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A Simple Derivation of the Equation of Time

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Abstract

The equation of time is the difference between the apparent (sun observed) time and the modern day uniform time. This paper explains what causes the Equation of Time to arise. It is shown that it arises from the Earth orbit ellipticity and its axial obliquity relative to the ecliptic (the orbit plane). The paper then derives its computing equations and compares the alternative algorithms for performing these computations. It also provides physical interpretation to its causes.

Keywords

Equation of Time, Sundial, Apparent Time, Solar Day, Mean Sun, Kepler Laws

1. Introduction

Sundial: Most probably you have seen one in a science museum or in an observatory. This very simple and amazing apparatus (do you call a stick buried in the ground an apparatus!) can be considered the man's second important invention after the wheel. With no gears or moving wheels it tells us, with sufficient accuracy, the time of the day. It basically consists of a rod that sticks out of a graduated base. The shadow of the rod reflected by the sun on the base indicates the time of the day. It comes in many forms and shapes. The simplest is the equatorial sundial whose face is parallel to the equatorial plane and the gnomon (a protruding pole from the dial center) is to it. Its dial perimeter is divided into equal 24 parts and configured so that its shadow points to 12 o'clock on Sun culmination. The Sun culmination (noon) is the epoch at which the Sun attains its highest altitude, and when the shadow of an upright pole would be the shortest of the day. It is the time when the Sun, pole and its shadow mark the local meridian (North-South) plane. With this setup, the Sun shadow indicates the hour of the day. History of sundials goes deep in the ancient Egyptian, Babylonian and ancient Greek times. In this

site [1] we may find a wealth of the history of this amazing tool. You may also be able to construct your own sundial. To convert to local time you will need to know the longitude of sundial position, as well as of course, the Equation of Time.

Of course the sundial has drawbacks, but—considering the times in which it was used—was very effective method for measuring the time during the day and indicating the start of a new day (days used to start at noon). And by a stretch of imagination we could also determine the day or the season of the year. The time elapsed between two culminations is called *solar day*. The time system based on this unit is called the *apparent solar time* (based on the *apparent Sun,* which is the celestial body that we observe daily revolving about the Earth). Its position in the sky is determined by its spherical angle with the local meridian and it is this angle that is measured by the Sundial.

An annoying issue: Since antiquity astronomers have observed that the period of the solar day varies throughout the year. That is with a "uniform" running watch; we find that the noon intervals vary throughout the course of the year. If this uniform watch is so mechanized that its days are the average of all the solar days throughout the year we arrive at our modern watch. As will be seen later, the variation in the solar day, defined earlier as the period between consecutive culminations, is not constant, but a result of the orbit eccentricity and obliquity. Now we have two time scales the apparent time *i.e.* sundial time (which we have no means for measuring it in modern times) and the mean solar time (which we can measure accurately by our clocks). The Equation of Time (*EoT*), is the difference between the *apparent solar time* and the *mean solar time*:

$$EoT = \text{apparent solar time}(\text{dial}) - \text{mean solar time}(\text{clock}) \tag{1.1}$$

It varies roughly between ±15 minutes. The word "equation" is meant to be the balance tip between the two times.

Despite its simple appearance, deriving the mathematical formula of the *EoT* requires a bit of math, analysis and a bit of imagination. Its two terms appear to be completely unrelated. It is because they miss the many angles that relate them. We aim to describe the relevant angles in both of the ecliptic and equatorial and tie them to a common computational frame. We start with the ecliptic plane.

2. The Ecliptic Plane

Herein, we describe the motion of Earth in the ecliptic plane. This motion is controlled by the Kepler laws. The first law states that the orbit of the Earth is an ellipse with Sun resides at its focus.

As shown in **Figure 1**, a and b are the semi major and semi minor axes of the ellipse, and its eccentricity is $e = \sqrt{1 - b^2/a^2}$. The end points of the major axis are called the perihelion, the closest point to the Sun, and the aphelion its farthest point.

Figure 2 shows the angle, ν, called the *true anomaly* that the Earth-Sun radius vector makes with the semi-major axis. In the figure, c is the center of the ellipse.

It also shows the *eccentric anomaly E* the angle between radius *cD* and the semi-major axis. *D* is the intersection of the orthogonal of the Earth to the semi-major axis and the circle formed by the semi-major axis. Ellipse geometry shows that the true anomaly v is related to the eccentric anomaly angle E, by

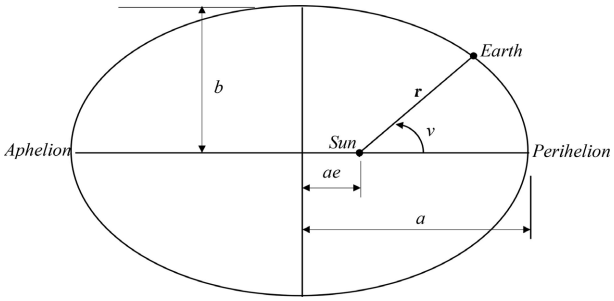

Figure 1. The earth's ecliptic.

$$\tan\frac{v}{2}=\frac{1}{\cos\alpha}\tan\frac{E}{2} \tag{1.2}$$

where

$$\cos\alpha=\sqrt{\frac{1-e}{1+e}} \tag{1.3}$$

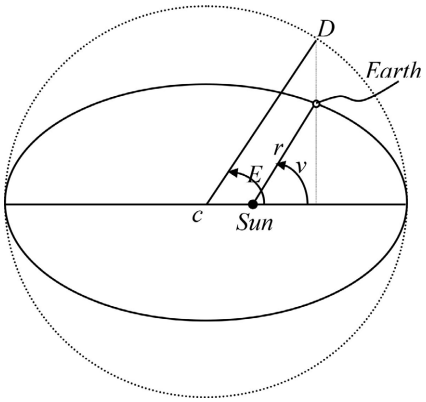

Figure 2. Heliocentric view of the earth's true, and eccentric anomalies in the ecliptic.

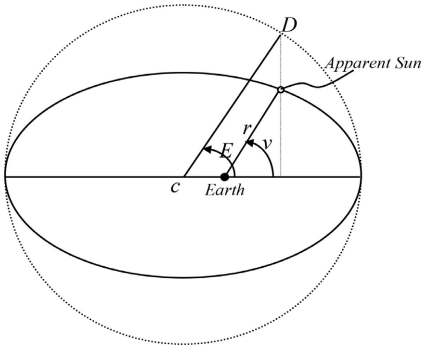

Figure 3. Geocentric view of the apparent sun and mean sun in the ecliptic.

As the Earth leaves the perihelion, its Earth-Sun radius vector would sweep an elliptic area (bounded by the semi-major axis, the radius vector and the elliptical arc that joins them). Kepler's 2nd law states that the Earth-Sun vector sweeps equal areas in equal times; *i.e.* the rate of swept area is constant. Thus the period T takes the Earth to sweep the entire ellipse is constant. If *Mean motion* is $n = 1/T$ rev/day; then swept area/ellipse area $= t/T = nt$. Thus Kepler's second law can be expressed by the more familiar equations

$$E - e\sin E = M \tag{1.4}$$

$$M = nt \tag{1.5}$$

where M is the Sun mean anomaly.

From now on, we will adopt the geocentric view for Sun-Earth motion, in which the Earth is at the center of the universe and the Sun describes an ellipse in the ecliptic, as shown in **Figure 3**. In it, we will be able to depict all the angles needed to continue with the mathematical derivations as shown in **Figure 4**. This figure—which is a variant of **Figure 3**—depicts a key player "the vernal equinox line". This line is the intersection of the ecliptic and the equator planes and to which all angles—called ecliptic longitudes—of the celestial bodies are referenced. It points to an almost a fixed position in the sky, which changes very slowly over time. The ecliptic is inclined to the equator by an angle, ε.

The linearity of M with time suggests the motion of an imaginary Sun m that orbits the ecliptic at a uniform rate. Astronomers [2]-[5] have invented a hypothetical Sun (called the dynamic Sun)—runs in the ecliptic—and orbits the Earth at constant angular rate. Its rate is selected such that it completes a 360 circle around the Earth in exactly the same time as the true Sun. This Sun is nothing but the imaginary Sun m described earlier.

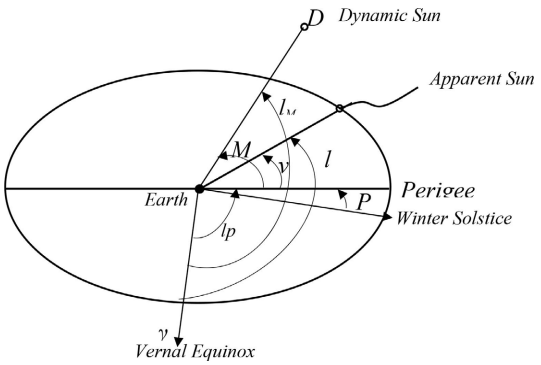

Figure 4. The sun motion in the ecliptic.

Figure 4 depicts l and l_p the ecliptic longitudes of the Apparent Sun and the perihelion respectively. It also shows l_M, defined to be the longitude of the mean Sun. Moreover, ν is the true anomaly, M is the mean Sun anomaly and P is the angle between the perihelion and the winter solstice. From **Figure 4** we get,

$$l_M = l_p + M \tag{1.6}$$

$$l = l_p + \nu \tag{1.7}$$

Even though the dynamic Sun has overcome the uniformity issue, yet we still face another problem. This dynamic Sun runs in the ecliptic that is inclined to the equator, hence its time on a sundial will not be uniform. Because the dynamic Sun runs in the ecliptic, its observed time will not appear to be uniform. Thus it can not be used as a time keeper. This is averted by inventing another hypothetical Sun [2], called the mean Sun that runs in the equator and has exactly the same angular rate as the dynamic Sun. Both will always meet, at the vernal equinox. Time is associated with the equator plane, and it is time, no pun, to discuss it and some of its relevant angles.

3. The Equator Plane

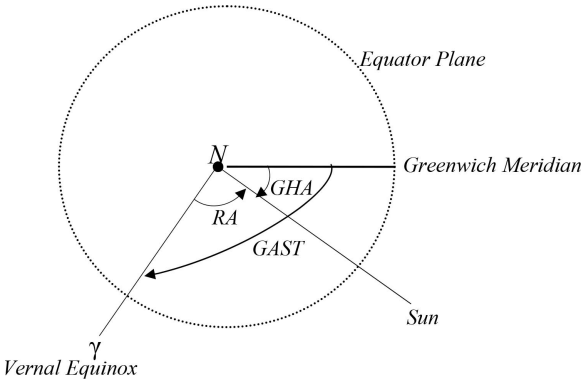

Figure 5. The equator plane.

Figure 5 depicts the plane and three relevant angles. If we imagine the projection of the Earth on the equator plane, we can see the radii that branch from the equator center are just the lines of global longitudes. These longitudes—similar to the ecliptic longitudes—provide alternative way to determine the position of celestial bodies. It uses two angles; the first is the right ascension (*RA*) the global longitude but referenced relative to the vernal equinox line, see Figure 5. The second angle, not shown in the figure, is determined by the angle of declination of the body relative to the equator, (its global latitude so to speak). Even though *RA* is a geometric angle whose value is determined by degrees, it is typically represented by hours by dividing it by (15 = 360 deg/24 hrs per day). The Greenwich Hour Angle (*GHA*) is the longitude of the celestial body. *GHA*, like the *RA*, is also given in hours. It should be noticed that in astronomy, longitudes are positive westward and negative eastwards. Finally, the angle between the Greenwich line and the vernal equinox is called Greenwich Apparent Sidereal Time (*GAST*).

With all these angles described in the ecliptic and equatorial planes, we can proceed to derive the *EoT* formula.

4. Derivation of the EoT

With no loss of generality, we shall use the Greenwich longitude as the line on

which observers are situated and to which all angles are referenced as indicated in Figure 6. If the apparent Sun is on the longitude NA, then the apparent time is determined by how the Sun is far from the Greenwich longitude. But this is exactly given by the central angle that is bounded by AG in the figure. (Do you still remember the sundial? If the gnomon is protruding from the center of the celestial sphere, the Sun shadow would fall along cA; and the angle GcA would indicate the Apparent Sun time.) This angle is the same as the hour angle h_A, or the GHA just described. Likewise, the mean Sun time is indicated by the hour angle h_M. Therefore Equation (1.1) becomes

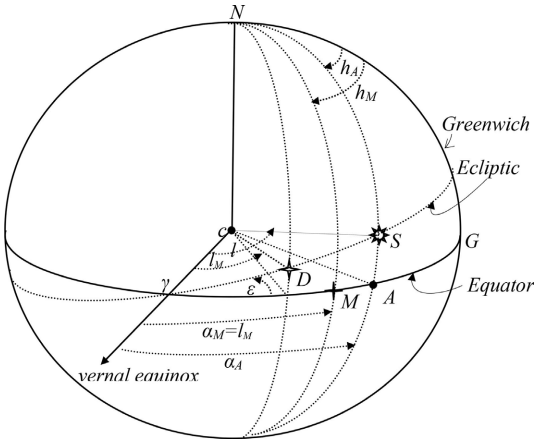

Figure 6. Hour angles on the celestial sphere.

$$EoT = h_A\left(\text{hour angle of Apparent Sun}\right) - h_M\left(\text{hour angle of Mean Sun}\right) \quad (1.8)$$

The geometry of Figure 6 implies

$$\alpha_A\left(\text{right ascension}\right) = GAST - h_A\left(\text{hour angle}\right) \quad (1.9)$$

and

$$\alpha_M\left(\text{right ascension}\right) = GAST - h_M\left(\text{hour angle}\right) \quad (1.10)$$

Substituting from Equations (1.9) and (1.10) in Equation (1.8) gives

$$\begin{aligned} EoT &= \left(GAST - RA \text{ of Apparent Sun}\right) - \left(GAST - RA \text{ of Mean Sun}\right) \\ &= \alpha_M - \alpha_A \end{aligned} \quad (1.11)$$

Since, by choice, $l_M = \alpha_M$, then the above becomes

$$EoT = l_M - \alpha_A \quad (1.12)$$

In Figure 6, γAS is a spherical triangle with right angle A; thus has the properties

$$\begin{aligned} \sin SA &= \sin\varepsilon \sin\gamma S \\ \tan\gamma A &= \cos\varepsilon \tan\gamma S \end{aligned}$$

Since $AS = \delta$; $\gamma A = \alpha_A$; $\gamma S = l$, where δ is the Sun declination angle, then:

$$\begin{aligned} \sin\delta &= \sin\varepsilon \sin l \\ \tan\alpha_A &= \cos\varepsilon \tan l \end{aligned} \quad (1.13)$$

Equation (1.13) implies

$$\alpha_A = \tan^{-1}\left(\cos\varepsilon \tan l\right) \tag{1.14}$$

Thus Equation (1.12) becomes:

$$EoT = l_M - \tan^{-1}\left(\cos\varepsilon \tan l\right) \tag{1.15}$$

Equation (1.15) is similar to Meeus [6] formula albeit in absence of the small terms that account for aberration and nutation.

By virtue of Equations (1.6) and (1.7) we get an alternate, yet identical form of the EoT

$$EoT = M + P - \tan^{-1}\frac{\tan\left(P+\nu\right)}{\cos\varepsilon} \tag{1.16}$$

5. Numerical Methods for Computing the EoT

The first step for solving the EoT is to compute the mean anomaly. For a given Julian day, JD, the mean anomaly M, is given by Meeus [6]

$$\begin{aligned} T &= \left(JD - 2451545\right)/36525 \\ M &= 357.52911 + 35999.05029T - 0.0001537T^2 \quad [\deg] \end{aligned} \tag{1.17}$$

The mean longitude, the first term of Equation (1.15), is given by

$$l_M = 280.46646 + 36000.76983T + 0.0003032T^2 \quad [\deg] \tag{1.18}$$

From Equations (1.4) and (1.2), the true anomaly, ν is computed. The celestial longitude l is then computed from,

$$l = l_M - M + \nu \tag{1.19}$$

Alternative to the mean longitude is the argument to perihelion, P from which

$$l_M = M + P + 3\pi/2 \tag{1.20}$$

Now, we have all the terms for computing the EoT. We have many methods for executing the computations. Common to all the methods are the executions of Equation (1.17) and Equation (1.18) or Equation (1.20). We start with the analytical method.

5.1. The Analytical Method

It is meant to be strict execution of the EoT equations without any approximations. We iteratively compute the eccentric angle E from Equation (1.4) and use it to compute the true anomaly, ν, from Equation (1.2). We then compute the celestial longitude l from Equation (1.19) and substitute in Equation (1.15) to compute EoT. Alternative to computing the celestial longitude is the determination of the argument of Perihelion P and substituting in Equation (1.16). Graph of the EoT is shown in **Figure 7**. Either approach requires use of trigonometric functions to solve Equation (1.2) and an iterative algorithm for solving Equation (1.4). The amount of computations is unjustified given the numerical circumstances of the problem at hand. In the following, we shall see that approximation of the analytical method yields simplified equations and yet provides a very acceptable accuracy.

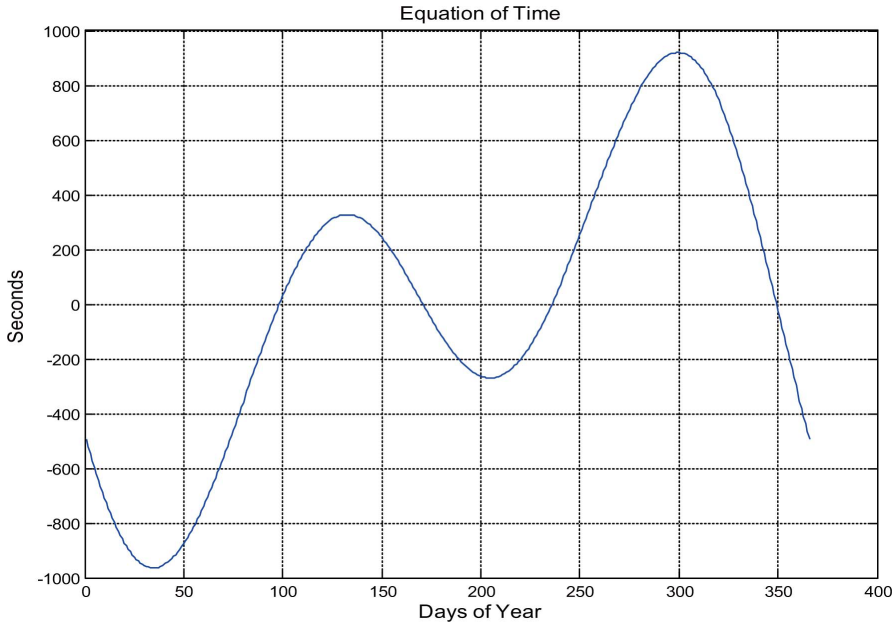

Figure 7. Equation of time in seconds.

5.2. Celestial Longitude Expansion

Our objective is to bypass the numerical complexities that come with the analytic approach. First: expand the true anomaly ν in terms of the eccentric anomaly E. Second: expand E in terms of the mean anomaly M. Third: combine these two steps to expand ν in terms of M. Fourth: expand right ascension α_A in terms of the celestial longitude l. In the sequel we illustrate the above steps.

Expand $\tan y = \dfrac{1}{\cos\alpha}\tan x$ as derived in **Appendix A**. In it we apply $\cos\alpha$ of Equation (1.3). The result is given in Equation (A.8). Replacing y by $\nu/2$ and x by $E/2$ makes Equation (1.2) expansion be

$$\nu = E + \left(e + \frac{1}{4}e^3\right)\sin E + \frac{1}{4}e^2\sin 2E + \frac{1}{12}e^3\sin 3E \tag{1.21}$$

We aim to get the above expansion in terms of M rather than E. This is done in two steps. First, **Appendix B** expands E in terms of M to give:

$$E = M + \left(e - \frac{1}{8}e^3\right)\sin M + \left(\frac{1}{2}e^2 - \frac{1}{6}e^4\right)\sin 2M + \frac{3}{8}e^3\sin 3M + \cdots \tag{1.22}$$

Second, expand $\sin E$, $\sin 2E$ and $\sin 3E$ as follows. From (B.3) and using (B.5) gives

$$\sin E = \frac{E-M}{e} = \left(1 - \frac{1}{8}e^2\right)\sin M + \frac{1}{2}\left(e - \frac{1}{3}e^3\right)\sin 2M + \frac{3}{8}e^2\sin 3M + O\left(e^3\right) \tag{1.23}$$

Keeping only terms $O\left(e^3\right)$, then Equation (1.22) approximates $\sin 2E$ and $\sin 3E$ to:

$$\begin{aligned} \sin 2E &= \sin\left(2M + 2e\sin M\right) \\ &\cong \sin 2M + 2e\sin M\cos 2M \\ \sin 3E &\cong \sin 3M \end{aligned} \tag{1.24}$$

Substituting from Equations (1.22) - (1.24) in Equation (1.21) gives

$$\nu = M + \left(2e - \frac{1}{4}e^3\right)\sin M + \frac{5}{4}e^2 \sin 2M + \frac{13}{12}e^3 \sin 3M \tag{1.25}$$

Finally, **Appendix C** expands $\alpha_A = \tan^{-1}(\cos\varepsilon \tan l)$ into:

$$\alpha_A = l - y\sin(2l) + 1/2\, y^2 \sin(4l) \tag{1.26}$$

where

$$y = \tan^2 \frac{\varepsilon}{2} \tag{1.27}$$

Substituting from Equations (1.19) (1.25) (1.26) in (1.15)

$$EoT = (l + M - \nu) - \left(l - y\sin(2l) + \frac{1}{2}y^2 \sin(4l)\right) \Rightarrow \tag{1.28}$$

$$\begin{aligned} EoT = &-\left(\left(2e - \frac{1}{4}e^3\right)\sin M + \frac{5}{4}e^2 \sin 2M + \frac{13}{12}e^3 \sin 3M\right) \\ &-\left(-y\sin(2l) + \frac{1}{2}y^2 \sin(4l)\right) \Rightarrow \end{aligned} \tag{1.29}$$

$$EoT = -C - R \tag{1.30}$$

C and R, denoted by "Equation of Center" and "Reduction to Equator" respectively, are:

$$C = \nu - M = \left(2e - \frac{1}{4}e^3\right)\sin M + \frac{5}{4}e^2 \sin 2M + e^3 \frac{13}{12}\sin 3M \tag{1.31}$$

$$R = \alpha_A - l = -y\sin 2l + \frac{1}{2}y^2 \sin 4l \tag{1.32}$$

Smart [4] simplified the equation further by computing it in terms of mean longitude, l_M, and retaining second order terms only to get

$$EoT = 2e\sin M + \frac{5}{4}e^2 \sin 2M - y\sin 2l_M - 2ey\sin M \cos 2l_M + \frac{1}{2}y^2 \frac{\varepsilon}{2}\sin 4l_M \tag{1.33}$$

5.3. Argument of Perihelion Expansion

Mueller [7] expanded Equation (1.16) in sine terms of P and M only—fourth and higher power terms are omitted—to get,

$$\begin{aligned} EoT = &-y\left(1 - 4e^2\right)\sin 2(M + P) - 2e\sin M + 2ey\sin(M + 2P) \\ &- 2ey\sin(3M + 2P) - 0.5y^2 \sin 4(M + P) - \frac{5}{4}e^2 \sin 2M \\ &+ 2ey^2 \sin(3M + 4P) - 2ey^2 \sin(5M + 4P) \\ &- \frac{13}{4}e^2 y\sin(4M + 2P) - \frac{1}{3}y^3 \sin 6(M + P) \end{aligned} \tag{1.34}$$

To compare the different computational methods, we plotted their differences with that of the analytical method as shown in **Figure 8**.

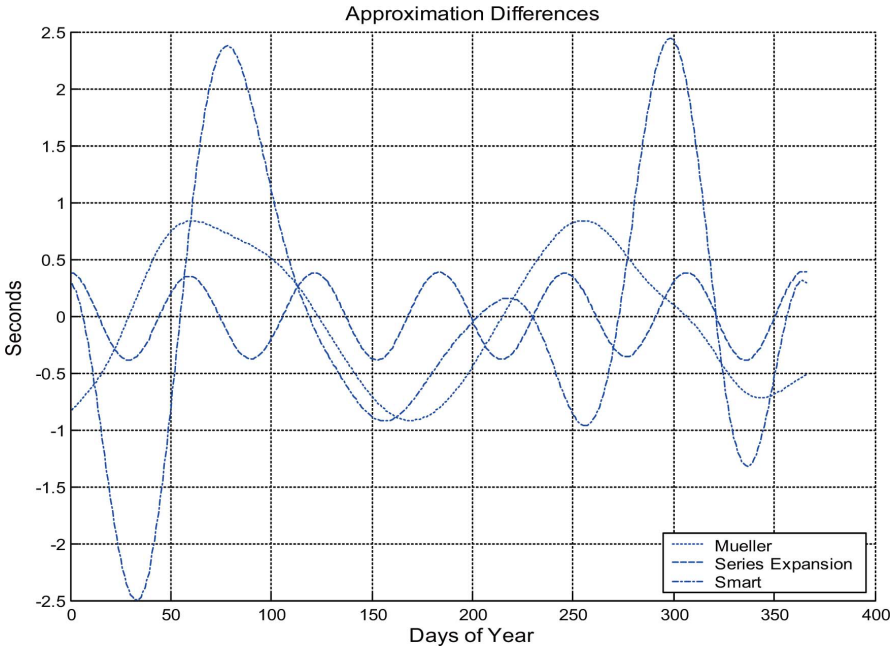

Figure 8. Approximation differences.

6. Physical Interpretation of the EOT

We shall see that the EoT is caused by the Earth orbit's ellipticity and its obliquity.

6.1. Ellipticity Effects

Initially, we suppress the Earth's obliquity, *i.e.* the Earth's axis is orthogonal to the ecliptic. Hence the ecliptic will coincide with the equator and the dynamic Sun will be the same as the mean Sun.

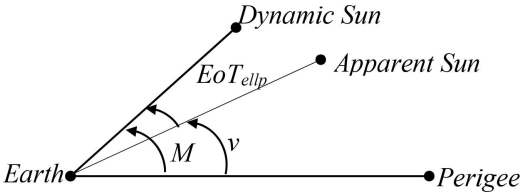

Figure 9. Equation of time due to ellipticity.

The EoT given by Equation (1.8) would become

$$EoT_{ellp} = h_A\left(\text{hour angle of Apparent Sun}\right) - h_M\left(\text{hour angle of Dynamic Sun}\right) \quad (1.35)$$

The geometry of the apparent and mean Suns will be as simple as that shown in Figure 9 and the EoT would simplify to

$$EoT_{ellp}\left(t\right) = M\left(t\right) - \nu\left(t\right) \quad (1.36)$$

Comparing with Equation (1.31) we see that

$$EoT_{ellp}\left(t\right) = -C \quad (1.37)$$

Now, we attempt to interpret the physical behavior of the above equation. We

first notice from Equation (1.4) that the mean Sun (MS) and the apparent Sun (AS) always meet at the perigee. On the orbital paths their radii vectors will make angles M and ν with the semi major axis.

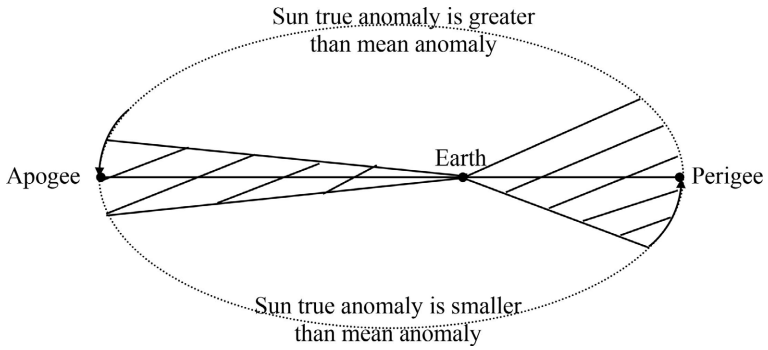

Figure 10. Equal scanned areas for equal times.

Kepler's 2nd law implies that AS sweeps larger areas when close to perigee than when close to the apogee, see **Figure 10**.

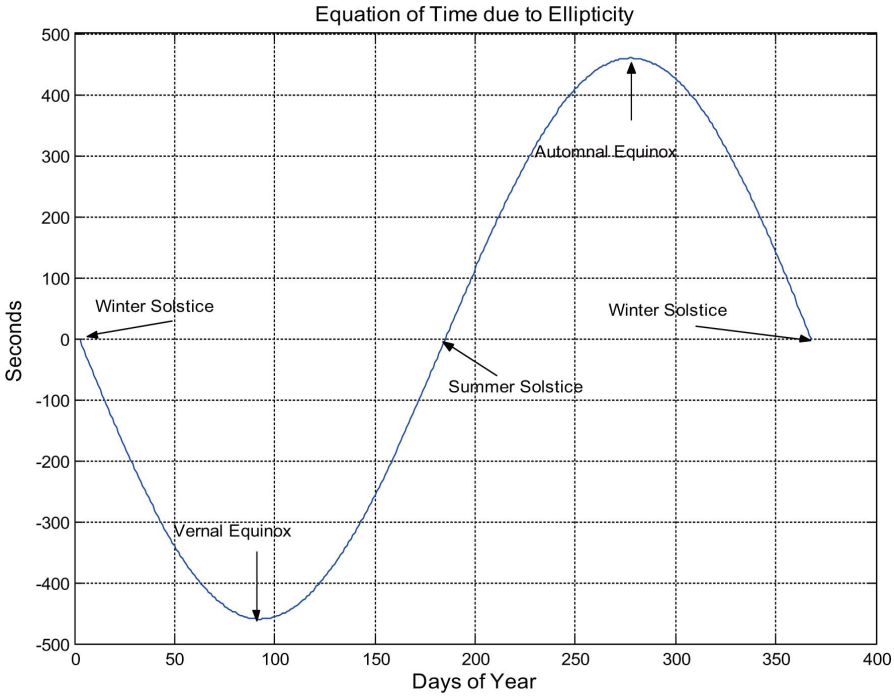

Figure 11. *EoT* due to Ellipticity.

Meanwhile, the rate of swept areas of the Ms is constant. When the AS and MS start their journeys starting at the perigee, the AS will advance the MS until it reaches the vernal equinox (a mid point between the perigee and apogee). The AS will then slow down and the MS will close the advance until they meet at the apogee. On the journey back from the apogee to the perigee the above behavior is reversed. The *EoT* time history due to ellipticity is depicted in **Figure 11**.

6.2. Obliquity

Obliquity is the Earth's polar axis tilt towards the ecliptic. Its orientation with respect to the ecliptic is described with two angles: first, the obliquity angle, ε, with respect to the normal of the ecliptic and is approximately 23˚.45. The second is the angle P between the projection of the polar axis and the major axis of the ecliptic and is approximately 12˚.25. We note that the projection of the polar axis indicates the solstices on the ecliptic, as shown in Figure 4.

To study the contribution of the obliquity on the EoT we suppress the ellipticity and envision circular orbit on which the Sun orbital angular velocity is constant. In this scenario, AS will coincide with the dynamic Sun (DS) as well as their hour angles. In view of Figure 4 $l_M = l$. EoT due to tilt, from Equation (1.12), becomes

$$EoT_{tilt} = l - \alpha_A \tag{1.38}$$

Comparing with Equation (1.32) we see that

$$EoT_{tilt} = -R \tag{1.39}$$

The journeys of MS and DS start at the autumnal equinox; as it is the point at which they coincide and thus the EoT is null. As evident from Figure 6, moving towards the winter solstice we see that the right ascension of the DS (same as AS) lags behind the MS. Equation (1.13) now becomes

$$\tan\alpha_D = \tan\alpha_A = \cos\varepsilon \tan l = \cos\varepsilon \tan\alpha_M \tag{1.40}$$

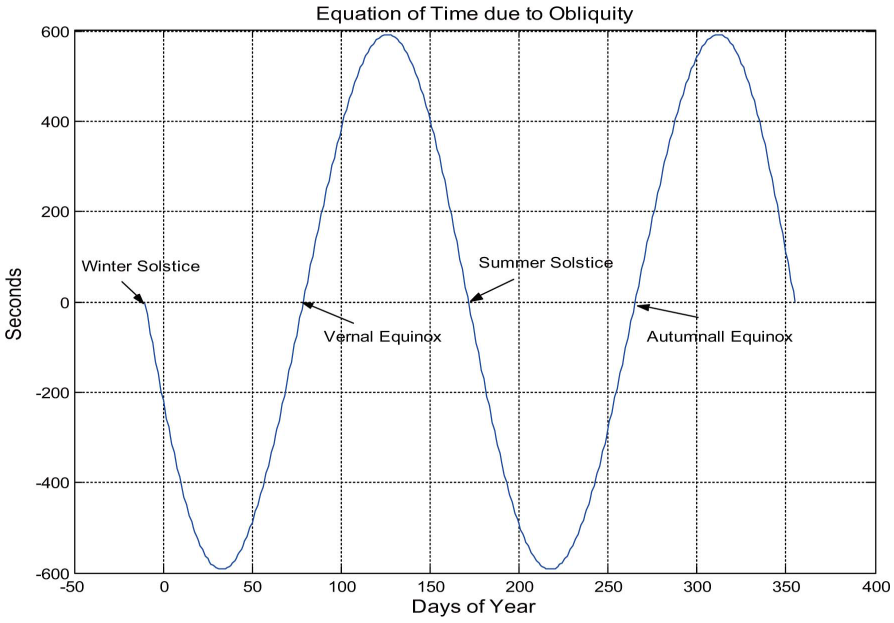

Figure 12. EoT due to Tilt.

Meanwhile, the DS starts to increase until it reaches a point in which the lines of longitudes get closer. This will increase the right ascension and cause it to decrease the lag with the MS. At the solstice the ecliptic longitude of both suns will be the same and the EoT becomes null again. Also, the AS declination becomes at

its maximum. This behavior will be repeated from winter solstice to vernal equinox albeit with EoT is on the negative side. Thus, the EoT due to tilt (obliquity) is periodic with two periods per year and its time history is depicted in **Figure 12**. Finally we observe from Equations (1.37), (1.39) and (1.30) that

$$EoT_{ellp}(t) + EoT_{tilt} = -C - R = EoT \tag{1.41}$$

We may find in [8] illustration of the behavior of the EoT. Python code for computing the EoT is listed in [9].

7. Conclusion

The paper explained the EoT and its relevance to astronomical measurements. It has presented the common algorithms for computing it. It should be noticed that all share the same root. They differ only on the approximation methods used and the available parameters needed for computing it.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix A: Series Expansion of $\tan y = \frac{1}{\cos\alpha}\tan x$

This equation implies:

$$y = \tan^{-1}\left(\frac{1}{\cos\alpha}\tan x\right) \tag{A.1}$$

The derivative of this equation gives

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos\alpha}\frac{\sec^2 x}{\frac{\tan^2 x}{\cos^2\alpha}+1} = \frac{\cos\alpha\sec^2 x}{\sec^2 x - 1 + \cos^2\alpha} = \frac{\cos\alpha}{1-\sin^2\alpha\cos^2 x} \Rightarrow$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos\alpha}{2-\sin^2\alpha(1+\cos 2x)} = \frac{2\cos\alpha}{1+\cos^2\alpha-\sin^2\alpha\cos 2x} \Rightarrow$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4\cos\alpha}{2(1+\cos^2\alpha)-\sin^2\alpha\left(\mathrm{e}^{i2x}+\mathrm{e}^{-i2x}\right)} = \frac{-4\cos\alpha/\sin^2\alpha}{\left(\mathrm{e}^{i2x}+\mathrm{e}^{-i2x}\right)-\frac{2(1+\cos^2\alpha)}{\sin^2\alpha}} \Rightarrow$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4\cos\alpha/\sin^2\alpha\,\mathrm{e}^{i2x}}{\mathrm{e}^{i4x}-\frac{2(1+\cos^2\alpha)}{\sin^2\alpha}\mathrm{e}^{i2x}+1} = \frac{-4\cos\alpha/\sin^2\alpha\,\mathrm{e}^{i2x}}{\left(\mathrm{e}^{i2x}-\beta\right)\left(\mathrm{e}^{i2x}-\beta^{-1}\right)} \tag{A.2}$$

In the above, the roots of the denominator β and β^{-1} are determined as follows

$$\begin{aligned}\beta &= \frac{1+\cos^2\alpha}{\sin^2\alpha} - \sqrt{\left(\frac{1+\cos^2\alpha}{\sin^2\alpha}\right)^2 - 1} \\ &= \frac{(1+\cos^2\alpha) - \sqrt{\left((1+\cos^2\alpha)-\sin^2\alpha\right)\left((1+\cos^2\alpha)+\sin^2\alpha\right)}}{\sin^2\alpha} \\ &= \frac{(1+\cos^2\alpha) - \sqrt{2\cos^2\alpha\cdot 2}}{\sin^2\alpha} = \frac{(1+\cos^2\alpha)-2\cos\alpha}{1-\cos^2\alpha} \\ &= \frac{(1-\cos\alpha)^2}{(1-\cos\alpha)(1+\cos\alpha)} = \frac{1-\cos\alpha}{1+\cos\alpha} = \frac{2\sin^2\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \tan^2\frac{\alpha}{2}\end{aligned} \tag{A.3}$$

Applying the partial fractions to (A.2) gives

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{-4\cos\alpha}{\sin^2\alpha\left(\beta-\beta^{-1}\right)}\left(\frac{\mathrm{e}^{i2x}}{\mathrm{e}^{i2x}-\beta} - \frac{\mathrm{e}^{i2x}}{\mathrm{e}^{i2x}-\beta^{-1}}\right) \\ &= \frac{-4\cos\alpha}{\sin^2\alpha\left(\beta-\beta^{-1}\right)}\left(\frac{1}{1-\beta\mathrm{e}^{-i2x}} + \frac{\beta\mathrm{e}^{i2x}}{1-\beta\mathrm{e}^{i2x}}\right)\end{aligned} \tag{A.4}$$

Equation (A.3) gives

$$\beta-\beta^{-1} = \frac{1-\cos\alpha}{1+\cos\alpha} - \frac{1+\cos\alpha}{1-\cos\alpha} = \frac{(1-\cos\alpha)^2-(1+\cos\alpha)^2}{1-\cos^2\alpha} = \frac{-4\cos\alpha}{\sin^2\alpha} \tag{A.5}$$

Supposing that $\beta<1$ and using (A.5), Equation (A.4) can be expanded into

$$\frac{dy}{dx}=\sum_{n=0}^{\infty}\beta^n e^{-i2nx}+\beta e^{i2x}\sum_{n=0}^{\infty}\beta^n e^{i2nx}=1+\sum_{n=1}^{\infty}\beta^n e^{-i2nx}+\sum_{n=1}^{\infty}\beta^n e^{i2nx}$$

$$=1+\sum_{n=1}^{\infty}\beta^n\left(e^{i2nx}+e^{-i2nx}\right)$$

$$\frac{dy}{dx}=1+\sum_{n=1}^{\infty}2\beta^n\cos(2nx) \tag{A.6}$$

Integrating (A.6) gives

$$y=x+\sum_{n=1}^{\infty}\frac{\beta^n}{n}\sin(2nx)+const$$

From (A.1), $y=0$ at $x=0$, thus the integration constant in the above is 0, therefore

$$y=x+\sum_{n=1}^{\infty}\frac{1}{n}\tan^{2n}\frac{\alpha}{2}\sin(2nx)\Rightarrow$$

$$y=x+\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)^n\sin(2nx) \tag{A.7}$$

Special Case: $\cos\alpha=\sqrt{\frac{1-e}{1+e}}$

The coefficients of (A.7) can be simplified as follows:

$$\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{(1-\cos\alpha)^2}{1-\cos^2\alpha}=\frac{1+\cos^2\alpha-2\cos\alpha}{1-\cos^2\alpha}=\frac{1+\frac{1-e}{1+e}-2\sqrt{\frac{1-e}{1+e}}}{1-\frac{1-e}{1+e}}$$

$$=\frac{1+e+1-e-2\sqrt{1-e^2}}{1+e-1+e}=\frac{1-\sqrt{1-e^2}}{e}$$

When e is small, then to an accuracy of e^3 the above expression is approximated to

$$\frac{1-\cos\alpha}{1+\cos\alpha}=\frac{1}{e}\left(1-1+\frac{1}{2}e^2+\frac{1}{8}e^4\right)=\frac{1}{2}e+\frac{1}{8}e^3$$

Hence

$$\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)^2=\frac{1}{4}e^2\quad\text{and}\quad\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)^3=\frac{1}{8}e^3$$

Consequently (A.7) is simplified to

$$y=x+\left(\frac{1}{2}e+\frac{1}{8}e^3\right)\sin 2x+\frac{1}{8}e^2\sin 4x+\frac{1}{24}e^3\sin 6x \tag{A.8}$$

Appendix B: Expanding Kepler Equation Using Lagrange Reversion Theorem

The Lagrange reversion theorem, Battin [10], Deakin [11], states that, if

$$z=y+xf(z) \tag{B.1}$$

x, y and z are arbitrary variables, then g(z) can be expanded into a series of x and y

$$g(z)=g(y)+xg'(y)f(y)+\frac{x^2}{2!}\frac{\partial}{\partial y}\left\{g'(y)\left[f(y)\right]^2\right\}+\frac{x^3}{3!}\frac{\partial^2}{\partial y^2}\left\{g'(y)\left[f(y)\right]^3\right\}+\cdots \tag{B.2}$$

Applying to the Kepler equation yields,

$$E=M+e\sin E \tag{B.3}$$

Comparing with (B.1) shows that expansion is obtained when:

$$z=E \quad y=M \quad x=e \quad f(z)=\sin(z) \quad g(z)=z \tag{B.4}$$

Substituting in the Lagrange equation yields

$$E=M+e\sin M+\frac{e^2}{2!}\frac{\partial}{\partial M}\sin^2 M+\frac{e^3}{3!}\frac{\partial^2}{\partial M^2}\sin^3 M+\cdots$$

$$E=M+e\sin M+\frac{e^2}{2}\sin 2M+\frac{e^3}{8}(3\sin 3M-\sin M)+\frac{e^4}{6}(2\sin 4M-\sin 2M)+\cdots$$

Rearranging terms yields the expansion of the Kepler equation

$$E=M+\left(e-\frac{1}{8}e^3\right)\sin M+\left(\frac{1}{2}e^2-\frac{1}{6}e^4\right)\sin 2M+\frac{3}{8}e^3\sin 3M+\frac{1}{3}e^4\sin 4M+\cdots \tag{B.5}$$

Appendix C: Series Expansion of $\tan\alpha_A=\cos\varepsilon\tan l$

Substituting for

$$\alpha_A=90-\alpha_c$$
$$l=90-l_c$$

in the above equation yields $\tan(90-\alpha_c)=\cos\varepsilon\tan(90-l_c)$

From the properties of the tangent function, the above becomes

$$1/\tan(\alpha_c)=\cos\varepsilon/\tan(l_c)\Rightarrow$$
$$\tan(\alpha_c)=(1/\cos\varepsilon)\tan(l_c)$$

for which the series expansion is given by

$$\alpha_c=l_c+y\sin(2l_c)+1/2\,y^2\sin(4l_c)$$

where $y=\tan^2\varepsilon/2$

Substituting back for α_A and l results in

$$90-\alpha_A=90-l+y\sin(180-2l)+1/2\,y^2\sin(360-4l)$$

which yields,

$$\alpha_A=l-y\sin(2l)+1/2\,y^2\sin(4l)$$