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Theoretically Expected Phase Portrait of an Initial Value Problem with Periodic Piecewise Continuous Functions

Hugo Ramón Olmedo Mareco

Independent Researcher, Luque, Paraguay
Email: hrom2010@gmail.com

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Abstract

This work predicts theoretically the phase portrait including the existence and uniqueness of ω-limit cycle for an initial value problem $y(0) = y_0$ with an ordinary differential equation $y' + py = f$ in which p and f are L-periodic piecewise continuous functions.

Keywords

Ordinary Linear Nonhomogeneous Differential Equation, One-Sided Limits

1. Introduction

In shortly words, Dynamical systems study the long-term behavior of evolving systems [1] [2]. The dominant way of modeling how such systems change is by means of differential equations and vice versa. The theory of Dynamical systems has applications in a wide variety of fields as Mathematics, Physics, Biology, Chemistry, Engineering, Economics, and Medicine [3]. Coupled mass-spring systems [4], the nonlinear pendulum, the homoclinic tangle [5], population growth and logistic equation [6], Van der Pol oscillator, Lorenz weather model [7], billiards, cellular automaton [8], model of chemical reactions [9] [10], Rössler equations and Chua's circuit [11] are some examples of dynamical systems. Most simple differential equation is the ordinary first order linear differential equations. This paper will concern about the initial value problem given in the form

$$\begin{aligned} y' + py &= f \\ y(0) &= y_0 \end{aligned} \tag{1.1}$$

Functions p and f are L-periodic piecewise continuous. To detail the problem, some prior concepts will be necessary. A Dynamical system is a system whose state is uniquely specified by a set of variables and whose behavior is described by predefined rules. According to our initial value problems at hands, the state will be given by some representation such that involves $y = y(x)$ and its total derivative or its rate of change $\frac{\mathrm{d}y}{\mathrm{d}x} =: y'$. Then, what's the state in our present problem? We will define the basic element of the set of variables, the vector state of

$$\boldsymbol{\psi} := \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \boldsymbol{\alpha}\psi_1 + \boldsymbol{\beta}, \quad \psi_1 := y, \quad \psi_2 := y'. \tag{1.2}$$

in which it was defined

$$\boldsymbol{\alpha} := \begin{pmatrix} 1 \\ -p \end{pmatrix}, \quad \boldsymbol{\beta} := \begin{pmatrix} 0 \\ f \end{pmatrix} \tag{1.3}$$

To the vector state $\boldsymbol{\psi}$ is associated the point (y, y') in some Cartesian real plane. This plane is often referred as Poincaré phase plane. More generally,

$$\boldsymbol{\psi}' = \begin{pmatrix} 0 & 1 \\ -p' & -p \end{pmatrix} \boldsymbol{\psi} + \boldsymbol{\beta}' \tag{1.4}$$

represents the rule of how this dynamical system's state changes with initial condition

$$\boldsymbol{\psi}(0) = \begin{pmatrix} y_0 \\ y_0'^{+} \end{pmatrix}. \tag{1.5}$$

Now the solutions $\boldsymbol{\psi}$ can be represented by curves in the Poincaré plane or phase space. The curve is called an orbit, trajectory, or path that satisfies the evolution's equation write before. Roughly, if the trajectory known as a ω-limit cycle exists, then trajectory is "closed" and isolated. Neighboring trajectories tend toward it in a spiral, either moving away from or toward the cycle. We will determine the existence and uniqueness of limit cycle. To understand the research, it will give the theoretical framework and mathematical model. Consider the set $\mathbb{R}$ of the real numbers, its subsets $\mathbb{N} = \{0, 1, 2, \cdots\}$, $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$, N a finite positive number of the set $\mathbb{N}$ and

$$(s_1, \cdots, s_N) \tag{1.6}$$

an N-tuple of points in $\mathbb{R}^+$. Let

$$(a_n) \tag{1.7}$$

a N-periodic sequence from $\mathbb{N}$ into $\mathbb{R}$ of minimal period N

$$a_{n+N} = a_n \tag{1.8}$$

such that

$$a_n = s_{n+1} \tag{1.9}$$

if the value of n is between 0 and $N-1$ inclusive of both values. Via the principle of mathematical induction together with the above equation, the next useful result may be deduced

$$a_{n+mN} = a_n \tag{1.10}$$

for all $m \in \mathbb{N}$. By induction on *m*. For $m = 0$ the before statement is trivially true. For each number *m*, $a_{n+(m+1)N} = a_{(n+mN)+N} = a_{n+mN} = a_n$ is true whenever $a_{n+mN} = a_n$. Now, let's look to the following basic tool in the sequel

$$(x_n) \tag{1.11}$$

a sequence from $\mathbb{N}$ into $\mathbb{R}$ such that

$$x_0 := 0 \tag{1.12}$$

and

$$x_n := x_{n-1} + a_{n-1} \tag{1.13}$$

if $n > 0$. Considering previous function (x_n), by induction on *n* any term x_n may be written as

$$x_n = \begin{cases} x_0 & n = 0 \\ \sum_{k=0}^{n-1} a_k & n > 0 \end{cases}. \tag{1.14}$$

To prove this last statement takes in account that the case $n = 0$ is trivially true. Assume that given statement is also true for the number $n = j$. Then to stablish the validity of statement for $n = j+1$: $x_{j+1} = x_j + a_j = \sum_{k=0}^{j-1} a_k + a_j = \sum_{k=0}^{(j+1)-1} a_k$. According to the last result, it may be deduced for all $m \in \mathbb{N}$

$$x_{n+mN} = x_n + mL. \tag{1.15}$$

where

$$L := x_N. \tag{1.16}$$

By induction technique on *m*:

$$\begin{aligned} x_{n+N} &= \sum_{k=0}^{n+N-1} a_k = \sum_{k=0}^{N-1} a_k + \sum_{k=N}^{n+N-1} a_k = \sum_{l=0}^{n-1} a_{l+N} + \sum_{k=0}^{N-1} a_k \\ &= \sum_{l=0}^{n-1} a_l + \sum_{k=0}^{N-1} a_k = x_n + x_N = x_n + L, \end{aligned} \tag{1.17}$$

$$x_{n+(m+1)N} = x_{(n+mN)+N} = x_{n+mN} + L = (x_n + mL) + L = x_n + (m+1)L. \tag{1.18}$$

It is appropriate to introduce open real intervals and the relation of congruence modulo *N* on $\mathbb{N}$. The open real intervals of length a_n will be denoted by

$$\langle a_n \rangle := \{ t \in \mathbb{R} : x_n < t < x_{n+1} \}. \tag{1.19}$$

The Euclid's division algorithm states that given integers n, N, there exist unique numbers $m, j \in \mathbb{N}$ such that

$$n = mN + j \tag{1.20}$$

in which the possible value of the remainder is

$$j = 0, 1, \cdots, N-1. \tag{1.21}$$

The quotient set of $\mathbb{N}$ by the congruence modulo *N*

$$\{[0], [1], \cdots, [j], [N-1]\} \tag{1.22}$$

is the set of congruence class modulo N in which an equivalence class is given by

$$[j] = \{n \in \mathbb{N} : n = mN + j \text{ for some } m \in \mathbb{N}\} \tag{1.23}$$

with the remainder j as its representative. Now, for every value of the representative j it will define L-periodic piecewise continuous functions

$$p_j, f_j \tag{1.24}$$

from

$$\bigcup_{n \in [j]} \langle a_n \rangle = \bigcup_{m=0}^{\infty} \langle a_{mN+j} \rangle \tag{1.25}$$

into the real number system such that if

$$x \in \langle a_{n \in [j]} \rangle \tag{1.26}$$

both functions p_j and f_j have one-sided limits at x_n and x_{n+1}

$$\begin{aligned} \lim_{x \to x_n^+} p_j(x) &=: p_j^+(x_n) \quad \lim_{x \to x_{n+1}^-} p_j(x) =: p_j^-(x_{n+1}) \\ \lim_{x \to x_n^+} f_j(x) &=: f_j^+(x_n) \quad \lim_{x \to x_{n+1}^-} f_j(x) =: f_j^-(x_{n+1}) \end{aligned} \tag{1.27}$$

Then the L-periodic piecewise continuous functions appear

$$p, f : \bigcup_{n \in \mathbb{N}} \langle a_n \rangle \to \mathbb{R} \tag{1.28}$$

such that

$$p(x) = p_j(x) \quad f(x) = f_j(x) \quad x \in \langle a_{n \in [j]} \rangle \tag{1.29}$$

subject to the condition

$$\int_0^L p(t)\,\mathrm{d}t := \lambda > 0. \tag{1.30}$$

So, this work will treat about the phase portrait for the periodic piecewise continuous boundary value problem

$$\begin{aligned} &y'(x) + p(x)y(x) = f(x) \\ &y(x_0) = y_0 \end{aligned} \tag{1.31}$$

where the first derivative is $y' = \dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $x_0 = 0$.

Making a change of variable via the variable ξ,

$$0 \le \xi := x - x_n \le a_n \tag{1.32}$$

the one-sided limits of the function p (similar for f) may be written as follow

$$p_n^+ = p^+(x_n) = \lim_{x \to x_n^+} p(x) = \lim_{x \to x_n^+} p_{j \in [n]}(x) = \lim_{\xi \to 0^+} p_j(x_n + \xi) = \tilde{p}_j^+(\xi = 0), \tag{1.33}$$

$$p_n^- = p^-(x_n) = \lim_{x \to x_n^-} p(x) = \lim_{x \to x_n^-} p_{j^* \in [n-1]}(x) = \lim_{\xi \to a_{n-1}^-} p_{j^*}(x_{n-1} + \xi) = \tilde{p}_{j^*}^-(\xi = a_{n-1}). \tag{1.34}$$

As immediate consequence of the periodicity any open intervals $\langle a_{n \in [j]} \rangle$ and $\langle a_j \rangle$ have the same length

$$a_n = x_{n+1} - x_n = a_{n = j + mN} = x_{(j+1)+mN} - x_{j+mN} = x_{j+1} - x_j = a_j, \tag{1.35}$$

the functions p_j, f_j satisfies that

$$p_j\left(x_n+\xi\right)=p_j\left(x_j+\xi\right) \quad f_j\left(x_n+\xi\right)=f_j\left(x_j+\xi\right) \tag{1.36}$$

and

$$\lambda=\int_{x_{mN}}^{x_{mN+N}} p(t)\mathrm{d}t . \tag{1.37}$$

Therefore for any open real interval $\langle a_n \rangle$ is equivalent to only one of the element of the set

$$\left\{\langle a_0\rangle,\langle a_1\rangle,\cdots,\langle a_j\rangle,\cdots,\langle a_{N-1}\rangle\right\}. \tag{1.38}$$

2. Discrete-Continuous Solution

In this section we will find the solution

$$\boldsymbol{\psi}=\boldsymbol{\alpha}y+\boldsymbol{\beta} . \tag{2.1}$$

However, we must do some observations more. First remarks, if we know the function $y(x)$ then it is determined the vector $\boldsymbol{\psi}$. Second point of view, our same problem

$$\begin{aligned} &y'(x)+p(x)y(x)=f(x)\\ &y(x_0)=y_0 \end{aligned} \tag{2.2}$$

may be formulated as initial value problem in every successive interval

$$\begin{aligned} &y'(x)+p(x)y(x)=f(x)\\ &y(x_n)=y_n \end{aligned}. \tag{2.3}$$

The general solution $y(x)$ of this well-known differential equation inside any open interval $\langle a_{n\in[j]}\rangle \ni x$ is given by the "**continuous solution**"

$$y(x)=y_n R(x)+S(x), \tag{2.4}$$

From this last equation for $x\in\langle a_{n\in[j]}\rangle$ we will use the integrals

$$R(x):=\mathrm{e}^{-\int_{x_n}^{x} p(t)\mathrm{d}t} \tag{2.5}$$

and

$$S(x):=\int_{x_n}^{x}\mathrm{e}^{-\int_s^x p(t)\mathrm{d}t} f(s)\mathrm{d}s . \tag{2.6}$$

The evaluation of piecewise *L*-periodic functions $R(x)$, $S(x)$ in $x=x_{n+1}=x_n+a_n$ gives us

$$y_{n+1}=y_n R_n+S_n , \tag{2.7}$$

where

$$R_{n=mN+j}:=\mathrm{e}^{-\int_{x_n}^{x_{n+1}} p(t)\mathrm{d}t}=\mathrm{e}^{-\int_0^{a_n} p(x_n+\omega)\mathrm{d}\omega}=\mathrm{e}^{-\int_0^{a_j} p(x_j+\omega)\mathrm{d}\omega}=\mathrm{e}^{-\int_{x_j}^{x_{j+1}} p(t)\mathrm{d}t}=R_j \tag{2.8}$$

together

$$\begin{aligned} S_{n=mN+j}&:=\int_{x_n}^{x_{n+1}}\mathrm{e}^{-\int_s^{x_{n+1}} p(t)\mathrm{d}t} f(s)\mathrm{d}s=\int_0^{a_n}\mathrm{e}^{-\int_\theta^{a_n} p(x_n+\lambda)\mathrm{d}\lambda} f\left(x_n+\theta\right)\mathrm{d}\theta\\ &=\int_0^{a_j}\mathrm{e}^{-\int_\theta^{a_j} p(x_j+\lambda)\mathrm{d}\lambda} f\left(x_j+\theta\right)\mathrm{d}\theta=S_j. \end{aligned} \tag{2.9}$$

To calculate the phase portrait it will need

$$\boldsymbol{\psi} := \begin{pmatrix} y \\ y' \end{pmatrix} = \boldsymbol{\alpha} y + \boldsymbol{\beta}, \tag{2.10}$$

but to know $y = y_n R + S$ we must find y_n, $R(x)$, $S(x)$ inside the interval $\langle a_{n\in[j]} \rangle \ni x$. To compute y_n it will require of every previous initial condition $y_{n-1}, \cdots, y_0$ using the recurrence tools $y_{n+1} = y_n R_n + S_n$. Therefore, the sequence "**discrete solution**"

$$\left(y_0, \cdots, y_j, \cdots, y_{N-1}, y_N, y_{N+1}, \cdots, y_{n=mN+j}, \cdots\right) = \left(y_n = y_n\left(y_0\right)\right) \tag{2.11}$$

will complete required information with the discrete function $y_n = y_n(y_0)$. Next sections will be dedicated to obtaining an expression for y_n. Also it will take in account the discontinuities. Using the one-sided limits to define the following quantities for $y(x)$,

$$y_n^+ = y^+\left(x_n\right) = \lim_{x \to x_n^+} y(x) = \lim_{\xi \to 0^+} y\left(x_n + \xi\right) = \tilde{y}^+(0), \tag{2.12}$$

$$y_{n>0}^- = y^-\left(x_n\right) = \lim_{x \to x_n^-} y(x) = \lim_{\eta \to a_{n-1}^-} y\left(x_{n-1} + \eta\right) = \tilde{y}^-\left(a_{n-1}\right). \tag{2.13}$$

therefore the continuity of this solution $y(x)$ is written as

$$y_0 = y_0^+ \tag{2.14}$$

$$y_n^- = y_n = y_n^+ \tag{2.15}$$

For the first derivative $y'(x)$ it is possible to deduce with the differential equation of the boundary value problem that

$$y_n'^+ = y'^+\left(x_n\right) = \lim_{x \to x_n^+} y'(x) = -p_n^+ y_n + f_n^+, \tag{2.16}$$

$$y_{n>0}'^- = y'^-\left(x_n\right) = \lim_{x \to x_n^-} y'(x) = -p_n^- y_n + f_n^-. \tag{2.17}$$

A Heuristic-Discrete Sequence

This section is dedicated to achieving

$$y_n = y_n\left(y_0\right) \tag{2.1.1}$$

using the recurrence relation

$$y_{n+1} = y_n R_n + S_n. \tag{2.1.2}$$

We begin for some first value of *n*, for example, for the first six $y_{n=1,2,\cdots,6}$ we have

$$y_1 = y_0 R_0 + S_0 = y_0 T_{01} + V_1 T_{11},$$

$$y_2 = y_1 R_1 + S_1 = y_0 R_0 R_1 + S_0 R_1 + S_1 = y_0 R_{01} + S_0 R_1 + S_1 1 = y_0 T_{02} + V_1 T_{12} + V_2 T_{22},$$

$$y_3 = y_0 R_0 R_1 R_2 + S_0 R_1 R_2 + S_1 R_2 + S_2 = y_0 R_{0.2} + S_0 R_{12} + S_1 R_2 + S_2 1 = y_0 T_{03} + V_1 T_{13} + V_2 T_{23} + V_3 T_{33},$$

$$y_4 = y_0 R_{0.3} + S_0 R_{1.3} + S_1 R_{23} + S_2 R_3 + S_3 1 = y_0 T_{04} + V_1 T_{14} + V_2 T_{24} + V_3 T_{34} + V_4 T_{44},$$

$$y_5 = y_0 T_{05} + V_1 T_{15} + V_2 T_{25} + V_3 T_{35} + V_4 T_{45} + V_5 T_{55},$$

$$y_6 = y_0 T_{06} + \sum_{\mu=1}^{5} V_\mu T_{\mu 6} + V_6 T_{66} = y_0 T_{06} + \sum_{\mu=1}^{6} V_\mu T_{\mu 6},$$

preceding steps suggests the expression

$$y_{n+1}^* = y_0 \overbrace{T_{0.n+1}}^{R_{0.n}} + \sum_{\mu=1}^{n} \overbrace{V_\mu}^{S_{\mu-1}} \overbrace{T_{\mu.n+1}}^{R_{\mu n}} + \overbrace{V_{n+1}}^{S_n} \overbrace{T_{n+1.n+1}}^{1} = y_0 T_{0.n+1} + V_\mu T_{\mu.n+1} \tag{2.1.3}$$

in which emerges the following expressions

$$\mu, \nu, \kappa \in \mathbb{N} \quad V_{\mu+1} := S_\mu \quad T_{\mu\nu} := \begin{cases} 0 & \mu > \nu \\ 1 & \mu = \nu \\ R_{\mu.\nu-1} := \prod_{\kappa=\mu}^{\nu-1} R_\kappa & \mu < \nu \end{cases}. \tag{2.1.4}$$

$$V_\mu T_{\mu\nu} = \begin{cases} 0 & \nu = 0 \\ \sum_{\mu=1}^{\nu} V_\mu T_{\mu\nu} & \nu > 0 \end{cases}$$

Without the application of periodicity of R_n and S_n, it holds using the Einstein's index mute's convention on summation

$$y_{l=0,1,\cdots,N-1,N} = y_0 T_{0l} + V_\mu T_{\mu l} = \begin{cases} y_{l=j=0,1,\cdots,N-1} = y_0 T_{0j} + V_\mu T_{\mu j} \\ y_{l=N} = y_0 r + V_\mu T_{\mu N} \end{cases}. \tag{2.1.5}$$

The case $y_{l=N} = y_0 T_{0N} + V_\mu T_{\mu N}$ is special relevant for the future, in which T_{0N} begin to appear the parameter r, very important for the existence of the limit cycle

$$0 < r := T_{0N} = \prod_{\kappa=0}^{N-1} \mathrm{e}^{-\int_{x_\kappa}^{x_{\kappa+1}} p(t)\mathrm{d}t} = \mathrm{e}^{-\sum_{\kappa=0}^{N-1}\int_{x_\kappa}^{x_{\kappa+1}} p(t)\mathrm{d}t} = \mathrm{e}^{-\int_{x_0=0}^{x_N=L} p(t)\mathrm{d}t} = \mathrm{e}^{-\lambda} < 1. \tag{2.1.6}$$

The proof that $y_l = y_0 T_{0l} + V_\mu T_{\mu l}$ will based in the induction on $j = 0,1,\cdots,N-1$ applied on $y_j^* = y_0 T_{0j} + V_\mu T_{\mu j}$,

$$y_0^* = y_0 \overbrace{T_{00}}^{1} + \overbrace{V_\mu T_{\mu 0}}^{0} = y_0$$

$$y_{j+1} = y_j R_j + S_j = y_j^* R_j + S_j = \left\{ \begin{matrix} \left(y_0 T_{0j} + V_\mu T_{\mu j}\right) R_j + S_j \\ y_0 R_{0.j-1} R_j + V_\mu T_{\mu j} R_j + V_{j+1} \\ y_0 R_{0.j} + \sum_{\mu=1}^{j} V_\mu T_{\mu.j+1} + V_{j+1} T_{j+1.j+1} \\ y_0 T_{0.j+1} + \sum_{\mu=1}^{j+1} V_\mu T_{\mu.j+1} \end{matrix} \right\}$$

$$= y_0 T_{0.j+1} + V_\mu T_{\mu.j+1} = y_{j+1}^*$$

Now it will search an expression for y_{N+l} in a similar way than before applying the periodicity of $R_n = R_j$ and $S_n = S_j$

$$y_{N+1} = \left\{ \begin{matrix} y_N R_N + S_N = \left(y_0 r + V_\mu T_{\mu N}\right) R_0 + S_0 = y_0 r R_0 + V_\mu T_{\mu N} R_0 + S_0 \\ y_0 r T_{01} + V_\mu T_{\mu N} R_0 + V_1 = y_0 r T_{01} + V_\mu T_{\mu N} R_0 + \sum_{\mu=1}^{N} V_\mu \delta_{\mu 1} T_{11} \\ y_0 r T_{01} + V_\mu \left(T_{\mu N} T_{01} + \delta_{\mu 1} T_{11}\right) \end{matrix} \right\},$$

$$y_{N+2} = \left\{ \begin{matrix} y_{N+1} R_{N+1} + S_{N+1} = \left(y_0 r R_0 + V_\mu \left(T_{\mu N} R_0 + \delta_{\mu 1}\right)\right) R_1 + S_1 = y_0 r R_{0.1} + V_\mu \left(T_{\mu N} R_{0.1} + \delta_{\mu 1} R_1\right) + S_1 \\ y_0 r T_{02} + V_\mu \left(T_{\mu N} T_{02} + \delta_{\mu 1} T_{12}\right) + V_2 = y_0 r T_{02} + V_\mu \left(T_{\mu N} T_{02} + \delta_{\mu 1} T_{12}\right) + \sum_{\mu=1}^{N} V_\mu \delta_{\mu 2} T_{22} \\ y_0 r T_{02} + V_\mu \left(T_{\mu N} T_{02} + \delta_{\mu 1} T_{12} + \delta_{\mu 2} T_{22}\right) \end{matrix} \right\},$$

$$\left\{\begin{array}{c} y_{N+3} \\ = \\ y_{N+2}R_{N+2}+S_{N+2} \\ \left(y_0rR_{0.1}+V_\mu\left(T_{\mu N}R_{0.1}+\delta_{\mu 1}R_1+\delta_{\mu 2}\right)\right)R_2+S_2=y_0rR_{0.2}+V_\mu\left(T_{\mu N}R_{0.2}+\delta_{\mu 1}R_{12}+\delta_{\mu 2}R_2\right)+S_2 \\ y_0rR_{0.2}+V_\mu\left(T_{\mu N}R_{0.2}+\delta_{\mu 1}R_{12}+\delta_{\mu 2}R_2\right)+V_3=y_0rR_{0.2}+V_\mu\left(T_{\mu N}R_{0.2}+\delta_{\mu 1}R_{12}+\delta_{\mu 2}R_2\right)+\sum_{\mu=1}^{N}V_\mu\delta_{\mu 3}T_{33} \\ y_0rT_{03}+V_\mu\left(T_{\mu N}T_{03}+\delta_{\mu 1}T_{13}+\delta_{\mu 2}T_{23}+\delta_{\mu 3}T_{33}\right) \end{array}\right\},$$

$$y^*_{N+n+1}=\left\{\begin{array}{c} y_0rT_{0.n+1}+V_\mu\left(T_{\mu N}T_{0.n+1}+\delta_{\mu 1}T_{1.n+1}+\delta_{\mu 2}T_{2.n+1}+.+\delta_{\mu.n+1}T_{n+1.n+1}\right) \\ y_0rT_{0.n+1}+V_\mu\left(T_{\mu N}T_{0.n+1}+\delta_{\mu 1}T_{1.n+1}+.+\delta_{\mu l}T_{l.n+1}+.+\delta_{\mu.n+1}T_{n+1.n+1}\right) \\ y_0rT_{0.n+1}+V_\mu\left(T_{\mu N}T_{0.n+1}+\delta_{\mu\kappa}T_{\kappa.n+1}\right) \end{array}\right\}.$$

Then by periodicity surges the expression

$$y_{N+l}=y_0rT_{0l}+V_\mu\left(T_{\mu N}T_{0l}+\delta_{\mu\kappa}T_{\kappa l}\right) \tag{2.1.7}$$

$$y_{N+l}=y_0rT_{0l}+V_\mu\left(T_{\mu N}T_{0l}+\delta_{\mu\kappa}T_{\kappa l}\right)=\begin{cases} y_{N+j}=y_0rT_{0j}+V_\mu\left(T_{\mu N}T_{0j}+\delta_{\mu\kappa}T_{\kappa j}\right) \\ y_{2N}=y_0r^2+V_\mu u_2T_{\mu N} \end{cases}. \tag{2.1.8}$$

The case $n=2N$ is deduced so

$$\begin{aligned} y_{N+N}=y_{2N}&=y_0rT_{0N}+V_\mu\left(T_{\mu N}T_{0N}+\delta_{\mu\kappa}T_{\kappa N}\right)=y_0rr+V_\mu\left(T_{\mu N}r+T_{\mu N}\right) \\ &=y_0r^2+V_\mu(r+1)T_{\mu N}=y_0r^2+V_\mu u_2T_{\mu N}, \end{aligned}$$

it was introduced $u_2=r+1$ and it begin to appear the Kronecker's delta

$$\delta_{\mu\kappa}=\begin{cases}0 & \mu\neq\kappa \\ 1 & \mu=\kappa\end{cases} \quad \delta_{\mu\kappa}T_{\kappa\nu}=\begin{cases}0 & \nu=0 \\ \sum_{\kappa=1}^{\nu}\delta_{\mu\kappa}T_{\kappa\nu} & \nu>0\end{cases}. \tag{2.1.9}$$

Proof of $y_{N+l}=y_0rT_{0l}+V_\mu\left(T_{\mu N}T_{0l}+\delta_{\mu\kappa}T_{\kappa l}\right)$ it is made by induction's method on the expression $y^*_{N+j}=y_0rT_{0j}+V_\mu\left(T_{\mu N}T_{0j}+\delta_{\mu\kappa}T_{\kappa j}\right)$. Then

$$y^*_N=y_0r\overbrace{T_{00}}^{1}+V_\mu\left(T_{\mu N}\overbrace{T_{00}}^{1}+\overbrace{\delta_{\mu\kappa}T_{\kappa 0}}^{0}\right)=y_0r+V_\mu T_{\mu N}=y_N$$

$$y_{N+(j+1)}=y_{N+j}R_{N+j}+S_{N+j}$$

$$=\left\{\begin{array}{c} \left(y_0rT_{0j}+V_\mu\left(T_{\mu N}T_{0j}+\delta_{\mu\kappa}T_{\kappa j}\right)\right)R_j+S_j \\ \left(y_0rR_{0.j-1}+V_\mu\left(T_{\mu N}R_{0.j-1}+\delta_{\mu\kappa}T_{\kappa j}\right)\right)R_j+S_j \\ y_0rR_{0.j-1}R_j+V_\mu\left(T_{\mu N}R_{0.j-1}+\delta_{\mu\kappa}T_{\kappa j}\right)R_j+S_j \\ y_0rR_{0.j-1}R_j+V_\mu\left(T_{\mu N}R_{0.j-1}R_j+\delta_{\mu\kappa}T_{\kappa j}R_j\right)+S_j \\ y_0rR_{0.j}+V_\mu\left(T_{\mu N}R_{0.j}+\delta_{\mu\kappa}T_{\kappa j}R_j\right)+V_{j+1} \\ y_0rT_{0.j+1}+V_\mu\left(T_{\mu N}T_{0.j+1}+\delta_{\mu\kappa}T_{\kappa j}R_j\right)+\sum_{\mu=1}^{N}V_\mu\delta_{\mu.j+1}T_{j+1.j+1} \\ y_0rT_{0.j+1}+V_\mu\left(T_{\mu N}T_{0.j+1}+\sum_{\kappa=1}^{j}\delta_{\mu\kappa}T_{\kappa.j+1}+\delta_{\mu.j+1}T_{j+1.j+1}\right) \\ y_0rT_{0.j+1}+V_\mu\left(T_{\mu N}T_{0.j+1}+\delta_{\mu\kappa}T_{\kappa.j+1}\right) \end{array}\right\}=y^*_{N+(j+1)}$$

Now it will search an expression for y_{2N+l}

$$y_{2N+1}=\begin{Bmatrix} y_{2N}R_{2N}+S_{2N}=\left(y_0r^2+V_\mu T_{\mu N}u_2\right)R_0+S_0=y_0r^2R_0+V_\mu u_2T_{\mu N}R_0+S_0 \\ y_0r^2T_{01}+V_\mu u_2T_{\mu N}R_0+V_1=y_0r^2T_{01}+V_\mu u_2T_{\mu N}R_0+V_\mu\delta_{\mu 1}T_{11} \\ y_0r^2T_{01}+V_\mu\left(T_{\mu N}u_2T_{01}+\delta_{\mu 1}T_{11}\right)\end{Bmatrix},$$

$$y_{2N+2}=\begin{Bmatrix} y_{2N+1}R_{2N+1}+S_{2N+1}=\left(y_0r^2R_0+V_\mu\left(T_{\mu N}u_2R_0+\delta_{\mu 1}\right)\right)R_1+S_1=y_0r^2R_{0.1}+V_\mu\left(T_{\mu N}u_2R_{0.1}+\delta_{\mu 1}R_1\right)+S_1 \\ y_0r^2T_{02}+V_\mu\left(T_{\mu N}u_2T_{02}+\delta_{\mu 1}T_{12}\right)+V_2=y_0r^2T_{02}+V_\mu\left(T_{\mu N}u_2T_{02}+\delta_{\mu 1}T_{12}\right)+V_\mu\delta_{\mu 2}T_{22} \\ y_0r^2T_{02}+V_\mu\left(T_{\mu N}u_2T_{02}+\delta_{\mu 1}T_{12}+\delta_{\mu 2}T_{22}\right)\end{Bmatrix},$$

$$\begin{Bmatrix} y_{2N+3} \\ = \\ y_{2N+2}R_{2N+2}+S_{2N+2} \\ \left(y_0r^2R_{0.1}+V_\mu\left(T_{\mu N}u_2R_{0.1}+\delta_{\mu 1}R_1+\delta_{\mu 2}\right)\right)R_2+S_2=y_0r^2R_{0.2}+V_\mu\left(T_{\mu N}u_2R_{0.2}+\delta_{\mu 1}R_{12}+\delta_{\mu 2}R_2\right)+S_2 \\ y_0r^2R_{0.2}+V_\mu\left(T_{\mu N}u_2R_{0.2}+\delta_{\mu 1}R_{12}+\delta_{\mu 2}R_2\right)+V_3=y_0r^2R_{0.2}+V_\mu\left(T_{\mu N}u_2R_{0.2}+\delta_{\mu 1}R_{12}+\delta_{\mu 2}R_2\right)+V_\mu\delta_{\mu 3}T_{33} \\ y_0r^2T_{03}+\sum_{\mu=1}^{N}V_\mu\left(T_{\mu N}u_2T_{03}+\delta_{\mu 1}T_{13}+\delta_{\mu 2}T_{23}+\delta_{\mu 3}T_{33}\right)\end{Bmatrix},$$

$$y^*_{2N+n+1}=\begin{Bmatrix} y_0r^2T_{0.n+1}+V_\mu\left(T_{\mu N}u_2T_{0.n+1}+\delta_{\mu 1}T_{1.n+1}+\delta_{\mu 2}T_{2.n+1}+\cdots+\delta_{\mu.n+1}T_{n+1.n+1}\right) \\ y_0r^2T_{0.n+1}+V_\mu\left(T_{\mu N}u_2T_{0.n+1}+\delta_{\mu 1}T_{1.n+1}+\cdots+\delta_{\mu\kappa}T_{\kappa.n+1}+\cdots+\delta_{\mu.n+1}T_{n+1.n+1}\right) \\ y_0r^2T_{0.n+1}+V_\mu\left(T_{\mu N}u_2T_{0.n+1}+\delta_{\mu\kappa}T_{\kappa.n+1}\right)\end{Bmatrix}.$$

Then the steps before suggests

$$y^*_{2N+l}=y_0r^2T_{0l}+V_\mu\left(T_{\mu N}u_2T_{0l}+\delta_{\mu\kappa}T_{\kappa l}\right). \tag{2.1.10}$$

Therefore the general expression that arise of the whole work until here is resumed in

$$y_{n=mN+l}=y_0r^mT_{0l}+V_\mu\left(T_{\mu N}u_mT_{0l}+\delta_{\mu\kappa}T_{\kappa l}\right) \tag{2.1.11}$$

where come up

$$u_m:=\begin{cases}0 & m=0 \\ \sum_{\kappa=0}^{m-1}r^\kappa=u\left(1-r^m\right) & m>0\end{cases} \tag{2.1.12}$$

that includes the sum of m term $\sum_{\kappa=0}^{m-1}r^\kappa$ of the geometric progression whose first term is the unity and common ratio r, $0<r=\mathrm{e}^{-\lambda}<1$. The set $\{0,1,1+r,1+r+r^2,\cdots,u_m,\cdots\}$ will converges to

$$u:=\lim_{m\to\infty}\frac{1-r^m}{1-r}=\frac{1}{1-r} \tag{2.1.13}$$

and the same number u represents the limit of the geometric power series $\sum_{\kappa=0}^{\infty}r^\kappa$. Proof of the validity of $y_{n=mN+l}=y_0r^mT_{0l}+V_\mu\left(T_{\mu N}u_mT_{0l}+\delta_{\mu\kappa}T_{\kappa l}\right)$ is via the induction's method on $y^*_{n=mN+j}=y_0r^mT_{0j}+V_\mu\left(T_{\mu N}u_mT_{0j}+\delta_{\mu\kappa}T_{\kappa j}\right)$:

$$y^*_{n=0}=y_0\overbrace{r^0}^{1}\overbrace{T_{00}}^{1}+V_\mu\left(T_{\mu N}\overbrace{u_0}^{0}T_{00}+\overbrace{\delta_{\mu\kappa}T_{\kappa 0}}^{0}\right)=y_0$$

$$y_{n+1} = y_{n=mN+j} R_{mN+j} + S_{mN+j}$$

$$= \left\{ \begin{array}{c} y^*_{mN+j} R_{mN+j} + S_{mN+j} \\ \left(y_0 r^m T_{0j} + V_\mu \left(T_{\mu N} u_m T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right) \right) R_j + S_j \\ \left(y_0 r^m R_{0.j-1} + V_\mu \left(T_{\mu N} u_m R_{0.j-1} + \delta_{\mu\kappa} T_{\kappa j} \right) \right) R_j + S_j \\ y_0 r^m R_{0.j-1} R_j + V_\mu \left(T_{\mu N} u_m R_{0.j-1} + \delta_{\mu\kappa} T_{\kappa j} \right) R_j + S_j \\ y_0 r^m R_{0.j-1} R_j + V_\mu \left(T_{\mu N} u_m R_{0.j-1} R_j + \delta_{\mu\kappa} T_{\kappa j} R_j \right) + S_j \\ y_0 r^m R_{0.j} + V_\mu \left(T_{\mu N} u_m R_{0.j} + \delta_{\mu\kappa} T_{\kappa j} R_j \right) + V_{j+1} \\ y_0 r^m T_{0.j+1} + V_\mu \left(T_{\mu N} u_m T_{0.j+1} + \delta_{\mu\kappa} T_{\kappa j} R_j \right) + \sum_{\mu=1}^{N} V_\mu \delta_{\mu.j+1} T_{j+1.j+1} \\ y_0 r^m T_{0.j+1} + V_\mu \left(T_{\mu N} u_m T_{0.j+1} + \sum_{\kappa=1}^{j} \delta_{\mu\kappa} T_{\kappa.j+1} + \delta_{\mu.j+1} T_{j+1.j+1} \right) \\ y_0 r^m T_{0.j+1} + V_\mu \left(T_{\mu N} u_m T_{0.j+1} + \delta_{\mu\kappa} T_{\kappa.j+1} \right) \end{array} \right\} = y^*_{n+1}$$

Nevertheless, in the equation $y_{n=mN+l} = y_0 r^m T_{0l} + V_\mu \left(T_{\mu N} u_m T_{0l} + \delta_{\mu\kappa} T_{\kappa l} \right)$ it is enough the restriction $l = j = 0,1,\cdots,N-1$

$$y_{n=mN+j} = y_0 r^m T_{0j} + V_\mu \left(T_{\mu N} u_m T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right) \tag{2.1.14}$$

to be all right because

$$y_{n=mN+l;l=N} = y_{n=(m+1)N;\, j=0}. \tag{2.1.15}$$

To prove this, consider separately $y_{n=mN+l;l=N}$ and $y_{n=(m+1)N;\, j=0}$.

$$\begin{aligned} y_{n=mN+l;l=N} &= y_0 r^m T_{0.l=N} + V_\mu \left(T_{\mu N} u_m T_{0.l=N} + \delta_{\mu\kappa} T_{\kappa.l=N} \right) \\ &= y_0 r^{m+1} + V_\mu \left(T_{\mu N} u_m r + T_{\mu N} \right) \\ &= y_0 r^{m+1} + V_\mu T_{\mu N} \left(u_m r + 1 \right) \end{aligned}$$

$$\begin{aligned} y_{n=(m+1)N;\, j=0} &= y_0 r^{m+1} T_{0.j=0} + V_\mu \left(T_{\mu N} u_{m+1} T_{0.j=0} + \delta_{\mu\kappa} T_{\kappa.j=0} \right) \\ &= y_0 r^{m+1} + V_\mu T_{\mu N} u_{m+1} \end{aligned}$$

But $u_m r + 1 = \frac{1-r^m}{1-r} r + 1 = \frac{r - r^{m+1} + 1 - r}{1-r} = \frac{1-r^{m+1}}{1-r} = u_{m+1}$. So the equation $y_{n=mN+l;l=N} = y_{n=(m+1)N;\, j=0}$ holds. Other observation more

$$y_{n=mN+j;m=0,\, j=0,1,\cdots,N-1} = y_0 T_{0j} + V_\kappa T_{\kappa j}. \tag{2.1.16}$$

The proof of this last statement is

$$y_{n=j=0,1,\cdots,N-1} = \left\{ \begin{array}{c} y_0 \overbrace{r^0}^{1} T_{0j} + V_\mu \left(T_{\mu N} \overbrace{u_0}^{0} T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right) \\ y_0 T_{0j} + V_\mu \delta_{\mu\kappa} T_{\kappa j} \\ y_0 T_{0j} + \sum_{\mu=1}^{N} V_\mu \sum_{\kappa=1}^{j} \delta_{\mu\kappa} T_{\kappa j} \\ y_0 T_{0j} + \sum_{\mu=1}^{j} V_\mu \sum_{\kappa=1}^{j} \delta_{\mu\kappa} T_{\kappa j} + \sum_{\mu=j+1}^{N} V_\mu \underbrace{\sum_{\kappa=1}^{j} \delta_{\mu\kappa} T_{\kappa j}}_{0} \\ y_0 T_{0j} + \sum_{\mu=1}^{j} V_\mu \sum_{\kappa=1}^{j} \delta_{\mu\kappa} T_{\kappa j} \end{array} \right\} = y_0 T_{0j} + V_\kappa T_{\kappa j}.$$

Note that it has been obtained the identity

$$V_\mu \delta_{\mu\kappa} T_{\kappa j} = V_\kappa T_{\kappa j} \,. \tag{2.1.17}$$

Too, for any $n = mN$ it is obtained

$$y_{n=mN} = y_0 r^m \overbrace{T_{00}}^{1} + V_\mu \left(T_{\mu N} u_m T_{00} + \overbrace{\delta_{\mu\kappa} T_{\kappa 0}}^{0} \right) = y_0 r^m + V_\mu u_m T_{\mu N} \,. \tag{2.1.18}$$

It is necessary one step more to arrive the desired equation. For this purpose it is suppose $m > 0$, replacing u_m by $u\left(1 - r^m\right)$ in $y_{n=mN+j}$ and rearranging

$$y_{n=mN+j} = \begin{cases} y_0 r^m T_{0j} + V_\mu \left(T_{\mu N} u_m T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right) \\ y_0 r^m T_{0j} + V_\mu T_{\mu N} u_m T_{0j} + V_\mu \delta_{\mu\kappa} T_{\kappa j} \\ y_0 r^m T_{0j} + V_\mu T_{\mu N} u \left(1 - r^m\right) T_{0j} + V_\mu \delta_{\mu\kappa} T_{\kappa j} \\ y_0 r^m T_{0j} + V_\mu \left(T_{\mu N} u T_{0j} - T_{\mu N} u r^m T_{0j} \right) + V_\mu \delta_{\mu\kappa} T_{\kappa j} \\ y_0 r^m T_{0j} + V_\mu \left(-T_{\mu N} u r^m T_{0j} + T_{\mu N} u T_{0j} \right) + V_\mu \delta_{\mu\kappa} T_{\kappa j} \\ y_0 r^m T_{0j} - V_\mu T_{\mu N} u r^m T_{0j} + V_\mu \left(T_{\mu N} u T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right) \\ \underbrace{\left(\overbrace{y_0 - u V_\mu T_{\mu N}}^{h} \right) T_{0j}}_{\theta_j} r^m + \underbrace{V_\mu \left(T_{\mu N} u T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right)}_{\omega_j} \\ \theta_j r^m + V_\mu T_{\mu N} u T_{0j} + V_\mu \delta_{\mu\kappa} T_{\kappa j} \\ \theta_j r^m + y_0 T_{0j} - \theta_j + V_\kappa T_{\kappa j} \\ \theta_j r^m - \theta_j + y_0 T_{0j} + V_\kappa T_{\kappa j} \\ \theta_j r^m + \underbrace{y_j - \theta_j}_{\omega_j} \end{cases}$$

Finally, we found the basic equation for the remainder work,

$$y_{n;\, j\in[n]} = \theta_j r^m + \omega_j \tag{2.1.19}$$

in which

$$\theta_j := h T_{0j} \quad h := y_0 - \rho \quad \rho := V_\mu u T_{\mu N} \quad \omega_j := V_\mu \left(T_{\mu N} u T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right) = y_j - \theta_j \tag{2.1.20}$$

Observe that the discrete dynamical of the sets $\{y_n\}$ has the property of convergence

$$\lim_{n\to\infty} y_n = \omega_j \,. \tag{2.1.21}$$

Now, taken in account that

$$y = y_n R + S = \left(\theta_j r^m + \omega_j \right) R + S = \theta_j R r^m + \omega_j R + S$$

it is convenient to define

$$\omega = \omega_j R + S \,. \tag{2.1.22}$$

according to this quantity $\omega = \omega(x)$ for $x \in \left\langle a_{n\in[j]} \right\rangle$

$$y = \left(y_0 - \rho \right) T_{0j} r^m R + \omega \,. \tag{2.1.23}$$

Substituting $y = \theta_j R r^m + \omega$ inside our vector state

$$\boldsymbol{\psi} = \boldsymbol{\alpha} y + \boldsymbol{\beta} = \boldsymbol{\alpha}\left(\theta_j R r^m + \omega\right) + \boldsymbol{\beta} = \boldsymbol{\alpha}\theta_j R r^m + \boldsymbol{\alpha}\omega + \boldsymbol{\beta}$$

we introduce

$$\lim_{x\to\infty} \boldsymbol{\psi} = \omega\boldsymbol{\alpha} + \boldsymbol{\beta} =: \boldsymbol{\Omega} \tag{2.1.24}$$

Then our vector state is

$$\boldsymbol{\psi} = \left(y_0 - \rho\right) T_{0j} r^m R \boldsymbol{\alpha} + \boldsymbol{\Omega} . \tag{2.1.25}$$

3. Stability and Periodicity of Solution. Limit Cycle

The theoretical methods previously developed predicts the existence and ensure the uniqueness of the ω-limit cycle. Phase portrait may be constructing associating to every point $\left(y, y'\right)$ via the components $\psi_1 = y$ and $\psi_2 = y'$ of the vector state

$$\boldsymbol{\psi} = \boldsymbol{\psi}\left(x\right) = \left(y_0 - \rho\right) T_{0j} r^m R \boldsymbol{\alpha} + \boldsymbol{\Omega} \tag{3.1}$$

in any interval $\left\langle a_{n\in[j]} \right\rangle \ni x$. In this way the points in the phase plane will trace a curve in such way that

$$\lim_{x\to x_n^+} \boldsymbol{\psi}\left(x\right) = \boldsymbol{\psi}_n^+ = \begin{pmatrix} y_n \\ y_n'^+ \end{pmatrix} \tag{3.2}$$

and

$$\lim_{x\to x_{n+1}^-} \boldsymbol{\psi}\left(x\right) = \boldsymbol{\psi}_{n+1}^- = \begin{pmatrix} y_{n+1} \\ y_{n+1}'^- \end{pmatrix} . \tag{3.3}$$

Now, let's consider the trajectory γ_0 for which the initial condition is

$$y_0 = \rho , \tag{3.4}$$

$$y_n = \omega_j^0 = \rho T_{0j} + V_\kappa T_{\kappa j} , \tag{3.5}$$

$$y = \omega_j^0 R + S , \tag{3.6}$$

and

$$\boldsymbol{\psi}_0 = \boldsymbol{\Omega} . \tag{3.7}$$

We will probe that γ_0 is the limit cycle and there is the only one. But, first we consider the functions previously defined when $x \in \left\langle a_{n\in[j]} \right\rangle$

$$R\left(x\right) := \mathrm{e}^{-\int_{x_n}^{x} p(t)\mathrm{d}t} , \tag{3.8}$$

$$S\left(x\right) := \int_{x_n}^{x} \mathrm{e}^{-\int_{s}^{x} p(t)\mathrm{d}t} f\left(s\right) \mathrm{d}s . \tag{3.9}$$

We will need to use some properties of them. First, these functions are L-periodic

$$R\left(x+L\right) = R\left(x\right) \quad S\left(x+L\right) = S\left(x\right) \tag{3.10}$$

as consequence of the fact that the functions p, f are L-periodic. Another helpful property is related to its lateral limits

$$\lim_{x\to x_n^+} R\left(x\right) = 1 \quad \lim_{x\to x_{n+1}^-} R\left(x\right) = R_n , \tag{3.11}$$

$$\lim_{x\to x_n^+} S(x) = 0 \quad \lim_{x\to x_{n+1}^-} S(x) = S_n . \tag{3.12}$$

To will show that the path γ_0 is L-periodic, we remember that

$$\boldsymbol{\Omega}(x) = \omega(x)\boldsymbol{\alpha}(x) + \boldsymbol{\beta}(x) = (\omega_j R(x) + S(x))\begin{pmatrix} 1 \\ -p(x) \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix}. \tag{3.13}$$

For γ_0

$$\boldsymbol{\psi}_0(x) = \boldsymbol{\Omega}(x) = (\omega_j^0 R(x) + S(x))\begin{pmatrix} 1 \\ -p(x) \end{pmatrix} + \begin{pmatrix} 0 \\ f(x) \end{pmatrix} \tag{3.14}$$

but because the functions p, f, R, S are L-periodic

$$\boldsymbol{\psi}_0(x+L) = \boldsymbol{\Omega}(x+L) = \boldsymbol{\Omega}(x) = \boldsymbol{\psi}_0(x). \tag{3.15}$$

Among the results it was also found that

$$\boldsymbol{\alpha}(x+L) = \boldsymbol{\alpha}(x) \quad \boldsymbol{\beta}(x+L) = \boldsymbol{\beta}(x). \tag{3.16}$$

There is no other orbit other than path γ_0 because the constant

$$\omega_j := V_\mu \left(T_{\mu N} u T_{0j} + \delta_{\mu\kappa} T_{\kappa j} \right) \tag{3.17}$$

depends only on the quantities V_μ and $T_{\mu\nu}$. Therefore every ω_j is the same for any other curve γ with y_0. If the curve γ_0, ω_j^0 is taken as our reference for others γ, ω_j

$$\omega_j = \rho T_{0j} + V_\kappa T_{\kappa j}, \tag{3.18}$$

$$\boldsymbol{\Omega} = (\omega_j R + S)\boldsymbol{\alpha} + \boldsymbol{\beta}, \tag{3.19}$$

$$\boldsymbol{\psi} = (y_0 - \rho) T_{0j} r^m R \boldsymbol{\alpha} + \boldsymbol{\Omega}. \tag{3.20}$$

Suppose x tends to be very large, then x will belong to an open real interval $a_{n=mN+j}$ in which the number $n = mN + j$ is too very large. If n is very large, then it is equivalent to large value of m. Let ψ any orbit, then $\boldsymbol{\psi}(x) - \boldsymbol{\Omega}(x) = (y_0 - \rho) T_{0j} r^m R(x) \boldsymbol{\alpha}(x)$ will tend to 0 for very large value of x. In summary, independently of the initial condition any orbit $\boldsymbol{\psi}$ will converge to $\boldsymbol{\Omega}$,

$$\lim_{x\to\infty} \boldsymbol{\psi} = \boldsymbol{\Omega} \tag{3.21}$$

Therefore the path γ_0 is the only one limit cycle or stable orbit ω-limit.

4. Applicability and Limitations

Remember, our initial value problem is formulated as

$$\begin{aligned} y'(x) + p(x) y(x) &= f(x) \\ y(0) &= y_0 \end{aligned} \tag{4.1}$$

is a linear and ordinary simple differential equation but functions p, f are periodically changing. For this reason, the same problem may be viewed as a chain sequential and repetitive, for every $\langle a_{n\in[j]} \rangle$ we have

$$\begin{aligned} y'(x) + p_j(x) y(x) &= f_j(x) \\ y(x_n) &= y_n \end{aligned}. \tag{4.2}$$

An advantage of this problem in any of its point of view is clear that we treat with a known solution at every open interval. Moreover, many differential equations can be reduced to the ordinary linear first-order differential equation using a change of the dependent variable or by Taylor expansion. Therefore, some applications may be included here. As follow we consider some models. For example, two simple differential models: the AIDS's model [12] [13], a price-adjustment demand and supply in Economy Dynamics [14] and the oscillations of first order linear retarded differential equations [15]. In order to model survivability with AIDS, let's consider

$$\begin{aligned} &\frac{\mathrm{d}S}{\mathrm{d}t} + kS = kS_i \\ &S(0) = 1 \end{aligned}. \tag{4.3}$$

The independent variable t is as usual the time. The time at which a host develops AIDS will be denoted by $t = 0$. In the equation k, S_i, S_0 are real constants. In this survival model for a cohort of AIDS patients, it is postulated that AIDS is not fatal condition for a fraction S_i of the cohort, to be called the immortal fraction, which is probably zero. For the remaining part of the cohort, the probability of dying per unit time at time t will be assumed to be a positive constant k. Thus the survival fraction

$$S(t) = S_i + (1 - S_i) 2^{-\frac{t}{T_{1/2}}} \tag{4.4}$$

is the solution in which time's unit is the survival half-life $T_{1/2} := \frac{\ln(2)}{k}$, defined as the time required for half of the cohort to die. Instead of the parameters k and $T_{1/2}$, a new parameter can be defined for a host whose AIDS is fatal: the average survival time T_{aver} given by $T_{aver} = \frac{1}{k}$. Now, in what way an initial value problem inspired in the survival fraction $\frac{\mathrm{d}S}{\mathrm{d}t} + kS = kS_i$ $S(0) = 1$ can be obtained. An idea, not exactly like the exposed in this paper, subject to medical opinion and others is explained via the proposed model applying two medical treatment symbolizes as $\mathcal{A}$, $\mathcal{B}$

$$\begin{aligned} &\frac{\mathrm{d}S}{\mathrm{d}t} + k_n S = k_n S_{in} \\ &S(t_n) = \left(\frac{1}{2}\right)^{n=0,1,\cdots} \end{aligned} \tag{4.5}$$

forming an infinite alternating sequence $\mathcal{ABAB}\cdots$ in the time. When the time $t = 0 := t_0$ the totality of the cohort of AIDS patients begins the treatment $\mathcal{A}$ until the survival fraction $S(t_1) = 1/2$. Immediately to the time t_1 begin the treatment $\mathcal{B}$ until $S(t_2) = 1/4$ and so on. If this $\mathcal{AB}$-treatments achieve $\{t_n : t_0 < t_1 < t_2 < \cdots\}$ or same thing like that, then the effort can be worth it because the life may be prolonged.

Let's take in mind the price-adjustment demand and supply in Economy

Dynamics whose equations take the form

$$\begin{aligned} q^d &= a + bp \qquad b < 0 \\ q^s &= c + dp \qquad d > 0 \, . \\ \frac{\mathrm{d}p}{\mathrm{d}t} &= \alpha\left(q^d - q^s\right) \quad \alpha > 0 \end{aligned} \tag{4.6}$$

What's about the meanings of the variables? Time is represented by *t*, q^d, q^s are defined as resources or quantities demanded and supply per period of time, and p symbolizes the price. Substituting the demand and supply equations into the price adjustment we derive

$$\begin{aligned} &\frac{\mathrm{d}p}{\mathrm{d}t} - \alpha(b-d)p = \alpha(a-c) \\ &p(0) = p_0 \end{aligned} . \tag{4.7}$$

According to our idea, we can do a pricing experiment online examining price elasticity of demand. Price elasticity is ratio that show how much demand a product changes when the price of the product changes. A ratio of greater than one indicates an elastic product; a ratio less than one indicates an inelastic product. Knowing the price of demand for goods allows someone selling that good to make informed decisions about pricing strategies. This metric provides sellers with information about consumer pricing sensitivity. It is also key for makers of goods to determine manufacturing plans, as well as governments to assess how to impose taxes on goods. Pricing experiments may obey the equations

$$\begin{aligned} &\frac{\mathrm{d}p}{\mathrm{d}t} - \alpha(b-d)p = \alpha\delta(a-c) \\ &p(t_n) = p_n \end{aligned} . \tag{4.8}$$

In this experiment in the market it appears the function

$$\delta = \delta(t) := \begin{cases} 1 & t \in \langle \tau \rangle \\ 0 & t \in \langle \varepsilon \rangle \end{cases} \quad t_0 = 0, t_1 = \tau, t_2 = \tau + \varepsilon = T, \cdots, t_{n=2m+j} = mT + t_j . \tag{4.9}$$

Simultaneously we can measure or deduce the value of the price elasticity to take decision and strategies in the future. There are other options included in this last equation. For example, we can observe how it affects the price elasticity when the time's interval τ or ε tends to be more and more small.

We shall conclude this section with an example inspired by the oscillations of first-order linear retarded differential equations

$$x'(t) + \sum_{k=1}^{n} p_k(t)\, x\left(t - \tau_k(t)\right) = 0 \tag{4.10}$$

where the functions p_k and τ_k are nonnegative continuous on an interval $[t_0, \infty)$, and

$$\lim_{t\to\infty}\left(t - \tau_k(t)\right) = \infty \quad k = 1, \cdots, n . \tag{4.11}$$

One variant to investigate may be stated in this way

$$\begin{aligned} &x'(t) + p_k(t)\, x\left(t - \tau_k(t)\right) = 0 \\ &x(t_l) = x_l \end{aligned} \tag{4.12}$$

in which $t_0 = 0$, $t \in \langle a_{l=mn+k-1} \rangle$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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