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Traveling Wave Solutions of a SIR Epidemic Model with Spatio-Temporal Delay

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Abstract

In this paper, we studied the traveling wave solutions of a SIR epidemic model with spatial-temporal delay. We proved that this result is determined by the basic reproduction number R_0 and the minimum wave speed c^* of the corresponding ordinary differential equations. The methods used in this paper are primarily the Schauder fixed point theorem and comparison principle. We have proved that when $R_0 > 1$ and $c > c^*$, the model has a non-negative and non-trivial traveling wave solution. However, for $R_0 < 1$ and $c \geq 0$ or $R_0 > 1$ and $0 < c < c^*$, the model does not have a traveling wave solution.

Keywords

Susceptible-Infected-Recovered Epidemic Model, Traveling Wave Solutions, Spatio-Temporal Delay, Schauder Fixed Point Theorem

1. Introduction

In the field of infectious diseases, the study of traveling wave solutions holds significant practical importance. The existence of traveling wave solutions implies that the infectious disease is spreading through space at a constant speed. By analyzing traveling waves in reaction-diffusion models, we can anticipate the conditions that lead to rapid disease outbreaks, enabling us to take timely preventive measures [1]-[3].

In this paper, we consider the traveling waves of the following SIR model with constant external supplies and spatio-temporal delay.

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\Delta S(x,t)+B-\mu_1 S(x,t)-\dfrac{\beta S(x,t)K*I(x,t)}{S(x,t)+K*I(x,t)},\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\Delta I(x,t)+\dfrac{\beta S(x,t)K*I(x,t)}{S(x,t)+K*I(x,t)}-(\mu_2+\gamma)I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\Delta R(x,t)+\gamma I(x,t)-\mu_3 R(x,t),\end{cases}\tag{1.1}$$

where S, I and R denote the sizes of susceptible, infected and removed individuals respectively, $d_i\,(i=1,2,3)$ refers to the spatial diffusion coefficient for each class, B is regarded as the rate of the inflow of newborns into the susceptible population, $\mu_j\,(j=1,2,3)$ represents the death rates of each class, β and γ denote the rates of disease transmission and the recovery rate of the infective individuals, and

$$\begin{aligned}(K*I)(x,t)&=\int_{-\infty}^{t}\int_{\mathbb{R}}K(x-y,t-s)I(y,s)\mathrm{d}y\mathrm{d}s\\ &=\int_{0}^{\infty}\int_{\mathbb{R}}K(y,s)I(x-y,t-s)\mathrm{d}y\mathrm{d}s.\end{aligned}$$

Moreover, the kernel function $K(y,s)$ describes the interaction between the infective and susceptible individuals at location x and time t which occurred at location y and at earlier time $t-s$, see [4]-[6]. Next, we list some assumptions on $K(y,s)$.

(K1) K is non-negative and integrable, and satisfies

$$K(y,s)=K(-y,s),\int_0^\infty\int_{\mathbb{R}}K(y,s)\mathrm{d}y\mathrm{d}s=1,(y,s)\in\mathbb{R}\times[0,\infty).$$

(K2) For each $c>0$, there exists $\lambda_c\le\infty$ such that $\int_0^\infty\int_{\mathbb{R}}K(x,t)\mathrm{e}^{-\lambda(x+ct)}\mathrm{d}x\mathrm{d}t<+\infty$ for any $\lambda\in(0,\lambda_c)$, and $\int_0^\infty\int_{\mathbb{R}}K(x,t)\mathrm{e}^{-\lambda(x+ct)}\mathrm{d}x\mathrm{d}t\to+\infty$ as $\lambda\to\lambda_{c^-}$.

(K3) For each $c>0$, there exists $\rho>0$ such that $\int_0^\infty\int_{\mathbb{R}}J(y,x)^{\rho s}\,\mathrm{d}y\mathrm{d}s<+\infty$.

In [7], Zhou *et al.* considered the diffusive SIR model with the standard incidence rate

$$\begin{cases}\dfrac{\partial S(x,t)}{\partial t}=d_1\dfrac{\partial^2 S(x,t)}{\partial x^2}+B-\mu S(x,t)-\dfrac{\beta S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)},\\ \dfrac{\partial I(x,t)}{\partial t}=d_2\dfrac{\partial^2 I(x,t)}{\partial x^2}+\dfrac{\beta S(x,t)I(x,t-\tau)}{S(x,t)+I(x,t-\tau)}-(\mu+\gamma)I(x,t),\\ \dfrac{\partial R(x,t)}{\partial t}=d_3\dfrac{\partial^2 R(x,t)}{\partial x^2}+\gamma I(x,t)-\mu R(x,t).\end{cases}\tag{1.2}$$

They find that the existence of traveling wave solutions of Equation (1.2) is determined by the basic reproduction number of the corresponding spatial-homogenous delay differential system and the minimal wave speed. They also investigated the existence and non-existence of traveling waves satisfying the asymptotic boundary conditions.

Time delays between infection and symptom onset, symptom onset and infectivity, and infection and death are significant [8]-[12]. However, it is unscientific that Equation (1.2) was dealt with by simply adding a diffusion term to the delayed differential equation models. In fact, time delay and diffusion are not independent as individuals have not been at the same location in space at previous times. Thus, the consideration of non-local interactions between infected and susceptible individuals has been integrated into epidemic models. In [4], Li *et al.* are concerned with the traveling wave solutions of a diffusive SIR system with spatio-temporal delay. Zhen *et al.* considered the following non-local dispersal SIR model with spatio-temporal delay in [11].

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1\left[J*S(x,t)-S(x,t)\right]-\dfrac{\beta S(x,t)K*I(x,t)}{S(x,t)+K*I(x,t)+R(x,t)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2\left[J*I(x,t)-I(x,t)\right]+\dfrac{\beta S(x,t)K*I(x,t)}{S(x,t)+K*I(x,t)+R(x,t)}, \\ \dfrac{\partial R(x,t)}{\partial t} = d_3\left[J*R(x,t)-R(x,t)\right]+\gamma I(x,t)-\mu R(x,t). \end{cases}$$

The authors obtained the existence and non-existence of the non-trivial and non-negative traveling wave solutions for the model, and the threshold dynamics of the model are determined by the basic reproduction number R_0 of the corresponding reaction system and minimal wave speed c^*.

Some research models largely ignore the influence of population structure, and these models exhibit rapid outbreak patterns, which can only be used to simulate rapidly developing diseases [13] [14]. However, many disease outbreaks are long-lasting, and the population size will change in reality [15]-[19]. Therefore, we consider models with population dynamics. Drawing inspiration from the preceding research, this paper aims to establish the existence and non-existence of traveling wave solutions for Equation (1.1). By constructing upper and lower solutions and Schauder's fixed point theorem, we obtain the existence of non-trivial solutions of (1.1). The non-existence of traveling waves for $R_0 > 1$ and any $0 < c < c^*$ or $R_0 < 1$ is proven through the comparison principle and the theory of asymptotic spreading.

The paper is organized as follows. In Section 2, we construct a pair of upper and lower solutions for system (1.1) and are concerned with the existence of the traveling wave solutions. In Section 3, we are devoted to the study of the non-existence of traveling wave solutions for system (1.1).

2. Existence of the Traveling Wave Solutions

We focus our analysis on the first two equations of (1.1), as the third equation is relatively independent.

Implement a scaling transformation

$$\tilde{S}(x,t) = \frac{\mu_1}{B} S(x,t), \quad \tilde{I}(x,t) = \frac{\mu_1}{B} I(x,t),$$

and dropping the tilde for convenience, the first two equations of (1.1) can be reduced to

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} = d_1 \Delta S(x,t) + \mu_1 - \mu_1 S(x,t) - \dfrac{\beta S(x,t) K * I(x,t)}{S(x,t) + K * I(x,t)}, \\ \dfrac{\partial I(x,t)}{\partial t} = d_2 \Delta I(x,t) + \dfrac{\beta S(x,t) K * I(x,t)}{S(x,t) + K * I(x,t)} - (\mu_2 + \gamma) I(x,t). \end{cases} \tag{2.1}$$

We look for the non-trivial traveling wave solutions $(S(x+ct), I(x+ct))$ of Equation (2.1) satisfying the following boundary conditions at infinity

$$S(-\infty) = 1, \; I(-\infty) = 0. \tag{2.2}$$

Let $\xi = x + ct$. Then the system describing travelling wave solutions is given as below

$$\begin{cases} cS'(\xi) = d_1 S''(\xi) + \mu_1 - \mu_1 S(\xi) - \dfrac{\beta S(\xi)(K * I)(\xi)}{S(\xi) + (K * I)(\xi)}, \\ cI'(\xi) = d_2 I''(\xi) + \dfrac{\beta S(\xi)(K * I)(\xi)}{S(\xi) + (K * I)(\xi)} - (\mu_2 + \gamma) I(\xi), \end{cases} \tag{2.3}$$

with

$$(K * I)(\xi) = \int_0^\infty \int_{\mathbb{R}} K(y,s) I(\xi - y - cs) \mathrm{d}y\mathrm{d}s.$$

Linearizing the second equation of (2.3) at the initial disease-free point $(1,0)$, we get

$$cI'(\xi) = d_2 I''(\xi) + \beta \int_0^\infty \int_{\mathbb{R}} K(y,s) I(\xi - y - cs) \mathrm{d}y\mathrm{d}s - (\mu_2 + \gamma) I(\xi).$$

Let $I(\xi) = \mathrm{e}^{\lambda \xi}$, then we establish a characteristic equation

$$\Delta(\lambda, c) = d_2 \lambda^2 - c\lambda + \beta \int_0^\infty \int_{\mathbb{R}} K(y,s) \mathrm{e}^{-\lambda(y+cs)} \mathrm{d}y\mathrm{d}s - (\mu_2 + \gamma). \tag{2.4}$$

It is easy to show the following lemma.

Lemma 2.1. *Assume* $R_0 := \dfrac{\beta}{\mu_2 + \gamma} > 1$. *Then there exists a positive pair* (λ^*, c^*) *such that* $\Delta(\lambda^*, c^*) = 0$ *and* $\dfrac{\partial}{\partial \lambda} \Delta(\lambda, c) \Big|_{(\lambda^*, c^*)} = 0$.

(1) *If* $0 < c < c^*$, *then* $\Delta(\lambda, c) > 0$ *for all* $[0, +\infty)$.

(2) *If* $c > c^*$, *then the equation* $\Delta(\lambda, c) = 0$ *has two positive roots* $\lambda_1(c)$ *and* $\lambda_2(c)$ *with* $0 < \lambda_1(c) < \lambda^* < \lambda_2(c)$ *such that*

$$\Delta(\lambda, c) \begin{cases} > 0 & \text{for all } \lambda \in [0, \lambda_1(c)) \cup (\lambda_2(c), \infty), \\ < 0 & \text{for all } \lambda \in (\lambda_1(c), \lambda_2(c)). \end{cases}$$

In the following, we always assume that $R_0 > 1$. In addition, we fix $c > c^*$ and denote $\lambda_i(c)$ by $\lambda_i, i = 1, 2$.

2.1. Construction of the Upper and Lower Solutions

Definition 2.1. *The continuous functions* $(\bar{S}, \bar{I})$ *and* $(\underline{S}, \underline{I})$ *are called a pair of upper and lower solutions of* (2.2), *if* $\bar{S}$, $\underline{S}$, $\bar{I}$, $\underline{I}$ *exist and satisfy the* *following inequalities*

$$d_1\overline{S}''(\xi)-c\overline{S}'(\xi)+\mu_1\left(1-\overline{S}(\xi)\right)-\frac{\beta\overline{S}(\xi)\left(K*\underline{I}\right)(\xi)}{\overline{S}(\xi)+\left(K*\underline{I}\right)(\xi)}\le 0, \tag{2.5}$$

$$d_1\underline{S}''(\xi)-c\underline{S}'(\xi)+\mu_1\left(1-\underline{S}(\xi)\right)-\frac{\beta\underline{S}(\xi)\left(K*\overline{I}\right)(\xi)}{\underline{S}(\xi)+\left(K*\overline{I}\right)(\xi)}\ge 0, \tag{2.6}$$

$$d_2\overline{I}''(\xi)-c\overline{I}'(\xi)-(\mu_2+\gamma)\overline{I}(\xi)+\frac{\beta\overline{S}(\xi)\left(K*\overline{I}\right)(\xi)}{\overline{S}(\xi)+\left(K*\overline{I}\right)(\xi)}\le 0, \tag{2.7}$$

$$d_2\underline{I}''(\xi)-c\underline{I}'(\xi)-(\mu_2+\gamma)\underline{I}(\xi)+\frac{\beta\underline{S}(\xi)\left(K*\underline{I}\right)(\xi)}{\underline{S}(\xi)+\left(K*\underline{I}\right)(\xi)}\ge 0, \tag{2.8}$$

hold except for finitely many points of $\xi\in\mathbb{R}$.

For a fixed $c>c^*$, we can find suitable $\lambda_1<\lambda^*<\lambda_2$ such that $\Delta(\lambda_1,c)=\Delta(\lambda_2,c)=0$. Moreover, it is possible to choose $\eta\in(0,\lambda^*)$ in such a way that it satisfies $\Delta(\eta,c)<0$ and $\lambda_1<\eta<\min\{\lambda_2,\lambda^*,\lambda_2-\lambda_1\}$. Now, we define four functions as follows

$$\begin{cases}\overline{S}(\xi)=1,\\ \overline{I}(\xi)=\min\left\{\mathrm{e}^{\lambda_1\xi},M\right\},\\ \underline{S}(\xi)=\max\left\{1-\sigma_0\mathrm{e}^{\alpha\xi},0\right\},\\ \underline{I}(\xi)=\max\left\{\mathrm{e}^{\lambda_1\xi}\left(1-\sigma_1\mathrm{e}^{\eta\xi}\right),0\right\},\end{cases} \tag{2.9}$$

where $M=\dfrac{\beta-\mu_2-\gamma}{\mu_2+\gamma}$.

Lemma 2.2. *The constants* α, σ_0, σ_1 *are chosen in the sequence such that the following assumptions* (1)-(3) *are held.*

(1) $\alpha>0$ *is small enough such that* $0<\alpha<\lambda_1$ *and* $-d_1\alpha^2+c\alpha+\mu_1>0$,

(2) $\sigma_0>\max\left\{1,\dfrac{\beta\int_0^\infty\int_{\mathbb{R}}K(y,s)\mathrm{e}^{-\lambda_1(y+cs)}\mathrm{d}y\mathrm{d}s}{-d_1\alpha^2+c\alpha+\mu_1}\right\}$,

(3) $\sigma_1>\max\left\{1,\dfrac{\beta\mathrm{e}^{(\lambda^1-\eta)\xi}\left[\int_0^\infty\int_{\mathbb{R}}K(y,s)\mathrm{e}^{-\lambda_1(y+cs)}\mathrm{d}y\mathrm{d}s\right]^2}{-\Delta(\lambda_1+\eta,c)}\right\}$.

Denote $\xi_j:=\dfrac{-\ln\sigma_1}{\eta}$, $\boxed{\xi}_j:=\dfrac{\ln M}{\lambda_1}$ and $\xi_0:=\dfrac{-\ln\sigma_0}{\alpha}$.

Lemma 2.3. *The functions* ($\overline{S}(\xi)$, $\overline{I}(\xi)$), ($\underline{S}(\xi)$, $\underline{I}(\xi)$) *defined by* (2.9) *is a pair of upper and lower solutions of system* (2.3).

Proof. Firstly, the function $\overline{S}(\xi)$ is of class $C^1(\mathbb{R})$ and inequality (2.5) holds on $\mathbb{R}$, since

$$d_1\overline{S}''(\xi)-c\overline{S}'(\xi)+\mu_1\left(1-\overline{S}(\xi)\right)-\frac{\beta\overline{S}(\xi)(K*\underline{I})(\xi)}{\overline{S}(\xi)+(K*\underline{I})(\xi)}=-\frac{\beta\overline{S}(\xi)(K*\underline{I})(\xi)}{\overline{S}(\xi)+(K*\underline{I})(\xi)}\le 0.$$

Then, we show that $\overline{I}(\xi)=\min\left\{\mathrm{e}^{\lambda_1\xi},M\right\}$ satisfies Equation (2.7). By the definition of $\overline{I}(\xi)$, we have

$$(K*I)(\xi)\le \min\left\{e^{\lambda_1\xi}\int_0^{\infty}\int_{\mathbb{R}}K(y,s)e^{-\lambda(y+cs)}\mathrm{d}y\mathrm{d}s, M\right\}.$$

When $\xi>\overset{\boxed{\xi}}{\xi}_j$, we have $\overline{I}(\xi)=M$ and

$$d_2\overline{I}''(\xi)-c\overline{I}'(\xi)-(\mu_2+\gamma)\overline{I}(\xi)+\frac{\beta\overline{S}(\xi)(K*\overline{I})(\xi)}{\overline{S}(\xi)+(K*\overline{I})(\xi)}=0.$$

When $\xi<\overset{\boxed{\xi}}{\xi}_j$, $\overline{I}(\xi)=e^{\lambda_1\xi}$, then

$$\begin{aligned}&d_2\overline{I}''(\xi)-c\overline{I}'(\xi)-(\mu_2+\gamma)\overline{I}(\xi)+\frac{\beta\overline{S}(\xi)(K*\overline{I})(\xi)}{\overline{S}(\xi)+(K*\overline{I})(\xi)}\\&\le d_2\overline{I}''(\xi)-c\overline{I}'(\xi)-(\mu_2+\gamma)\overline{I}(\xi)+\beta(K*\overline{I})(\xi)\\&=\Delta(\lambda_1,c)e^{\lambda_1\xi}\\&=0.\end{aligned}$$

Since the function $\underline{S}(\xi)$ is continuous in $\mathbb{R}$ and of class $C^1(\mathbb{R}\setminus\{\xi_0\})$, next, we show that Equation (2.6) holds for $\xi\ne\xi_0$.

When $\xi>\xi_0$, we have $\underline{S}(\xi)=0$, and it is easy to show that

$$d_1\underline{S}''(\xi)-c\underline{S}'(\xi)+\mu_1(1-\underline{S}(\xi))-\frac{\beta\underline{S}(\xi)(K*\overline{I})(\xi)}{\underline{S}(\xi)+(K*\overline{I})(\xi)}=\mu_1>0.$$

When $\xi<\xi_0$, we have $\underline{S}(\xi)=1-\sigma_0e^{\alpha\xi}$, $\overline{I}(\xi)=e^{\lambda_1\xi}$ and

$$\begin{aligned}&d_1\underline{S}''(\xi)-c\underline{S}'(\xi)+\mu_1(1-\underline{S}(\xi))-\frac{\beta\underline{S}(\xi)(K*\overline{I})(\xi)}{\underline{S}(\xi)+(K*\overline{I})(\xi)}\\&\ge d_1\underline{S}''(\xi)-c\underline{S}'(\xi)+\mu_1(1-\underline{S}(\xi))-\beta(K*\overline{I})(\xi)\\&=e^{\alpha\xi}\left[\sigma_0(-d_1\alpha^2+c\alpha+\mu_1)-\beta e^{(\lambda_1-\alpha)\xi}\int_0^{\infty}\int_{\mathbb{R}}K(y,s)e^{-\lambda_1(y+cs)}\mathrm{d}y\mathrm{d}s\right]\\&\ge 0.\end{aligned}$$

Finally, we show that Equation (2.8) holds. When $\xi>\xi_j$, we have $\underline{I}(\xi)=0$, which implies that Equation (2.8) holds. When $\xi<\xi_j$, $\underline{I}(\xi)=e^{\lambda_1\xi}(1-\sigma_1e^{\eta\xi})$ and $\underline{S}(\xi)=1-\sigma_0e^{\alpha\xi}$. Then

$$\begin{aligned}&d_2\underline{I}''(\xi)-c\underline{I}'(\xi)-(\mu_2+\gamma)\underline{I}(\xi)+\frac{\beta\underline{S}(\xi)(K*\underline{I})(\xi)}{\underline{S}(\xi)+(K*\underline{I})(\xi)}\\&=d_2\underline{I}''(\xi)-c\underline{I}'(\xi)-(\mu_2+\gamma)\underline{I}(\xi)+\beta(K*\underline{I})(\xi)\\&\quad+\frac{\beta\underline{S}(\xi)(K*\underline{I})(\xi)}{\underline{S}(\xi)+(K*\underline{I})(\xi)}-\beta(K*\underline{I})(\xi)\\&=-\sigma_1e^{(\lambda_1+\eta)\xi}\Delta(\lambda_1+\eta,c)-\beta\frac{\left[(K*\underline{I})(\xi)\right]^2}{\underline{S}(\xi)+(K*\underline{I})(\xi)}\\&\ge-\sigma_1e^{(\lambda_1+\eta)\xi}-\beta e^{2\lambda_1\xi}\left[\int_0^{\infty}\int_{\mathbb{R}}K(y,s)e^{-\lambda_1(y+cs)}\mathrm{d}y\mathrm{d}s\right]^2.\end{aligned}$$

The choice of σ_1 leads to the validity of Equation (2.8), we complete the proof.

2.2. The Verification of the Schauder Fixed Point Theorem

Choosing two constants α_1, α_2 such that

$$G(S,I)(x)=\mu_1-\mu_1 S(x)-\frac{\beta S(x)(K*I)(x)}{S(x)+(K*I)(x)}+\alpha_1 S(x)$$

is nondecreasing in $S(x)\in[0,1]$ and nonincreasing in $I(x)\in[0,M]$ for all $x\in\mathbb{R}$, and

$$Q(S,I)(x)=-(\mu_2+\gamma)I(x)+\frac{\beta S(x)(K*I)(x)}{S(x)+(K*I)(x)}+\alpha_2 I(x)$$

is nondecreasing in S,I for $(S,I)\in[0,1]\times[0,M]$.

Obviously, Equation (2.3) is equal to

$$d_1 S''(x)-cS'(x)-\alpha_1 S(x)+G(S,I)(x)=0,$$
$$d_2 I''(x)-cI'(x)-\alpha_2 I(x)+Q(S,I)(x)=0.$$

Define

$$\Gamma=\left\{(S,I)\in C(\mathbb{R},\mathbb{R}^2)\,|\,\underline{S}(x)\le S(x)\le\overline{S}(x),\underline{I}(x)\le I(x)\le\overline{I}(x)\right\},$$

and

$$\Lambda_{11}=\frac{c-\sqrt{c^2+4d_1\alpha_1}}{2d_1},\ \Lambda_{12}=\frac{c+\sqrt{c^2+4d_1\alpha_1}}{2d_1},\ \rho_1=d_1(\Lambda_{12}-\Lambda_{11}),$$
$$\Lambda_{21}=\frac{c-\sqrt{c^2+4d_2\alpha_2}}{2d_2},\ \Lambda_{22}=\frac{c+\sqrt{c^2+4d_2\alpha_2}}{2d_2},\ \rho_2=d_2(\Lambda_{22}-\Lambda_{21}).$$

Furthermore, define an operator $F:\Gamma\to C(\mathbb{R},\mathbb{R}^2)$ by

$$F(S,I)(\xi)=\begin{pmatrix}F_1(S,I)(\xi)\\F_2(S,I)(\xi)\end{pmatrix},$$

where

$$\begin{aligned}F_1(S,I)(\xi)&=\frac{1}{\rho_1}\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}G(S,I)(x)\mathrm{d}x+\frac{1}{\rho_1}\int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}G(S,I)(x)\mathrm{d}x,\\F_2(S,I)(\xi)&=\frac{1}{\rho_2}\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{21}(\xi-x)}Q(S,I)(x)\mathrm{d}x+\frac{1}{\rho_2}\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{22}(\xi-x)}Q(S,I)(x)\mathrm{d}x.\end{aligned}\tag{2.10}$$

Lemma 2.4. *The set* Γ *is nonempty, closed and convex in* $C(\mathbb{R},\mathbb{R}^2)$.

It is easy to prove, so we omit the details here.

Lemma 2.5. *The operator* F *maps* Γ *into* Γ.

Proof. For $(S,I)\in\Gamma$, we only need to prove that

$$\underline{S}(\xi)\le F_1(S,I)(\xi)\le\overline{S}(\xi),\ \underline{I}(\xi)\le F_2(S,I)(\xi)\le\overline{I}(\xi),\ \forall\xi\in\mathbb{R}.$$

Thanks to the choice of the constants α_1 and α_2, it suffices to prove that for any $\xi\in\mathbb{R}$,

$$\begin{aligned}&\underline{S}(\xi)\le F_1(\underline{S},\overline{I})(\xi)\le F_1(S,I)(\xi)\le F_1(\overline{S},\underline{I})(\xi)\le\overline{S}(\xi),\\&\underline{I}(\xi)\le F_2(\underline{S},\underline{I})(\xi)\le F_2(S,I)(\xi)\le F_2(\overline{S},\overline{I})(\xi)\le\overline{I}(\xi).\end{aligned}\tag{2.11}$$

Firstly, we consider $F_1(S,I)(\xi)$. For any $\xi\in\mathbb{R}$, we have

$$\begin{aligned}F_1(S,I)(\xi)&\le F_1(\overline{S},\underline{I})(\xi)\\&=\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)G(\overline{S},\underline{I})(x)\mathrm{d}x\\&\le\frac{\alpha_1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)\mathrm{d}x\\&=\overline{S}.\end{aligned}$$

Here, we use the fact that $\dfrac{\beta\overline{S}(\xi)(K*\underline{I})(\xi)}{\overline{S}(\xi)+(K*\underline{I})(\xi)}\ge d_1\overline{S}''(\xi)-c\overline{S}'(\xi)+\mu_1(1-\overline{S}(\xi))$.

Similarly, we get

$$\begin{aligned}F_1(S,I)(\xi)&\ge F_1(\underline{S},\overline{I})(\xi)\\&=\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)G(\underline{S},\overline{I})(x)\mathrm{d}x\\&\ge\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)\left[-d_1S''(x)+cS'(x)+\alpha_1S(x)\right]\mathrm{d}x.\end{aligned}$$

When $\xi\ge\xi_0$, $\underline{S}(\xi)=0$, then we obtain

$$\begin{aligned}&F_1(S,I)(\xi)\\&\ge\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)\left[-d_1\underline{S}''(x)+c\underline{S}'(x)+\alpha_1\underline{S}(x)\right]\mathrm{d}x\\&=\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi_0}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi_0}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)\left[-d_1\underline{S}''(x)+c\underline{S}'(x)+\alpha_1\underline{S}(x)\right]\mathrm{d}x\\&=\underline{S}(\xi)+\frac{d_1\mathrm{e}^{\Lambda_{11}(\xi-\xi_0)}}{\rho_1}\left[\underline{S}'(\xi_0+0)-\underline{S}'(\xi_0-0)\right]\\&\quad+\frac{d_1\Lambda_{11}-c}{\rho_1}\mathrm{e}^{\Lambda_{11}(\xi-\xi_0)}\left[\underline{S}(\xi_0+0)-\underline{S}(\xi_0-0)\right]\\&\ge\underline{S}(\xi).\end{aligned}$$

For $\xi<\xi_0$, $\underline{S}(\xi)=1-\sigma_0e^{\alpha\xi}$, then we have

$$\begin{aligned}&F_1(S,I)(\xi)\\&\ge\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)\left[-d_1\underline{S}''(x)+c\underline{S}'(x)+\alpha_1\underline{S}(x)\right]\mathrm{d}x\\&=\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{\xi_0}\mathrm{e}^{\Lambda_{12}(\xi-x)}+\int_{\xi_0}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)\left[-d_1\underline{S}''(x)+c\underline{S}'(x)+\alpha_1\underline{S}(x)\right]\mathrm{d}x\\&=\underline{S}(\xi)+\frac{d_1\mathrm{e}^{\Lambda_{12}(\xi-\xi_0)}}{\rho_1}\left[\underline{S}'(\xi_0+0)-\underline{S}'(\xi_0-0)\right]\\&\quad+\frac{d_1\Lambda_{12}-c}{\rho_1}\mathrm{e}^{\Lambda_{12}(\xi-\xi_0)}\left[\underline{S}(\xi_0+0)-\underline{S}(\xi_0-0)\right]\\&\ge\underline{S}(\xi).\end{aligned}$$

Next, we consider $F_2(S,I)(\xi)$. For any $\xi\in\mathbb{R}$, we can show

$$\begin{aligned}F_2(S,I)(\xi)&\geq F_2(\underline{S},\underline{I})(\xi)\\&=\frac{1}{\rho_2}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{21}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{22}(\xi-x)}\right)Q(\underline{S},\underline{I})(x)\mathrm{d}x\\&\geq\frac{1}{\rho_2}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{21}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{22}(\xi-x)}\right)\left[-d_2\underline{I}''(x)+c\underline{I}'(x)+\alpha_2\underline{I}(x)\right]\mathrm{d}x\\&\geq\underline{I}(\xi),\end{aligned}$$

and

$$\begin{aligned}F_2(S,I)(\xi)&\leq F_2(\overline{S},\overline{I})(\xi)\\&=\frac{1}{\rho_2}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{21}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{22}(\xi-x)}\right)Q(\overline{S},\overline{I})(x)\mathrm{d}x\\&\leq\frac{1}{\rho_2}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{21}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{22}(\xi-x)}\right)\left[-d_2\overline{I}''(x)+c\overline{I}'(x)+\alpha_2\overline{I}(x)\right]\mathrm{d}x\\&=\overline{I}(\xi).\end{aligned}$$

Further applying the continuity, Equation (2.11) is held. We complete the proof.

Define

$$B_\tau(\mathbb{R},\mathbb{R}^2):=\left\{\Phi=(\phi_1,\phi_2)\in C(\mathbb{R},\mathbb{R}^2)\mid\sup_{\xi\in\mathbb{R}}|\phi_1(\xi)|\mathrm{e}^{-\tau|\xi|}<+\infty,\sup_{\xi\in\mathbb{R}}|\phi_2(\xi)|\mathrm{e}^{-\tau|\xi|}<+\infty\right\},$$

with norm

$$|\Phi|_\tau=\max\left\{\sup_{\xi\in\mathbb{R}}|\phi_1(\xi)|\mathrm{e}^{-\tau|\xi|},\sup_{\xi\in\mathbb{R}}|\phi_2(\xi)|\mathrm{e}^{-\tau|\xi|}\right\},$$

where $\tau>0$ is a constant such that $\tau<\min\{-\Lambda_{11},-\Lambda_{21}\}$.

Lemma 2.6. *The operator* $F:\Gamma\to\Gamma$ *is continuous with respect to the norm* $|\cdot|_\tau$ *in* $B_\tau(\mathbb{R},\mathbb{R}^2)$.

Proof. For any $\Phi_1=(S_1(\cdot),I_1(\cdot))\in\Gamma$ and $\Phi_2=(S_2(\cdot),I_2(\cdot))\in\Gamma$, we have

$$\begin{aligned}&\left|F_1[(S_1(\cdot),I_1(\cdot)](\xi)-F_1[(S_2(\cdot),I_2(\cdot)](\xi)\right|\mathrm{e}^{-\tau|\xi|}\\&=\frac{\mathrm{e}^{-\tau|\xi|}}{\rho_1}\left|\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)\left[G(S_1,I_1)(x)-G(S_2,I_2)(x)\right]\mathrm{d}x\right|\\&\leq\frac{\mathrm{e}^{-\tau|\xi|}}{\rho_1}\left|\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right|\\&\quad\left[\alpha_1|S_1(x)-S_2(x)|+\mu_1|S_1(x)-S_2(x)|+\beta|(K*I_1)(x)-(K*I_2)(x)|\right]\mathrm{d}x\\&=\frac{\mathrm{e}^{-\tau|\xi|}}{\rho_1}\left|\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right|\left[\alpha_1|S_1(x)-S_2(x)|\mathrm{e}^{-\tau|x|}+\mu_1|S_1(x)-S_2(x)|\mathrm{e}^{-\tau|x|}\right.\\&\quad\left.+\beta|S_1(x)-S_2(x)|\mathrm{e}^{-\tau|x|}+\beta|(K*I_1)(\xi)-(K*I_2)(\xi)|\mathrm{e}^{-\tau|x|}\right]\mathrm{e}^{\tau|x|}\mathrm{d}x\\&\leq\frac{(\alpha_1+\mu_1+\beta)\left[\left(|\Phi_1-\Phi_2|_\tau+|\Phi_1-\Phi_2|_\tau\int_0^\infty\int_R\mathrm{e}^{\tau|y+cs|}K(y,s)\mathrm{d}y\mathrm{d}s\right)\mathrm{e}^{-\tau|\xi|}\right]}{\rho_1}\\&\quad\left|\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}\mathrm{e}^{\tau|x|}\mathrm{d}x+\int_{\xi}^{\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\mathrm{e}^{\tau|x|}\mathrm{d}x\right|.\end{aligned}$$

When $\xi\geq0$, we have

$$
\begin{aligned}
&\left|F_1\left[\left(S_1(\cdot),I_1(\cdot)\right](\xi)-F_1\left[\left(S_2(\cdot),I_2(\cdot)\right](\xi)\right|\mathrm{e}^{-\tau|\xi|}\\
&\le\frac{(\alpha_1+\mu_1+\beta)\left[\left(\left|\Phi_1-\Phi_2\right|_\tau+\left|\Phi_1-\Phi_2\right|_\tau\int_0^\infty\int_R\mathrm{e}^{\tau|y+cs|}K(y,s)\mathrm{d}y\mathrm{d}s\right)\mathrm{e}^{-\tau|\xi|}\right]}{\rho_1}\\
&\quad\left|\int_{-\infty}^0\mathrm{e}^{\Lambda_{11}(\xi-x)}\mathrm{e}^{\tau|x|}\mathrm{d}x+\int_0^\xi\mathrm{e}^{\Lambda_{11}(\xi-x)}\mathrm{e}^{\tau|x|}\mathrm{d}x+\int_\xi^\infty\mathrm{e}^{\Lambda_{12}(\xi-x)}\mathrm{e}^{\tau|x|}\mathrm{d}x\right|\\
&=\frac{(\alpha_1+\mu_1+\beta)\left[\left(\left|\Phi_1-\Phi_2\right|_\tau+\left|\Phi_1-\Phi_2\right|_\tau\int_0^\infty\int_R\mathrm{e}^{\tau|y+cs|}K(y,s)\mathrm{d}y\mathrm{d}s\right)\right]}{\rho_1}\\
&\quad\left(\frac{1}{\tau-\Lambda_{11}}+\mathrm{e}^{-\tau\xi+\Lambda_{11}\xi}\frac{2\tau}{\Lambda_{11}^2-\tau^2}-\frac{1}{\tau-\Lambda_{12}}\right)\\
&\le\frac{\alpha_1+\mu_1+\beta}{\rho_1}\left[\frac{\Lambda_{11}-\Lambda_{12}}{(\tau-\Lambda_{11})(\tau-\Lambda_{12})}+\frac{2\tau}{\Lambda_{11}^2-\tau^2}\right]\\
&\quad\left[\left(\left|\Phi_1-\Phi_2\right|_\tau+\left|\Phi_1-\Phi_2\right|_\tau\int_0^\infty\int_R\mathrm{e}^{\tau|y+cs|}K(y,s)\mathrm{d}y\mathrm{d}s\right)\right].
\end{aligned}
$$

Similarly, for $\xi<0$, we have

$$
\begin{aligned}
&\left|F_1\left[\left(S_1(\cdot),I_1(\cdot)\right](\xi)-F_1\left[\left(S_2(\cdot),I_2(\cdot)\right](\xi)\right|\mathrm{e}^{-\tau|\xi|}\\
&\le\frac{\alpha_1+\mu_1+\beta}{\rho_1}\left[\frac{\Lambda_{11}-\Lambda_{12}}{(\tau+\Lambda_{11})(\tau+\Lambda_{12})}+\frac{2\tau}{\Lambda_{12}^2-\tau^2}\right]\\
&\quad\left[\left(\left|\Phi_1-\Phi_2\right|_\tau+\left|\Phi_1-\Phi_2\right|_\tau\int_0^\infty\int_R\mathrm{e}^{\tau|y+cs|}K(y,s)\mathrm{d}y\mathrm{d}s\right)\right].
\end{aligned}
$$

Then, it follows that the mapping $F_1:\Gamma\to\Gamma$ is continuous with respect to the norm $|\cdot|_\tau$ in $B_\tau\left(\mathbb{R},\mathbb{R}^2\right)$. Similarly, we can prove that $F_2:\Gamma\to\Gamma$ is continuous with respect to the norm $|\cdot|_\tau$ in $B_\tau\left(\mathbb{R},\mathbb{R}^2\right)$.

Lemma 2.7. *The operator* $F:\Gamma\to\Gamma$ *is compact with respect to the norm* $|\cdot|_\tau$ *in* $B_\tau\left(\mathbb{R},\mathbb{R}^2\right)$.

Proof. For any $(S,I)\in\Gamma$, we have

$$
\begin{aligned}
\left|G(S,I)(\xi)\right|&\le\mu_1+\mu_1\bar{S}(\xi)+\alpha_1\bar{S}(\xi)+\frac{\beta S(\xi)(K*I)(\xi)}{S(\xi)+(K*I)(\xi)}\\
&\le2\mu_1+\alpha_1+\beta\sup_{\xi\in\mathbb{R}}\left\{S(\xi),(K*I)(\xi)\right\}\\
&<2\mu_1+\alpha_1+\frac{\beta^2}{\mu_2+\gamma}.
\end{aligned}
$$

Then, we can obtain

$$
\begin{aligned}
\left|\frac{\mathrm{d}}{\mathrm{d}\xi}F_1(S,I)(\xi)\right|&\le\frac{|\Lambda_{11}|}{\rho_1}\int_{-\infty}^\xi\mathrm{e}^{\Lambda_{11}(\xi-x)}G\left(\bar{S},\underline{I}\right)\mathrm{d}x+\frac{\Lambda_{12}}{\rho_1}\int_\xi^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}G\left(\bar{S},\underline{I}\right)\mathrm{d}x\\
&<\frac{|\Lambda_{11}|}{\rho_1}\int_{-\infty}^\xi\mathrm{e}^{\Lambda_{11}(\xi-x)}\left(2\mu_1+\alpha_1+\frac{\beta^2}{\mu_2+\gamma}\right)\mathrm{d}x\\
&\quad+\frac{\Lambda_{12}}{\rho_1}\int_\xi^{+\infty}\mathrm{e}^{\Lambda_{11}(\xi-x)}\left(2\mu_1+\alpha_1+\frac{\beta^2}{\mu_2+\gamma}\right)\mathrm{d}x\\
&=\frac{1}{\rho_1}\left(2\mu_1+\alpha_1+\frac{\beta^2}{\mu_2+\gamma}\right).
\end{aligned}
$$

Similarly, we can obtain that $\frac{\mathrm{d}}{\mathrm{d}\xi}F_2(S,I)(\xi)$ is also bounded.

For each integer $n \in \mathbb{N}$, define an operator F^n by

$$F^n[S,I](\xi)=\begin{cases} F[S,I](\xi), & \xi \in [-n,n], \\ F[S,I](-n), & \xi \in (-\infty,-n], \\ F[S,I](-n), & \xi \in [n,+\infty). \end{cases}$$

By the Ascoli-Arzela lemma, it is obvious that $F^n:\Gamma \to \Gamma$ is compact with respect to the supremum norm in $C(\mathbb{R},\mathbb{R}^2)$. Hence, $F^n:\Gamma \to \Gamma$ is compact with respect to the norm $|\cdot|_\tau$ in $B_\tau(\mathbb{R},\mathbb{R}^2)$. Furthermore, since $\{F^n\}_0^\infty$ is a compact series, it follows that

$$\begin{aligned}
&\left|F^n[S,I](\xi)-F[S,I](\xi)\right|_\tau \\
&=\sup_{\xi\in\mathbb{R}}\left|F^n[S,I](\xi)-F[S,I](\xi)\right|\mathrm{e}^{-\tau|\xi|} \\
&=\sup_{\xi\in(-\infty,-n]\cup[n,+\infty)}\left|F^n[S,I](\xi)-F[S,I](\xi)\right|\mathrm{e}^{-\tau|\xi|} \\
&\le \sup_{\xi\in(-\infty,-n]\cup[n,+\infty)}\max\left\{1,\frac{\beta}{\mu_2+\gamma}-1\right\}\mathrm{e}^{-\tau|\xi|} \\
&<\frac{\beta}{\mu_2+\gamma}\mathrm{e}^{-\tau n}\to 0 \ \text{ as } n\to+\infty.
\end{aligned}$$

By proposition 2.1 in [20], we know that $\{F^n\}_0^\infty$ converges to F in Γ with respect to the norm $|\cdot|_\tau$. Then, we obtain that F is compact with respect to the norm $|\cdot|_\tau$ in $B_\tau(\mathbb{R},\mathbb{R}^2)$.

2.3. The Proof of the Existence Theorem

Theorem 2.1. *Assume* $R_0>1$. *For any* $c>c^*$, *system* (2.1) *admits a traveling wave solution* $(S(x+ct),I(x+ct))$ *such that*

(1) $0<S(\xi)<1$, $0<I(\xi)<M$ for all $\xi\in\mathbb{R}$.

(2) $S(-\infty)=1$, $I(-\infty)=0$, $S'(-\infty)=0$, $I'(-\infty)=0$. Furthermore, $\lim_{\xi\to-\infty}\mathrm{e}^{-\lambda_1\xi}I(\xi)=1$.

Proof. When $c>c^*$, Schauder's fixed point theorem implies that there exists a pair of $(S,I)\in\Gamma$, which is a fixed point of the operator F. Consequently, the solution $(S(x+ct),I(x+ct))$ is a traveling wave solution of Equation (2.1) and for any $\xi\in\mathbb{R}$,

$$0\le S(\xi)\le 1, 0\le I(\xi)\le M.$$

Next, we show that strict inequalities hold. Indeed, note that $(S,I)\in\Gamma$ is a fixed point of the operator F, then $S(\xi)=F_1[S,I](\xi)$, $I(\xi)=F_2[S,I](\xi)$. Finally,

$$\begin{aligned}
S(\xi)&=F_1[S,I](\xi) \\
&\ge F_1[\underline{S},\overline{I}](\xi) \\
&=\frac{1}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\right)G(\underline{S},\overline{I})(x)\mathrm{d}x \\
&\ge\frac{\alpha_1-\mu_1-\beta}{\rho_1}\left(\int_{-\infty}^{\xi}\mathrm{e}^{\Lambda_{11}(\xi-x)}\mathrm{d}x+\int_{\xi}^{+\infty}\mathrm{e}^{\Lambda_{12}(\xi-x)}\mathrm{d}x\right)\underline{S}(x)>0,
\end{aligned}$$

and

$$\begin{aligned} I(\xi) &= F_2[S,I](\xi) \\ &\geq F_2[\underline{S},\underline{I}](\xi) \\ &= \frac{1}{\rho_1}\left(\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{21}(\xi-x)} + \int_{\xi}^{+\infty} \mathrm{e}^{\Lambda_{22}(\xi-x)}\right) Q(\underline{S},\underline{I})(x)\mathrm{d}x \\ &\geq \frac{\alpha_2-\mu_2-\gamma}{\rho_1}\left(\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)} + \int_{\xi}^{+\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)}\right) \underline{I}(x)\mathrm{d}x > 0. \end{aligned}$$

Because that $\underline{S}(x)$ and $\underline{I}(x)$ are continuous and not identically zero, we have that $S(\xi)$ and $I(\xi)$ are both positive for any $\xi \in \mathbb{R}$.

Similarly, we can prove that other inequalities are also strict ones. Note that

$$1-\sigma_0\mathrm{e}^{\alpha\xi} \leq S(\xi) < 1, \quad \mathrm{e}^{\lambda_1\xi} - \sigma_1\mathrm{e}^{\eta\xi} \leq I(\xi) \leq \mathrm{e}^{\lambda_1\xi}, \quad \forall \xi \in \mathbb{R},$$

it is easy to see that $S(-\infty)=1, I(-\infty)=0, \lim\limits_{\xi\to-\infty} \mathrm{e}^{-\lambda_1\xi} I(\xi)=1$ by the squeeze theorem.

Note that $(S,I)\in\Gamma$ is a fixed point of the operator of F. By the first equation of (2.10), we have

$$\begin{aligned} S'(\xi) &= \frac{\Lambda_{11}}{\rho_1}\int_{-\infty}^{\xi} \mathrm{e}^{\Lambda_{11}(\xi-x)} G(S,I)(x)\mathrm{d}x + \frac{\Lambda_{12}}{\rho_1}\int_{\xi}^{+\infty} \mathrm{e}^{\Lambda_{12}(\xi-x)} G(S,I)(x)\mathrm{d}x \\ &= \frac{\Lambda_{11}}{\rho_1}\int_{0}^{+\infty} \mathrm{e}^{\Lambda_{11}t} G[S,I](\xi-t)\mathrm{d}t + \frac{\Lambda_{12}}{\rho_1}\int_{-\infty}^{0} \mathrm{e}^{\Lambda_{12}t} G[S,I](\xi-t)\mathrm{d}t. \end{aligned} \tag{2.12}$$

Note that for any $t>0$,

$$\lim_{\xi\to-\infty} G[S,I](\xi-t) = \alpha_1.$$

Hence, from the Equation (2.12), we get

$$\lim_{\xi\to-\infty} S'(\xi) = \frac{\alpha_1}{\rho_1}\left[\mathrm{e}^{\Lambda_{11}t}\Big|_0^{+\infty} + \mathrm{e}^{\Lambda_{12}t}\Big|_{-\infty}^{0}\right] = 0.$$

Similarly, we have $I'(-\infty)=0$. Hence we have shown that

$$S(-\infty)=1, I(-\infty)=0, S'(-\infty)=0, I'(-\infty)=0.$$

3. Non-Existence of Traveling Wave Solutions

In this section, we will establish the non-existence of traveling wave solutions for system (2.1), for $R_0<1$ and $c\geq 0$, or $R_0>1$ and $c\in(0,c^*)$.

Theorem 3.1. *Assume that* $R_0>1$ *and* $c\in(0,c^*)$, *there exists no traveling wave solutions* $(S(x+ct), I(x+ct))$ *of* (2.1) *satisfying Equation* (2.2).

Proof. Assume that there exists non-trivial travelling wave solution $(S(x+ct), I(x+ct))$ of system (2.1) satisfying (2.2) for some $c_1\in(0,c^*)$. Let $\varepsilon>0$ and $c\in\left(0,\frac{c+c^*}{2}\right)$ such that equation

$$\lambda^2 - c\lambda + \beta(1-2\varepsilon)\int_0^{\infty}\int_R K(y,s)\mathrm{e}^{-\lambda(y+cs)}\mathrm{d}y\mathrm{d}s - (\mu_2+\gamma) = 0$$

has no real solution.

By Equation (2.2), we can choose $M_\varepsilon > 0$ large enough such that

$$1-\varepsilon \le S(\xi) < 1, \text{ for any } \varepsilon < -M_\varepsilon.$$

Furthermore,

$$c_1 I'(\xi) \ge d_2 I''(\xi) + \frac{\beta(1-\varepsilon)\int_0^\infty \int_R K(y,s) I(\xi - y - c_1 s)\mathrm{d}y\mathrm{d}s}{1+\int_0^\infty \int_R K(y,s) I(\xi - y - c_1 s)\mathrm{d}y\mathrm{d}s} - (\mu_2 + \gamma) I(\xi). \tag{3.1}$$

According to Theorem 2.1 and Equation (2.2), there exists a sufficiently large constant $h > 1$ such that

$$\begin{aligned}&\frac{\beta(1-\varepsilon)\int_0^\infty \int_R K(y,s) I(\xi - y - c_1 s)\mathrm{d}y\mathrm{d}s}{\left(1+I(\xi - c_1 s)\right)^{h+1}} \\ &\le \frac{\beta S(\xi)\int_0^\infty \int_R K(y,s) I(\xi - y - c_1 s)\mathrm{d}y\mathrm{d}s}{1+I(\xi - c_1 s)}, \xi \ge -M_\varepsilon.\end{aligned}$$

In fact, it is equivalent to the following inequality

$$\frac{1-\varepsilon}{\left(1+I(\xi - cs)\right)^h} \le S(\xi), \xi \ge -M_s.$$

Then, for $\xi \ge -M_s$,

$$c_1 I'(\xi) \ge d_2 I''(\xi) + \frac{\beta(1-\varepsilon)\int_0^\infty \int_R K(y,s) I(\xi - y - c_1 s)\mathrm{d}y\mathrm{d}s}{\left(1+I(\xi - c_1 s)\right)^{h+1}} - (\mu_2 + \gamma) I(\xi). \tag{3.2}$$

Define

$$\Phi(u(x,t)) = \inf \frac{\beta(1-\varepsilon) v(x,t)}{\left(1+v(x,t)\right)^{h+1}}. \tag{3.3}$$

Combining Equations (3.1)-(3.3), we can obtain that $u(x,t) = I(x + c_1 t)$ satisfies

$$\begin{cases}\dfrac{\partial u(x,t)}{\partial t} \ge d_2 \dfrac{\partial^2 u(x,t)}{\partial x^2} + \int_0^\infty \int_R K(y,s)\Phi(x-y,t-s)\mathrm{d}y\mathrm{d}s - (\mu_2+\gamma)u(x,t), x \in R, t > 0, \\ u(x,s) = I(x + c_1 s) > 0, s \in (-\infty, 0], x \in R.\end{cases}$$

By the comparison principle [21], $u(x,t)$ is an upper solution of the following initial value problem

$$\begin{cases}\dfrac{\partial v(x,t)}{\partial t} \ge d_2 \dfrac{\partial^2 v(x,t)}{\partial x^2} + \int_0^\infty \int_R K(y,s)\Phi(x-y,t-s)\mathrm{d}y\mathrm{d}s - (\mu_2+\gamma)v(x,t), x \in R, t > 0, \\ v(x,s) = I(x + c_1 s) > 0, s \in (-\infty, 0], x \in R.\end{cases}$$

Applying the theory of asymptotic spreading [22] [23], we obtain that

$$\liminf_{t\to+\infty} v(x,t) > 0, \ |x| < \frac{(c_1 + c^*)t}{2}.$$

Then,

$$\liminf_{t\to+\infty} u(x,t) \geq \liminf_{t\to+\infty} v(x,t) > 0,\ |x| < \frac{(c_1+c^*)t}{2}.$$

Let $-x = \frac{(c_1+c^*)t}{2}$, then $x + c_1 t = \frac{(c_1-c^*)t}{2}$. Note that $x + c_1 t \to -\infty$ if $t \to +\infty$, so we have

$$\lim_{t\to+\infty} u(x,t) = \lim_{t\to+\infty} I(x+c_1 t) = 0.$$

This is contradicted with $I(\xi) > 0$. We complete the proof.

Theorem 3.2. *Assume that* $R_0 < 1$ *and* $c > 0$, *there exists no non-trivial travelling wave solutions* $(S(x+ct), I(x+ct))$ *of Equation* (2.1) *satisfying* (2.2).

Proof. Suppose that there exists non-trivial travelling wave solution $(S(x+ct), I(x+ct))$ of system (2.1) satisfying Equation (2.2) for some $c > 0$. It follows that

$$\begin{aligned} cI'(\xi) &= d_2 I''(\xi) - \mu_2 I(\xi) + \frac{\beta S(\xi)(K*I)(\xi)}{S(\xi)+(K*I)(\xi)} - \gamma I(\xi) \\ &\leq d_2 I''(\xi) + \beta (K*I)(\xi) - (\mu_2+\gamma) I(\xi). \end{aligned}$$

Then $\omega(x,t) = I(x+ct) > 0$ satisfies

$$\begin{cases} \dfrac{\partial \omega(x,t)}{\partial t} \leq d_2 \Delta \omega(x,t) + \beta \int_0^\infty \int_R K(y,s) \Phi(x-y,t-s) \mathrm{d}y\mathrm{d}s - (\mu_2+\gamma)\omega(x,t), x \in R, t > 0, \\ \omega(x,s) = I(x+cs) > 0, s \in (-\infty, 0], x \in R. \end{cases}$$

Let $\omega_0 = \sup_{\xi\in\mathbb{R}} I(\xi)$, $\xi = x + cs$ and it is easy to see that $\omega_0 > 0$. Next, we consider the following initial value problem

$$\begin{cases} \dfrac{\partial \omega(x,t)}{\partial t} = d_2 \Delta \omega(x,t) + \beta \int_0^\infty \int_R K(y,s) \Phi(x-y,t-s) \mathrm{d}y\mathrm{d}s - (\mu_2+\gamma)\omega(x,t), x \in R, t > 0, \\ \omega(x,s) = \omega_0, s \in (-\infty, 0], x \in R. \end{cases}$$

Define

$$\bar{\omega}(t) := \min\left\{2\omega_0, 2\omega_0 \mathrm{e}^{-\rho t}\right\}, t > 0.$$

If $\rho > 0$, then

$$\beta \int_0^\infty \int_R K(y,s) \mathrm{e}^{\rho s} \mathrm{d}y\mathrm{d}s < \mu_2 + \gamma.$$

If $\rho < 0$, then

$$\beta < \mu_2 + \gamma.$$

In view of the comparison principle, we obtain that

$$\limsup_{t\to\infty} \sup_{x\in R} \omega(x,t) \leq \lim_{t\to\infty} \bar{\omega}(t) = 0.$$

Moreover,

$$I(x,t) \leq \omega(x,t), \quad t > 0. \tag{3.4}$$

By Equation (3.4) and the invariant form of $I(\xi)$, we obtain $I(\xi) \equiv 0$. This leads to a contradiction. We complete the proof.

4. Conclusions

In this paper, we investigate the existence and non-existence of traveling wave solutions of a SIR model with external supplies and non-local delays. This result is determined by the basic reproduction number R_0 [24]-[26] and the minimum wave speed c^* [27] [28] of the corresponding ordinary differential equations. We prove the existence of traveling wave solutions for system (2.1) using the upper and lower solutions method combined with Schauder's fixed-point theorem. We prove the non-existence of traveling wave solutions by using the comparison principle and the asymptotic propagation theory.

By the theory of limits, we proved the asymptotic behavior of the traveling wave solution $\left(S(\xi),I(\xi)\right)$ when $\xi\to-\infty$. However, due to the difficulty in constructing Lyapunov functions [29], we have not been able to prove the asymptotic behavior of the traveling wave solution $\left(S(\xi),I(\xi)\right)$ at $\xi\to+\infty$ and the stability of traveling wave solutions [30]. We leave it for future research.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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