorname{vec}(D)$$

can be obtained. Therefore, using the operator $\operatorname{vec}(\cdot)$ for the last expression of the Equation (4), we get

$$\frac{l}{\alpha k}\mathrm{vec}\left(A\left(X\otimes \mathbf{1}_\alpha\right)\left(B\otimes \mathbf{1}_{\frac{k}{l}}\right)\right)=\frac{l}{\alpha k}\left(B^{\mathrm{T}}\otimes \mathbf{1}_{\frac{k}{l}}\right)\mathrm{vec}(D)$$
$$=\frac{l}{\alpha k}\left(\left(B^{\mathrm{T}}\otimes \mathbf{1}_{\frac{k}{l}}\right)\otimes I_m\right)\left(I_q\otimes A'\right)\mathrm{vec}(X)$$
$$=\mathrm{vec}(C).$$

4.2. General Situation

Lemma 4.2. [6] If $m\neq h$, and the matrix second semi-tensor product Equation (2) has a solution vector of $p\times q$ dimension, then $\frac{h}{m},\frac{k}{l}$ must be a positive integer and satisfy

$$\gcd\left(\frac{h}{m\beta},\frac{\alpha}{\beta}\right)=1,\gcd\left(\frac{h}{m},\frac{k}{l}\right)=1,\beta\,|\,r,p=\frac{nh}{m\alpha},q=\frac{rk}{l\alpha},$$

where α is any common divisor of $\frac{nh}{m}$ and $\frac{rk}{l}$, $\beta=\gcd\left(\frac{h}{m},\alpha\right)$.

Similarly, if the compatibility condition is satisfied, just let $A'=\frac{m}{h}\left(A\otimes \mathbf{1}_{\frac{h}{m}}\right)$, then the matrix second semi-tensor product Equation (2) is in the form of $m=h$.

Theorem 4.2. If the matrix second semi-tensor product Equation (2) has a solution when $m\neq h$, then the matrix $C=\left(c_{ij}\right),i=1,2,\cdots,m;j=1,2,\cdots,k$ must have the following form:

$$C=\begin{bmatrix} Block_{11}(C) & \cdots & Block_{1l}(C)\\ \vdots & \ddots & \vdots \\ Block_{m1}(C) & \cdots & Block_{ml}(C)\end{bmatrix},$$

where,

$$Block_{ij}(C)=\begin{bmatrix} c'_{ij} & \cdots & c'_{ij}\\ \vdots & \ddots & \vdots\\ c'_{ij} & \cdots & c'_{ij}\end{bmatrix}\in\mathbb{C}^{\frac{h}{m}\times\frac{k}{l}},$$

and $c'_{ij}=c_{(i-1)\frac{h}{m}+1,(j-1)\frac{k}{l}+1},i=1,2,\cdots,m;j=1,2,\cdots,l$.

Proof. A proof similar to theorem 0.4 can be obtained

$$Row_i(A)\circ_l X\circ_l B=\left[w_1,w_2,\cdots,w_q\right]\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\circ_l B$$
$$=\left[w_1,w_2,\cdots,w_q\right]\cdot\mathbf{1}_{\frac{h}{m}\times\frac{nh}{mp}}\cdot\left(B\otimes\mathbf{1}_{\frac{k}{l}}\right)$$
$$=\begin{bmatrix} w_1 & \cdots & w_1 & \cdots & w_q & \cdots & w_q\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \\ w_1 & \cdots & w_1 & \cdots & w_q & \cdots & w_q\end{bmatrix}$$

$$\cdot \begin{bmatrix} b_{11} & \cdots & b_{11} & \cdots & b_{1l} & \cdots & b_{1l} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{11} & \cdots & b_{11} & \cdots & b_{1l} & \cdots & b_{1l} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{r1} & \cdots & b_{r1} & \cdots & b_{rl} & \cdots & b_{rl} \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ b_{r1} & \cdots & b_{r1} & \cdots & b_{rl} & \cdots & b_{rl} \end{bmatrix}$$

$$= \left[Block_{i1}(C) \quad Block_{i2}(C) \quad \cdots \quad Block_{il}(C) \right],$$

where

$$Block_{ij}(C) = \begin{bmatrix} c'_{ij} & \cdots & c'_{ij} \\ \vdots & \ddots & \vdots \\ c'_{ij} & \cdots & c'_{ij} \end{bmatrix} \in \mathbb{C}^{\frac{h}{m} \times \frac{k}{l}},$$

$$\begin{aligned} w_j = \frac{m^2 p}{nh^2} \Bigg[& x_{1,j} \left(\frac{h}{m} a_{i,1} + \frac{h}{m} a_{i,2} + \cdots + \frac{h}{m} a_{i,l_1} + l_2 a_{i,l_1+1} \right) \\ & + x_{2,j} \left(\left(\frac{h}{m} - l_2 \right) a_{i,l_1+1} + \frac{h}{m} a_{i,l_1+2} + \cdots + \operatorname{mod}\left(\frac{2n}{p} \right) a_{i,\left[\frac{2n}{p}\right]+1} \right) \\ & + \cdots + x_{\frac{h}{m},j} \left(l_2 a_{i,\frac{nh}{mp}-l_1} + \frac{h}{m} a_{i,\frac{nh}{mp}-l_1+1} + \cdots + \frac{h}{m} a_{i,\frac{nh}{mp}} \right) \\ & + x_{\frac{h}{m}+1,j} \left(\frac{h}{m} a_{i,\frac{nh}{mp}+1} + \cdots + \frac{h}{m} a_{i,\frac{nh}{mp}+l_1} + l_2 a_{i,\frac{nh}{mp}+l_1+1} \right) \\ & + \cdots + x_{p,j} \left(l_2 a_{i,n-l_1} + \frac{h}{m} a_{i,n-l_1+1} + \cdots + \frac{h}{m} a_{i,n} \right) \Bigg]. \end{aligned}$$

The theorem was proven.

Now we give the concrete steps to solve the matrix second semi-tensor product Equation (2):

Step 1. First, we examine whether the matrix second semi-tensor product Equation (2) satisfies the compatibility condition, that is,

$$\gcd\left(\frac{h}{m\beta}, \frac{\alpha}{\beta} \right) = 1, \gcd\left(\frac{h}{m}, \frac{k}{l} \right) = 1, \beta \mid r, p = \frac{nh}{m\alpha}, q = \frac{rk}{l\alpha},$$

and C satisfies

$$C = \begin{bmatrix} Block_{11}(C) & \cdots & Block_{1l}(C) \\ \vdots & \ddots & \vdots \\ Block_{m1}(C) & \cdots & Block_{ml}(C) \end{bmatrix},$$

where,

$$Block_{ij}(C) = \begin{bmatrix} c'_{ij} & \cdots & c'_{ij} \\ \vdots & \ddots & \vdots \\ c'_{ij} & \cdots & c'_{ij} \end{bmatrix} \in \mathbb{C}^{\frac{h}{m} \times \frac{k}{l}},$$

and $c'_{ij} = c_{(i-1)\frac{h}{m}+1,(j-1)\frac{k}{l}+1}, i = 1, 2, \cdots, m; j = 1, 2, \cdots, l$.

Step 2. Find all allowable dimensions $p \times q$ that satisfy Lemma 4.2.

Step 3. Let $A' = \frac{m}{h}\left(A \otimes \mathbf{1}_{\frac{h}{m}}\right)$.

Step 4. Find $B' = \frac{l}{\alpha k}\left(\left(\left(B^{\mathrm{T}} \otimes \mathbf{1}_{\frac{k}{l}}\right) \otimes I_m\right)\left(I_q \otimes A'\right)\right)$, $\dot{X} = \operatorname{vec}(X)$, $\dot{C} = \operatorname{vec}(C)$ we can get: $B'\dot{X} = \dot{C}$.

Step 5. Solving $p \times q$ equations yields *X*.

Since the solution obtained by this method is definite and more limited, we can try to find an approximate solution for the equation when it does not meet the conditions.

Example 4.1. Consider the matrix second semi-tensor product equation $A \circ_l X \circ_l B = C$, where A, B, C are as follows: (For convenience, let's say $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{r \times l}$, $C \in \mathbb{C}^{h \times k}$, $X \in \mathbb{R}^{p \times q}$.)

(1)

$$A = \begin{bmatrix} 4 & 0 & 0 & 4 \\ 0 & 4 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 10 & 10 & 10 & 10 & 6 & 6 & 4 & 4 \\ 10 & 10 & 10 & 10 & 6 & 6 & 4 & 4 \\ 5 & 5 & 5 & 5 & 3 & 3 & 2 & 2 \end{bmatrix}.$$

There are $m = h, k/l = 2, n = 4, \frac{rk}{l} = 8$, and note that the possible values of α are 1, 2, 4, so the matrix equation may have 1 × 2, 2 × 4, 4 × 8 solutions. It is easier to find the 1 × 2 solution of the equation as $X = \begin{bmatrix} 1 & 1 \end{bmatrix}$. In addition, $\frac{1}{2}(X \otimes \mathbf{1}_2), \frac{1}{4}(X \otimes \mathbf{1}_4)$ are also solutions of matrix equations.

(2) Let

$$C = \begin{bmatrix} 1 & 0 & 9 & 7 & 3 & 6 & 2 & 4 \\ 9 & 3 & 1 & 0 & 1 & 8 & 5 & 4 \\ 4 & 8 & 2 & 5 & 9 & 3 & 4 & 3 \end{bmatrix},$$

The matrices *A* and *B* are the same as in (1), and it is not difficult to verify that the matrices A, B, C are compatible, but the second semi-tensor product equation of the matrix (2) has no solution. It can be seen that the condition of lemma 4.1 is only necessary.

(3)

$$A = \begin{bmatrix} 3 & 1 & 2 & 2 & 3 & 7 \\ 9 & 7 & 4 & 3 & 6 & 8 \\ 3 & 1 & 2 & 2 & 3 & 7 \\ 9 & 7 & 4 & 3 & 6 & 8 \\ 1 & 9 & 7 & 3 & 2 & 8 \\ 7 & 7 & 6 & 8 & 9 & 9 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

$$C=\begin{bmatrix} 66 & 66 & 66 & 30 & 30 & 30 & 48 & 48 & 48 & 84 & 84 & 84 \\ 214 & 214 & 214 & 97 & 97 & 97 & 154 & 154 & 154 & 271 & 271 & 271 \\ 66 & 66 & 66 & 30 & 30 & 30 & 48 & 48 & 48 & 84 & 84 & 84 \\ 214 & 214 & 214 & 97 & 97 & 97 & 154 & 154 & 154 & 271 & 271 & 271 \\ 179 & 179 & 179 & 81 & 81 & 81 & 128 & 128 & 128 & 226 & 226 & 226 \\ 232 & 232 & 232 & 106 & 106 & 106 & 172 & 172 & 172 & 298 & 298 & 298 \end{bmatrix}.$$

Through calculation, we can get that the allowable dimensions of the solution are 2×3, 6×9, and

$$X_1=\begin{bmatrix} 6 & 9 & 3 \\ 3 & 0 & 3 \end{bmatrix}, X_2=\frac{1}{3}\left(X_1 \otimes \mathbf{1}_3\right).$$

(4)

$$A=\begin{bmatrix} 4 & 4 \\ 8 & 4 \end{bmatrix}, B=\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{bmatrix},$$

$$C=\begin{bmatrix} 42 & 42 & 42 & 34 & 34 & 34 & 34 & 34 & 34 & 42 & 42 & 42 \\ 42 & 42 & 42 & 34 & 34 & 34 & 34 & 34 & 34 & 42 & 42 & 42 \\ 42 & 42 & 42 & 34 & 34 & 34 & 34 & 34 & 34 & 42 & 42 & 42 \\ 42 & 42 & 42 & 34 & 34 & 34 & 34 & 34 & 34 & 42 & 42 & 42 \\ 66 & 66 & 66 & 53 & 53 & 53 & 53 & 53 & 53 & 66 & 66 & 66 \\ 66 & 66 & 66 & 53 & 53 & 53 & 53 & 53 & 53 & 66 & 66 & 66 \\ 66 & 66 & 66 & 53 & 53 & 53 & 53 & 53 & 53 & 66 & 66 & 66 \\ 66 & 66 & 66 & 53 & 53 & 53 & 53 & 53 & 53 & 66 & 66 & 66 \end{bmatrix}.$$

Through calculation, we can get that the allowable dimensions of the solution are 2×3, 4×6, 8×12, and

$$X_1=\begin{bmatrix} 3 & 6 & 3 \\ 3 & 3 & 3 \end{bmatrix}, X_2=\frac{1}{2}\left(X_1 \otimes \mathbf{1}_2\right), X_3=\frac{1}{4}\left(X_1 \otimes \mathbf{1}_4\right).$$

(5)

$$A=\begin{bmatrix} 4 & 0 \\ 8 & 0 \end{bmatrix}, B=\begin{bmatrix} 1 & 0 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 1 & 1 & 1 \\ 2 & 3 & 2 & 1 \end{bmatrix},$$

$$C=\begin{bmatrix} 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 \\ 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 \\ 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 \\ 6 & 6 & 6 & 5 & 5 & 5 & 5 & 5 & 5 & 6 & 6 & 6 \\ 12 & 12 & 12 & 10 & 10 & 10 & 10 & 10 & 10 & 12 & 12 & 12 \\ 12 & 12 & 12 & 10 & 10 & 10 & 10 & 10 & 10 & 12 & 12 & 12 \\ 12 & 12 & 12 & 10 & 10 & 10 & 10 & 10 & 10 & 12 & 12 & 12 \\ 12 & 12 & 12 & 10 & 10 & 10 & 10 & 10 & 10 & 12 & 12 & 12 \end{bmatrix}.$$

Through calculation, we can get that the allowable dimensions of the solution are 2 × 3, 4 × 6, 8 × 12, and

$$X_1 = \begin{bmatrix} 3 & 6 & 3 \\ k_1 & k_2 & k_3 \end{bmatrix}, k_1, k_2, k_3 \in \mathbb{R}, X_2 = \frac{1}{2}\left(X_1 \otimes \mathbf{1}_2\right), X_3 = \frac{1}{4}\left(X_1 \otimes \mathbf{1}_4\right).$$

5. Conclusions and Suggestions

In this paper, we have investigated the solution of the matrix second semi-tensor product equation $A \circ_l X \circ_l B = C$. The second semi-tensor product of the matrix is a new matrix multiplication constructed by Professor Cheng in 2019. Firstly, the solvability of the matrix-vector second semi-tensor product equation has been considered. At the same time, the compatibility conditions, sufficient and necessary conditions and the specific solution methods have been given. Then, the solvability of the second semi-tensor product equation of matrix has been studied. For each part, several examples are given to illustrate the validity of the results.

We expect the conclusions in this article to be useful and believe that they have broader application prospects in control systems, parameter identification, dynamic science, biology, dynamic analysis, nonlinear programming, and other fields.

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Conflicts of Interest

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