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Motion of an Infinitesimal Mass in the Field of a Cyclic Kite Configuration of the Second Kind

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Abstract

This paper investigates the equations of motion of each of the four-point masses that form a kite configuration of the second kind, relative to the remaining three, which are established in a synodic frame in terms of position vectors, mass parameter and mean motion. By eliminating the position vectors from the equations of four masses, a functional relation between the mean motion and mass parameter has been established, in which the mean motion of the synodic frame can be expressed as a function of the mass parameter. Using this relation, the governing equations for the motion of an infinitesimal mass influenced by the gravitational field of the kite configuration are formulated for all admissible values of the mass parameter and mean motion within their respective domains. These equations are further applied to the Sitnikov problem, where the resulting dynamics are analyzed. Finally, a periodic series solution is constructed through the iteration process of Green's function.

Keywords

Cyclic Kite Configuration, Synodic Frame, Dirac-Delta Function, Green's Function, Iteration Process, Periodic Series Solution, Trajectory of Periodic Series Solution

1. Introduction

Researchers of celestial mechanics used to discuss the particular cases of both branches, such as three-body, four-body and five-body configurations or problems. In the field of four-body configuration, MacMillan *et al.* [1] proved two theorems in detail for the existence of a quadrilateral configuration. Brumberg [2] studied the permanent configuration of the four-body problem. Albouy [3] discussed the symmetric central configuration of equal masses. Long *et al.* [4] analyzed the four-body central configuration with two pairs of equal masses and three equal masses. In the second system, they proved that a convex non-collinear planar four-body central configuration with three equal masses must be a kite.

Chavela *et al.* [5] proved that if in a convex four-body central configuration, two equal masses are located at opposite vertices of a quadrilateral and at most the mass of one of the remaining particles is larger than the equal masses, then the configuration must be a kite-shaped quadrilateral. Albouy *et al.* [6] established the relationship between the masses and some geometric properties of the quadrilateral configuration. Furthermore, they proved that the planar four-body problem is a convex central configuration and is symmetric concerning one diagonal if and only if the masses of the two particles on the other diagonal are equal. During discussions of central configuration for the planar Newtonian four-body problem, Pina *et al.* [7] developed an algorithm to construct a general four-body configuration by dividing the directed areas by the corresponding scalar areas.

These algorithms became important tools for finding the new properties of symmetric and non-symmetric central configurations. Pina [8] developed the new coordinates of the four particles forming the kite configuration in terms of principal moments of inertia and Eulerian angles independently. Cors *et al.* [9] studied the cyclic central configuration of point masses in the Newtonian four-body problem by using six mutual distances of the four-point masses as their coordinates and showed that the four-point masses form a two-dimensional plane surface. They extensively investigated two symmetric families, the kite configuration and the isosceles trapezoidal configuration.

Furthermore, they have demonstrated that if the masses of any two bodies in a four-body cyclic central configuration are equal, then the configuration possesses a line of symmetry. Balint *et al.* [10] extended the work of Cors *et al.* [9] in three cases (one in the convex case and two in concave cases) by expressing the masses of the central configuration in terms of angle coordinates. Additionally, they claimed that the derived formulas represent the exact analytical solutions of the four-body problem. Deng *et al.* [11] used mutual distances as the coordinates and proved that the diagonals of a cyclic central quadrilateral cannot be perpendicular except that the configuration is a kite. Corbera *et al.* [12] proved that the diagonals of a four-body central configuration are perpendicular; then the configuration must be a kite. Further, they verified the same theorem in the four-vortex convex central configuration.

Hassan [13] has proved three theorems about the existence of cyclic kite configurations, which are discussed in the next section. By using the results of the first and second theorems, he verified all the results of previous authors. Alam *et al.* [14] discussed the existence and stability of collinear and non-collinear libration points of a kite configuration of the first kind by using the first theorem.

Presently, we aim to express the mean motion of the kite in terms of the mass parameter and to establish the equations of motion for an infinitesimal mass moving in the gravitational field generated by the four-point masses forming a kite configuration of the second kind. Furthermore, we intend to apply these equations of motion to the Sitnikov problem.

2. Kite Configuration

Kite configuration means a quadrilateral with two pairs of equal adjacent sides, and a cyclic kite configuration is a kite whose vertices lie on the circumference of a common circle. It is also to be noted that a quadrilateral is a kite if and only if its diagonals are perpendicular to each other and at least one of the two diagonals is a line of symmetry. In some cases, both diagonals may be the lines of symmetry. Thus, the cyclic kite configuration can be classified into three classes as follows:

2.1. Theorem 1

When the cyclic kite configuration is formed by the combination of one equilateral triangle and one isosceles triangle with the common base P_2NP_4 (First Kind), as shown in Figure 1.

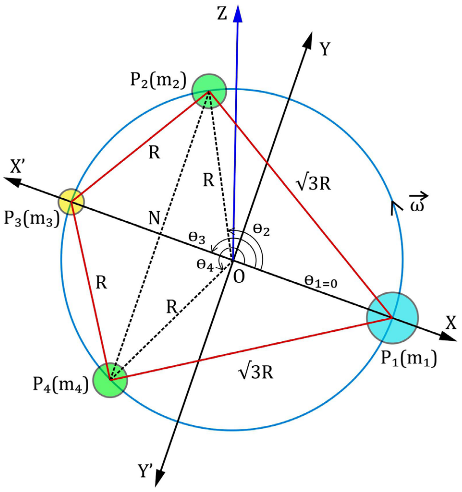

Figure 1. Kite configuration of the first kind.

Statement: The necessary and sufficient conditions for the existence of a cyclic kite configuration with four positive point masses $m_k\ (k=1,2,3,4)$ located at the respective vertices P_k of the kite are the existence of a non-dimensional mass parameter $\mu \in]0,1/3[$ such that $m_2 = m_4 = M\mu$, $m_1 = M/2(1-\mu)$, $m_3 = M/2(1-3\mu)$ where $m_1+m_2+m_3+m_4 = M$ and $P_1P_2 = P_1P_4 = P_2P_4 = \sqrt{3}R$, $P_2P_3 = P_3P_4 = R$, $P_1P_3 = 2R$, where R is the radius of the common circular orbit.

Proof: Let $P_1P_2P_3P_4$ be a cyclic kite formed by the combination of one equilateral triangle $P_1P_2P_4$ and one isosceles triangle $P_2P_3P_4$ with the common base P_2NP_4 and axis of symmetry $P_1OP_3 = 2R$. Let P_k be the positions of the four positive point masses m_k moving on a common circular orbit with centre at O, radius R and diameter $P_1OP_3 = 2R$ as the axis of symmetry. Considering the centre of the circle as the origin, the axis of symmetry as the X-axis and a line YOY'

parallel to P_2NP_4 and perpendicular to P_1OP_3 as the Y-axis. Using the properties of a cyclic quadrilateral $P_1P_2P_3P_4$ in the form of a cyclic kite configuration and the cyclic equilateral triangle $P_1P_2P_4$, the coordinates of the four point masses can be written as $P_k\left(R\cos\theta_k, R\sin\theta_k\right)$, where $\theta_1 = 0$, $\theta_2 = 2\pi/3$, $\theta_3 = \pi$, $\theta_4 = 4\pi/3$.

Thus, the position vectors of the four-point masses are given by
$\vec{r}_k = R\cos\theta_k\hat{i} + R\sin\theta_k\hat{j} = \overrightarrow{OP_k}$

$$\Rightarrow \vec{r}_1 = R\hat{i},\ \vec{r}_2 = -\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j},\ \vec{r}_3 = -R\hat{i},\ \vec{r}_4 = -\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j}. \tag{1}$$

The position vector of the centre of mass of the system of four-point masses is given by

$$\vec{c} = M^{-1}\sum_{k=1}^{4} m_k\vec{r}_k. \tag{2}$$

As our kite is a central configuration, we consider the centre of mass as the origin, then $\vec{c} = \hat{0}$

$$\Rightarrow m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 = \hat{0},$$

$$\Rightarrow m_1R\hat{i} + m_2\left(-\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j}\right) + m_3\left(-R\hat{i}\right) + m_4\left(-\frac{R}{2}\hat{i} - \frac{\sqrt{3}}{2}R\hat{j}\right) = 0,$$

$$\Rightarrow R\left\{2\left(m_1 - m_3\right) - \left(m_2 + m_4\right)\right\}\hat{i} + \sqrt{3}R\left\{m_2 - m_4\right\}\hat{j} = 0.$$

Taking the scalar product of $\hat{i}$ and $\hat{j}$ with the above equation, we get

$$\left.\begin{aligned} 2\left(m_1 - m_3\right) - \left(m_2 + m_4\right) = 0, \\ m_2 = m_4. \end{aligned}\right\} \tag{3}$$

Also, we know that

$$m_1 + m_2 + m_3 + m_4 = M. \tag{4}$$

Equations (3) and (4) yield

$$m_1 - m_2 - m_3 = 0, \tag{5}$$

$$m_1 + 2m_2 + m_3 = M. \tag{6}$$

Let us introduce a non-dimensional mass parameter $\mu \in \left]0, 1/3\right[$ such that $m_2 = m_4 = M\mu$ then from Equations (5) and (6), we get

$$m_1 = \frac{M\left(1-\mu\right)}{2},\quad m_3 = \frac{M\left(1-3\mu\right)}{2}.$$

Also,

$$\left|\overrightarrow{P_1P_2}\right| = \left|\overrightarrow{OP_2} - \overrightarrow{OP_1}\right| = \left|-\frac{R}{2}\hat{i} + \frac{\sqrt{3}}{2}R\hat{j} - R\hat{i}\right| = \sqrt{3}R = \left|\overrightarrow{P_1P_4}\right| = \left|\overrightarrow{P_2P_4}\right|,$$

$$\left|\overrightarrow{P_2P_3}\right| = \left|\overrightarrow{P_3P_4}\right| = R,\ \left|\overrightarrow{P_1P_3}\right| = 2R.$$

Thus, the necessary conditions for four positive point masses m_k situated at the vertices of a cyclic kite configuration formed by an equilateral triangle $P_1P_2P_4$

and an isosceles triangle $P_2P_3P_4$ are

$$\left.\begin{array}{ll}\text{(a) } m_2 = m_4 = \mu,\ m_1 = M - \mu/2,\ m_3 = M - 3\mu/2, & \\ \text{(b) } m_1 = m_3 + 2\mu, & \\ \text{(c) } P_1P_2 = P_2P_4 = P_1P_4 = \sqrt{3}R & \text{i.e., } r_{12} = r_{24} = r_{14} = \sqrt{3}R, \\ \text{(d) } P_2P_3 = P_3P_4 = R & \text{i.e., } r_{23} = r_{34} = R, \\ \text{(e) } P_1P_3 = 2R & \text{i.e., } r_{13} = 2R. \end{array}\right\} \qquad (7)$$

Conversely, the results of Equation (7a), $m_2 = m_4 = M\mu$ represents that the diagonal P_1OP_3 is the line of symmetry. Equation (7b) shows that $m_1 > m_3$ and from the results of Equations (7c), (7d) and (7e) $P_1P_2^2 + NP_2^2 = P_1N^2$ *i.e.*, the diagonals are perpendicular to each other. Thus, the results of Equation (7) represent the sufficient conditions for the existence of a cyclic kite configuration.

2.2. Theorem 2

When the kite is formed by two congruent isosceles right-angled triangles (Second Kind), as shown in **Figure 2**.

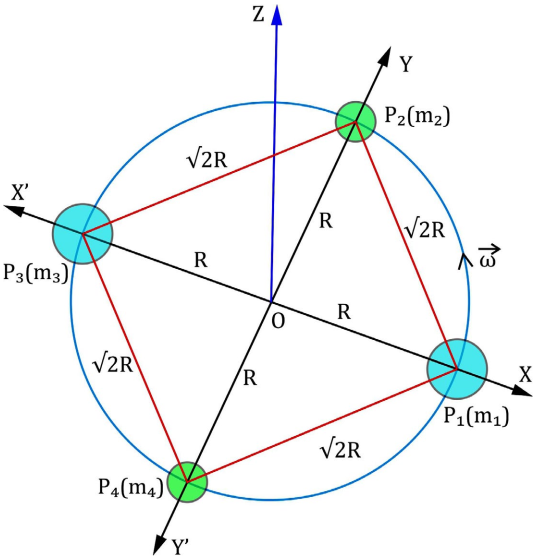

Figure 2. Kite configuration of the second kind.

Statement: The necessary and sufficient conditions for the existence of a cyclic kite configuration with four positive point masses $m_k\,(k = 1,2,3,4)$ at the respective vertices P_k of the kite are that there exists a non-dimensional mass parameter $\mu \in \left]0,1/2\right[$ such that $m_2 = m_4 = M\mu$, $m_1 = M(1/2 - \mu) = m_3$, where $m_1 + m_2 + m_3 + m_4 = M$ and $P_1P_2 = P_2P_3 = P_3P_4 = P_1P_4 = \sqrt{2}R$, $P_1P_3 = P_2P_4 = 2R$, where R is the radius of the common circular orbit.

Proof: Let $P_1P_2P_3P_4$ be the cyclic kite formed by the combination of two congruent isosceles right-angled triangles $P_1P_2P_4$ and $P_2P_3P_4$ with the common base $P_2OP_4 = 2R$ and axis of symmetry $P_1OP_3 = 2R$. Let P_k be the positions of the four positive point masses m_k on the cyclic kite. Taking the centre O of the

circle as the origin and two perpendicular diameters P_1OP_3 and P_2OP_4 as the X-axis and Y-axis, respectively, then the position vectors of the four point masses are $\overrightarrow{OP_1} = R\hat{i} = \vec{r}_1$, $\overrightarrow{OP_2} = R\hat{j} = \vec{r}_2$, $\overrightarrow{OP_3} = -R\hat{i} = \vec{r}_3$, $\overrightarrow{OP_4} = -R\hat{j} = \vec{r}_4$, where $\hat{i}$ and $\hat{j}$ are the unit vectors along the X-axis and Y–axis respectively. If we take the centre of mass as the origin, then by using Theorem 1, we have,
$m_1R\hat{i} + m_2R\hat{j} - m_3R\hat{i} - m_4R\hat{j} = \hat{0}$, Here either of P_1OP_3 or P_2OP_4 may be taken as the axis of symmetry. Taking the scalar product of $\hat{i}$ and $\hat{j}$, we get

$$m_1 = m_3 \text{ and } m_2 = m_4. \tag{8}$$

Introduction of Equation (8) in Equation (4) gives

$$m_1 + m_2 = M/2 \tag{9}$$

Let us introduce a non-dimensional mass parameter $\mu \in \left]0, 1/2\right[$ such that $m_2 = m_4 = M\mu$, then from Equations (2) and (7) $m_1 = m_3 = M(1/2 - \mu)$. Both the masses of each pair are equal, so both the diagonals are the axes of symmetry of the cyclic kite configuration. Thus, the necessary conditions for the existence of a cyclic kite configuration are

$$\left.\begin{array}{ll}
\text{(a) } m_2 = m_4 = \mu,\ m_1 = m_3 = M(1/2 - \mu), & \\
\text{(b) } m_1 = m_3 \geq m_2 = m_4, & \\
\text{(c) } P_1P_2 = P_2P_3 = P_3P_4 = P_1P_4 = \sqrt{2}R & \text{i.e., } r_{12} = r_{23} = r_{34} = r_{14} = \sqrt{2}R, \\
\text{(d) } P_1P_3 = P_2P_4 = 2R & \text{i.e., } r_{13} = r_{24} = 2R.
\end{array}\right\} \tag{10}$$

Conversely, the results of Equation (10a) (similar to Long *et al.* [4]) and Equation (10b) justify that both the diagonals are the axes of symmetry, and Equations (10c), and (10d) are the sufficient conditions for the perpendicularity of the diagonals.

2.3. Theorem 3

When the kite is formed by the combination of two non-congruent arbitrary isosceles triangles (Generalized Kind), as shown in **Figure 3**.

Statement: The necessary and sufficient conditions for the existence of a cyclic kite configuration are that there exists a parameter $\mu \in \left[0, 1/2(1-\cos\theta)\right]$ such that $m_2 = m_4 = M\mu$, $m_1 = M\left[1/2 - \mu(1+\cos\theta)\right]$, $m_3 = M\left[1/2 - \mu(1-\cos\theta)\right]$ where $\theta \in (0, \pi)$, $m_1 + m_2 + m_3 + m_4 = M$ and $P_1P_2 = P_1P_4 = 2R\sin\theta/2$, $P_2P_4 = 2R\sin\theta$, $P_2P_3 = P_3P_4 = 2R\cos\theta/2$, where $P_1P_3 = 2R$.

Proof: Let $P_1P_2P_3P_4$ be an arbitrary cyclic kite configuration formed by the combination of two non-congruent arbitrary isosceles triangles $P_1P_2P_4$ and $P_2P_3P_4$ with the common base P_2NP_4. Let the four point masses m_1, m_2, m_3, m_4 be moving on a common circular orbit of radius R and centre O. Considering $P_1P_2 = P_1P_4$ and $P_2P_3 = P_3P_4$ as the pair of adjacent equal sides. Let
$\angle P_1OP_2 = \angle P_1OP_4 = \theta$ then $\angle P_2OP_3 = \angle P_4OP_3 = \pi - \theta$. Taking the axis of symmetry P_1OP_3 as the X-axis, O as the origin (the barycentre) and the line YOY' as the Y-axis, then the position vectors of the four-point masses are given by

$$\overrightarrow{OP_1} = R\hat{i} = \vec{r}_1, \qquad \overrightarrow{OP_2} = R\cos\theta\hat{i} + R\sin\theta\hat{j} = \vec{r}_2,$$
$$\overrightarrow{OP_3} = -R\hat{i} = \vec{r}_3, \qquad \overrightarrow{OP_4} = R\cos\theta\hat{i} - R\sin\theta\hat{j} = \vec{r}_4.$$

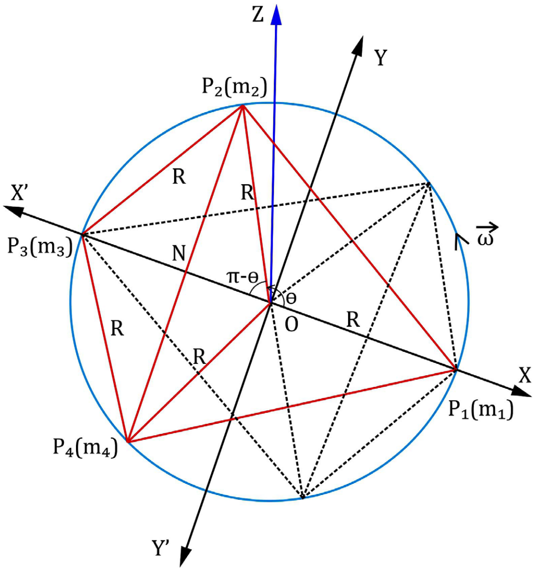

Figure 3. Generalized kite configuration.

As in previous theorems, $m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_4\vec{r}_4 = \hat{0}$

$$\Rightarrow m_1R\hat{i} + m_2\left(R\cos\theta\hat{i} + R\sin\theta\hat{j}\right) + m_3\left(-R\hat{i}\right) + m_4\left(R\cos\theta\hat{i} - R\sin\theta\hat{j}\right) = \hat{0},$$

$$\Rightarrow R\left(m_1 + m_2\cos\theta - m_3 + m_4\cos\theta\right)\hat{i} + R\left(m_2 - m_4\right)\sin\theta\hat{j} = \hat{0}.$$

By taking the scalar products of $\hat{i}$ and $\hat{j}$, we get

$$m_1 + \left(m_2 + m_4\right)\cos\theta - m_3 = 0. \tag{11}$$

$$m_2 = m_4. \tag{12}$$

The collaboration of Equations (11) and (12) gives

$$m_1 + 2m_2\cos\theta - m_3 = 0. \tag{13}$$

Introduction of Equation (12) in Equation (4) gives

$$m_1 + 2m_2 + m_3 = M. \tag{14}$$

The addition of Equations (13) and (14) gives

$$m_1 + m_2\left(1 + \cos\theta\right) = M/2. \tag{15}$$

Let us introduce a parameter $\mu \in \left[0, 1/2\left(1 - \cos\theta\right)\right]$ such that $m_2 = m_4 = M\mu$ where $\theta \in \left(0, \pi\right)$ then from Equations (14) and (15), we get

$$m_1 = M\left[1/2 - \mu\left(1 + \cos\theta\right)\right] \quad \text{and} \quad m_3 = M\left[1/2 - \mu\left(1 - \cos\theta\right)\right].$$

Thus, the necessary conditions for the existence of a cyclic kite configuration are

$$\left.\begin{array}{ll}
\text{(a)}\ m_2 = m_4 = \mu, & \\
\text{(b)}\ m_1 \geq m_2 = m_4 \geq m_3, & \\
\text{(c)}\ m_3 = m_1 + 2\mu\cos\theta, & \\
\text{(d)}\ m_1 = (M/2) - \mu(1+\cos\theta), & m_3 = (M/2) - \mu(1-\cos\theta), \\
\text{(e)}\ P_1P_2 = P_1P_4 = 2R\sin\theta/2, & P_2P_3 = P_3P_4 = 2R\cos\theta/2, \\
\quad\ \ P_2P_4 = 2R\sin\theta, & P_1P_3 = 2R.
\end{array}\right\} \tag{16}$$

Conversely, the results of Equations (16a) and (16b) will justify that the line P_1OP_3 is a line of symmetry, and the results of Equation (16e) will justify the perpendicularity of the diagonals of the kite.

2.4. Collapse of the Kite Configuration

Suppose the equal point masses lying on opposite sides of the axis of symmetry of the kite configuration are very close to either of the point masses of the axis of symmetry. In that case, a collision may happen, and hence the kite configuration may collapse. So, there should be a restricted domain of θ for the existence of a kite configuration. The necessary condition may be taken as $2P_2P_3 \geq R$

i.e., $4R\cos\theta/2 \geq R \Rightarrow \cos\theta/2 \geq 1/4 \Rightarrow \cos\theta/2 \geq 0.25 \Rightarrow 14.5^\circ \leq \theta/2 \leq 75.5^\circ$

$$\Rightarrow 29^\circ \leq \theta \leq 151^\circ.$$

Thus, the kite configuration will exist for $\theta \in [\pi/6, 5\pi/6]$ and will collapse if $\theta \leq 29^\circ$ and $\theta \geq 151^\circ$ approximately.

2.5. Justification of Theorems

The foregoing results of the three theorems represent the sufficient conditions for the existence of at least one line of symmetry and for the perpendicularity of the diagonals of the kite. In particular, we have verified and extended certain results established by earlier researchers, including Long *et al.* [4], Chavela *et al.* [5], Albouy *et al.* [6], Pina *et al.* [7], Pina [8], Cors *et al.* [9], and Corbera *et al.* [12]. Specifically, the results expressed in Equations (7a), (7b) and (7c) corroborate the theories of Chavela *et al.* [5], Albouy *et al.* [6], Pina *et al.* [7], and Cors *et al.* [9]. Furthermore, the results of Equations (7c), (7d) and (7e) are consistent with the conditions provided by Long *et al.* [4] as

$$\left(r_{24}^3 - r_{12}^3\right)\left(r_{23}^3 - r_{34}^3\right)\left(r_{13}^3 - r_{14}^3\right) = \left(r_{12}^3 - r_{14}^3\right)\left(r_{13}^3 - r_{34}^3\right)\left(r_{24}^3 - r_{23}^3\right). \tag{17}$$

This is the necessary and sufficient condition of a four-body planar central configuration that the six mutual distances r_{ij} determine a geometrically realizable planar configuration. The results of Equation (7) vanish the Caley-Menger determinant.

$$\text{i.e.,}\ V(r) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & r_{12}^2 & r_{13}^2 & r_{14}^2 \\ 1 & r_{12}^2 & 0 & r_{23}^2 & r_{24}^2 \\ 1 & r_{13}^2 & r_{23}^2 & 0 & r_{34}^2 \\ 1 & r_{14}^2 & r_{24}^2 & r_{34}^2 & 0 \end{vmatrix} = 3R^6 \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 0 \\ -1 & -1 & 1 & 2 \\ 1 & -2 & -1 & 1 \end{vmatrix} = -3R^6 \begin{vmatrix} -2 & 0 & 1 \\ 0 & -2 & -1 \\ 3 & -1 & -2 \end{vmatrix} = 0. \tag{18}$$

$V(r)=0$ can also be satisfied by the results of Equations (7) and (10). Thus, the square of six mutual distances r_{ij} satisfy $V(r)=0$ *i.e.*, the volume of the tetrahedron formed by four-point masses is zero, *i.e.*, a cyclic kite configuration is planar.

Again,

$$P(r)=r_{12}r_{34}+r_{14}r_{23}-r_{24}r_{13}=0.$$

Thus, the four-point masses forming a cyclic kite configuration and numbered sequentially lie on a common circle with diagonals, r_{13} and r_{24}. where $r_{24}\le r_{13}$. The twice of the directed areas of the four parts of the kite configuration $P_1P_2P_3P_4$ are given by

$$\vec{S}_1=\vec{r}_2\wedge\vec{r}_3+\vec{r}_3\wedge\vec{r}_4+\vec{r}_4\wedge\vec{r}_2,\quad \vec{S}_2=\vec{r}_1\wedge\vec{r}_4+\vec{r}_4\wedge\vec{r}_3+\vec{r}_3\wedge\vec{r}_1,$$

$$\vec{S}_3=\vec{r}_1\wedge\vec{r}_2+\vec{r}_2\wedge\vec{r}_4+\vec{r}_4\wedge\vec{r}_1,\quad \vec{S}_4=\vec{r}_1\wedge\vec{r}_3+\vec{r}_3\wedge\vec{r}_2+\vec{r}_2\wedge\vec{r}_1,$$

$$\Rightarrow \vec{S}_1=\left(-\frac{R}{2}\hat{i}+\frac{\sqrt{3}}{2}R\hat{j}\right)\wedge\left(-R\hat{i}\right)+\left(-R\hat{i}\right)\wedge\left(-\frac{R}{2}\hat{i}-\frac{\sqrt{3}}{2}R\hat{j}\right)$$

$$+\left(-\frac{R}{2}\hat{i}-\frac{\sqrt{3}}{2}R\hat{j}\right)\wedge\left(-\frac{R}{2}\hat{i}+\frac{\sqrt{3}}{2}R\hat{j}\right)=\frac{\sqrt{3}}{2}R^2,$$

$$\Rightarrow \vec{S}_1=\frac{\sqrt{3}}{2}R^2\hat{k},\quad \vec{S}_2=-\sqrt{3}R^2\hat{k},\quad \vec{S}_3=3\frac{\sqrt{3}}{2}R^2\hat{k},\quad \vec{S}_4=-\sqrt{3}R^2\hat{k}. \tag{19}$$

Thus, the directed areas $\vec{S}_1$ and $\vec{S}_3$ are normals to the plane of motion of the point masses, whereas $\vec{S}_2$ and $\vec{S}_4$ are the anti-normals to the plane of motion that is all directed areas parallel to each other.

Also,

$$\sum_{k=1}^{4}\vec{S}_k=\vec{S}_1+\vec{S}_2+\vec{S}_3+\vec{S}_4=0. \tag{20}$$

as $P_k\left(x_k,y_k\right)\Rightarrow P_1\equiv\left(R,0\right),\ P_2\equiv\left(-\frac{R}{2},\frac{\sqrt{3}}{2}R\right),\ P_3\equiv\left(-R,0\right),\ P_4\equiv\left(-\frac{R}{2},-\frac{\sqrt{3}}{2}R\right),$

then $\sum_{k=1}^{4}S_kx_k=\sum_{k=1}^{4}S_ky_k=0$,

and consequently

$$\sum_{k=1}^{4}S_k\vec{r}_k=0,$$

$$\sum_{m=1}^{4}\sum_{n=1}^{4}r_{mn}^2S_mS_n=\sum_{m=1}^{4}\sum_{n=1}^{4}\left|\vec{r}_m-\vec{r}_n\right|^2S_mS_n=\sum_{m=1}^{4}\sum_{n=1}^{4}\left(\vec{r}_m^{\,2}+\vec{r}_n^{\,2}-2\vec{r}_m\cdot\vec{r}_n\right)S_mS_n,$$

$$=\sum_{m=1}^{4}S_m\vec{r}_m^{\,2}\sum_{n=1}^{4}S_n+\sum_{m=1}^{4}S_m\sum_{n=1}^{4}S_n\vec{r}_n^{\,2}-2\sum_{m=1}^{4}S_m\vec{r}_m\cdot\sum_{n=1}^{4}S_n\vec{r}_n,$$

$$\sum_{m=1}^{4}\sum_{n=1}^{4}r_{mn}^2S_mS_n=\left(\sum_{m=1}^{4}S_m\vec{r}_m{}^2\right)0+0\left(\sum_{n=1}^{4}S_n\vec{r}_n{}^2\right)-2\left(\sum_{m=1}^{4}S_m\vec{r}_m\right)\cdot\left(\sum_{n=1}^{4}S_n\vec{r}_n\right)=0 \tag{21}$$

where $\vec{r}_m=x_m\hat{i}+y_m\hat{j}$.

Using Equations (7c), (7d), (10c) and (10d), we get

$$\left.\begin{aligned}\left|\vec{S}_1\right| &= \frac{r_{23}r_{34}r_{24}}{4R} = \frac{3\sqrt{3}}{2}R^2 = \Delta_1, \quad \left|\vec{S}_2\right| = \frac{r_{13}r_{34}r_{14}}{4R} = \sqrt{3}R^2 = \Delta_2, \\ \left|\vec{S}_3\right| &= \frac{r_{12}r_{24}r_{14}}{4R} = \frac{\sqrt{3}}{2}R^2 = \Delta_3, \quad \left|\vec{S}_4\right| = \frac{r_{12}r_{23}r_{13}}{4R} = \sqrt{3}R^2 = \Delta_4.\end{aligned}\right\} \tag{22}$$

$$\Rightarrow \Delta_1 \geq \Delta_2 = \Delta_4 \geq \Delta_3. \tag{23}$$

Now using the results of Equations (7c), (7d), (7e) and a matrix *A*, we have

$$\left.\begin{aligned}\left[r_{23}^2 r_{34}^2 r_{24}^2\right] A \left[r_{23}^2 r_{34}^2 r_{24}^2\right]^{\mathrm{T}} &= \left[4\Delta_1^2\right] \\ \left[r_{13}^2 r_{34}^2 r_{14}^2\right] A \left[r_{13}^2 r_{34}^2 r_{14}^2\right]^{\mathrm{T}} &= \left[4\Delta_2^2\right] \\ \left[r_{12}^2 r_{24}^2 r_{14}^2\right] A \left[r_{12}^2 r_{24}^2 r_{14}^2\right]^{\mathrm{T}} &= \left[4\Delta_3^2\right] \\ \left[r_{12}^2 r_{23}^2 r_{13}^2\right] A \left[r_{12}^2 r_{23}^2 r_{13}^2\right]^{\mathrm{T}} &= \left[4\Delta_4^2\right]\end{aligned}\right\} \text{where } A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \tag{24}$$

Thus, the results of Equations (7c), (7d) and (7e) justified the results of previous authors. Similarly, all the results from Equations (17) to (24) can be verified by the results of Equations (10) and (16) also. The partial derivatives of the Caley-Menger determinant with respect to the variables $r_{ij}^2\left(i \neq j = 1,2,3,4\right)$ are given by

$$\frac{\partial V(r)}{\partial r_{ij}^2} = \left(-1\right)^{i+j-1} 32\Delta_i\Delta_j. \tag{25}$$

3. Mean Motion of the Synodic Frame

Let the plane of motion of the four point masses rotate with a constant angular velocity $\vec{\varpi} = n\hat{k}$ about the Z-axis *OZ*; $\hat{r}_{ij}$ be the unit vector along the vector $\overrightarrow{P_iP_j} = \vec{r}_{ij}$. Joining the i^{th} and j^{th} point masses, then the equation of motion of the i^{th} point mass relative to the other three point masses is given by

$$-m_i n^2 \vec{r}_i = \sum_{j=1}^{4} \frac{\partial V_{ij}}{\partial r_{ij}} \hat{r}_{ij}, \ \left(\hat{i} \neq \hat{j}\right) \tag{26}$$

where the potential V between the i^{th} and j^{th} point masses and the gravitational constant G are given by the relation

$$-V_{ij} = G\frac{m_i m_j}{\left|\vec{r}_{ij}\right|}, \tag{27}$$

and the unit vector is given by

$$\hat{r}_{ij} = \frac{\vec{r}_{ij}}{\left|\vec{r}_{ij}\right|} = \frac{\overrightarrow{OP_j} - \overrightarrow{OP_i}}{\left|\overrightarrow{OP_j} - \overrightarrow{OP_i}\right|} = \frac{\vec{r}_j - \vec{r}_i}{\left|\vec{r}_j - \vec{r}_i\right|}. \tag{28}$$

Equations (26), (27) and (28) yield

$$n^2\vec{r}_i + G\sum_{j=1}^{4}\frac{m_j\vec{r}_{ij}}{\left|\vec{r}_{ij}\right|^3} = 0, \quad \left(i \neq j\right) \Rightarrow n^2\vec{r}_i + G\sum_{j=1}^{4}\frac{m_j\left(\vec{r}_j - \vec{r}_i\right)}{\left|\vec{r}_{ij}\right|^3} = 0. \tag{29}$$

Putting $i = 1$ and expanding the summation on j, we get

$$n^2\vec{r}_1 + G\left[\frac{m_2\left(\vec{r}_2-\vec{r}_1\right)}{\left|\vec{r}_{12}\right|^3}+\frac{m_3\left(\vec{r}_3-\vec{r}_1\right)}{\left|\vec{r}_{13}\right|^3}+\frac{m_4\left(\vec{r}_4-\vec{r}_1\right)}{\left|\vec{r}_{14}\right|^3}\right]=0,$$

$$n^2\vec{r}_2 + G\left[\frac{m_1\left(\vec{r}_1-\vec{r}_2\right)}{\left|\vec{r}_{21}\right|^3}+\frac{m_3\left(\vec{r}_3-\vec{r}_2\right)}{\left|\vec{r}_{23}\right|^3}+\frac{m_4\left(\vec{r}_4-\vec{r}_2\right)}{\left|\vec{r}_{24}\right|^3}\right]=0,$$

$$n^2\vec{r}_3 + G\left[\frac{m_1\left(\vec{r}_1-\vec{r}_3\right)}{\left|\vec{r}_{31}\right|^3}+\frac{m_2\left(\vec{r}_2-\vec{r}_3\right)}{\left|\vec{r}_{32}\right|^3}+\frac{m_4\left(\vec{r}_4-\vec{r}_3\right)}{\left|\vec{r}_{34}\right|^3}\right]=0,$$

and

$$n^2\vec{r}_4 + G\left[\frac{m_1\left(\vec{r}_1-\vec{r}_4\right)}{\left|\vec{r}_{41}\right|^3}+\frac{m_2\left(\vec{r}_2-\vec{r}_4\right)}{\left|\vec{r}_{42}\right|^3}+\frac{m_3\left(\vec{r}_3-\vec{r}_4\right)}{\left|\vec{r}_{43}\right|^3}\right]=0.$$

Thus, the equations of motion of the system of four-point masses with position vectors $\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4$ of the cyclic kite configuration are

$$\Rightarrow\quad n^2\vec{r}_1 + G\left[\frac{M\mu\left(\vec{r}_2-\vec{r}_1\right)}{2\sqrt{2}R^3}+\frac{M\left(1/2-\mu\right)\left(\vec{r}_3-\vec{r}_1\right)}{8R^3}+\frac{M\mu\left(\vec{r}_4-\vec{r}_1\right)}{2\sqrt{2}R^3}\right]=0,$$

$$n^2\vec{r}_2 + G\left[\frac{M\left(1/2-\mu\right)\left(\vec{r}_1-\vec{r}_2\right)}{2\sqrt{2}R^3}+\frac{M\left(1/2-\mu\right)\left(\vec{r}_3-\vec{r}_2\right)}{2\sqrt{2}R^3}+\frac{M\mu\left(\vec{r}_4-\vec{r}_2\right)}{2\sqrt{2}R^3}\right]=0,$$

$$n^2\vec{r}_3 + G\left[\frac{M\left(1/2-\mu\right)\left(\vec{r}_1-\vec{r}_3\right)}{8R^3}+\frac{M\mu\left(\vec{r}_2-\vec{r}_3\right)}{2\sqrt{2}R^3}+\frac{M\mu\left(\vec{r}_4-\vec{r}_3\right)}{2\sqrt{2}R^3}\right]=0,$$

$$\text{and}\quad n^2\vec{r}_4 + G\left[\frac{M\left(1/2-\mu\right)\left(\vec{r}_1-\vec{r}_4\right)}{2\sqrt{2}R^3}+\frac{M\mu\left(\vec{r}_2-\vec{r}_4\right)}{8R^3}+\frac{M\left(1/2-\mu\right)\left(\vec{r}_3-\vec{r}_4\right)}{2\sqrt{2}R^3}\right]=0.$$

Choosing units of mass, units of force in such a way that $M=m_1+m_2+m_3+m_4=1,\ G=1$ and the maximum distance between two-point masses as unity, then $2R=1$. Thus, $P_1\equiv\left(1/2,0,0\right)$, $P_2\equiv\left(0,1/2,0\right)$, $P_3\equiv\left(-1/2,0,0\right)$, $P_4\equiv\left(0,-1/2,0\right)$, $m_1=1/2-\mu=m_3$, $m_2=\mu=m_4$, where $\mu\in\left]0,1/2\right[$ and hence, from the above system of equations, we get

$$n^2\vec{r}_1+2\sqrt{2}\mu\left(\vec{r}_2-\vec{r}_1\right)+\left(1/2-\mu\right)\left(\vec{r}_3-\vec{r}_1\right)+2\sqrt{2}\mu\left(\vec{r}_4-\vec{r}_1\right)=0,$$

$$n^2\vec{r}_2+2\sqrt{2}\left(1/2-\mu\right)\left(\vec{r}_1-\vec{r}_2\right)+2\sqrt{2}\left(1/2-\mu\right)\left(\vec{r}_3-\vec{r}_2\right)+2\sqrt{2}\mu\left(\vec{r}_4-\vec{r}_2\right)=0,$$

$$n^2\vec{r}_3+\left(1/2-\mu\right)\left(\vec{r}_1-\vec{r}_3\right)+2\sqrt{2}\mu\left(\vec{r}_2-\vec{r}_3\right)+2\sqrt{2}\mu\left(\vec{r}_4-\vec{r}_3\right)=0,$$

$$\text{and}\quad n^2\vec{r}_4+2\sqrt{2}\left(1/2-\mu\right)\left(\vec{r}_1-\vec{r}_4\right)+\mu(\vec{r}_2-\vec{r}_4)+2\sqrt{2}\left(1/2-\mu\right)\left(\vec{r}_3-\vec{r}_4\right)=0.$$

Arranging the terms of the above equations as the linear combinations of four position vectors $\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4,$ as

$$\left.\begin{aligned}
&\left[n^2-\left(4\sqrt{2}-1\right)\mu-1/2\right]\vec{r}_1+2\sqrt{2}\mu\vec{r}_2+\left(1/2-\mu\right)\vec{r}_3+2\sqrt{2}\mu\vec{r}_4=0,\\
&\sqrt{2}\left(1-2\mu\right)\vec{r}_1+\left[n^2+\left(4\sqrt{2}-1\right)\mu-2\sqrt{2}\right]\vec{r}_2+\sqrt{2}\left(1-2\mu\right)\vec{r}_3+\mu\vec{r}_4=0,\\
&\left(1/2-\mu\right)\vec{r}_1+2\sqrt{2}\mu\vec{r}_2+\left[n^2-\left(4\sqrt{2}-1\right)\mu-1/2\right]\vec{r}_3+2\sqrt{2}\mu\vec{r}_4=0,\\
&\text{and}\quad\sqrt{2}\left(1-2\mu\right)\vec{r}_1+\mu\vec{r}_2+\sqrt{2}\left(1-2\mu\right)\vec{r}_3+\left[n^2+\left(4\sqrt{2}-1\right)\mu-2\sqrt{2}\right]\vec{r}_4=0.
\end{aligned}\right\}\quad(30)$$

Here, the above system of Equation (30) represents the equations of motion of four-point masses forming a cyclic kite in synodic frame.

Eliminating $\vec{r}_1,\vec{r}_2,\vec{r}_3,\vec{r}_4$ from the system of Equation (30), we get

$$\begin{vmatrix} a & 2\sqrt{2}\mu & 1/2-\mu & 2\sqrt{2}\mu \\ \sqrt{2}(1-2\mu) & b & \sqrt{2}(1-2\mu) & \mu \\ 1/2-\mu & 2\sqrt{2}\mu & a & 2\sqrt{2}\mu \\ \sqrt{2}(1-2\mu) & \mu & \sqrt{2}(1-2\mu) & b \end{vmatrix} = 0 \tag{31}$$

where $a = n^2 - \left(4\sqrt{2}-1\right)\mu - 1/2$, $b = n^2 + \left(4\sqrt{2}-1\right)\mu - 2\sqrt{2}$.

Equation (31) can give the values of the mean motion n in terms of μ that is $n = f(\mu)$. With the variation of μ; n will vary. Only positive real values slightly greater than one are considered. Expanding Equation (31) and putting the values of *a* and *b*, we get an eight-degree equation in n as

$$n^2\left[n^2 - 2\left(2\sqrt{2}-1\right)\mu - 1\right]\left[n^4 + 2\left(2\sqrt{2}\mu - 2\sqrt{2} - \mu\right)n^2 + 4\sqrt{2}\left(\mu - 2\sqrt{2} + 1\right)\right] = 0.$$

$$\Rightarrow n = 0,$$

$$\Rightarrow \left[n^2 - 2\left(2\sqrt{2}-1\right)\mu - 1\right] = 0, \tag{32}$$

$$\Rightarrow \left[n^4 + 2\left(2\sqrt{2}\mu - 2\sqrt{2} - \mu\right)n^2 + 4\sqrt{2}\left(\mu - 2\sqrt{2} + 1\right)\right] = 0. \tag{33}$$

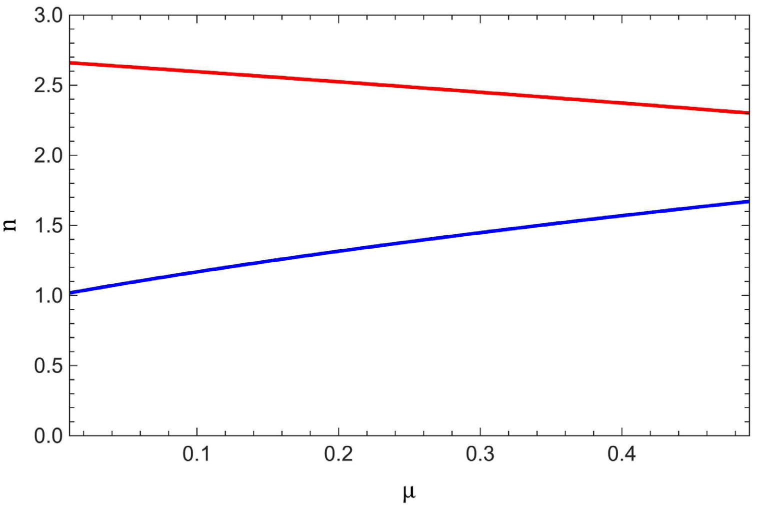

Figure 4. Graph of mass parameter μ versus mean motion $n = f(\mu)$.

The mean motion n can't be zero, so the other two equations will provide the data for n corresponding to the values of $\mu \in]0, 1/2[$ as shown in **Figure 4**. Equation (32) represents the graph of μ versus n (Blue Plot) in which n increases very slowly with the increase of μ. Equation (33) represents the graph of μ versus n (Red Plot), in which n decreases slowly with the increase of μ.

4. Exactness of Domain of the Mass Parameter

If we put $\mu = 0$, then $m_2 = m_4 = 0$ and $m_1 = m_3 = 1/2$ and the four-body cyclic kite configuration will be reduced to a two-body straight line configuration lying on the axis of symmetry P_1OP_3. From Equation (31), we get

$$\begin{vmatrix} n^2-1/2 & 0 & 1/2 & 0 \\ \sqrt{2} & n^2-2\sqrt{2} & \sqrt{2} & 0 \\ 1/2 & 0 & n^2-1/2 & 0 \\ \sqrt{2} & 0 & \sqrt{2} & n^2-2\sqrt{2} \end{vmatrix} = 0 \Rightarrow n = 0, 1, 1.682.$$

Here $n = 0, 1.682$ are invalid, and so the two-body straight-line configuration lying on the X-axis rotating with the mean motion $n = 1$.

If we put $\mu = 1/2$, then $m_2 = m_4 = 1/2$ and $m_1 = m_3 = 0$ then the cyclic kite configuration again will be reduced to the two-body straight-line configuration lying on the Y-axis. From Equation (31), we get

$$\begin{vmatrix} n^2-2\sqrt{2} & \sqrt{2} & 0 & \sqrt{2} \\ 0 & n^2-1/2 & 0 & 1/2 \\ 0 & \sqrt{2} & n^2-2\sqrt{2} & \sqrt{2} \\ 0 & 1/2 & 0 & n^2-1/2 \end{vmatrix} = 0 \Rightarrow n = 0, 1, 1.674,$$

where $n = 0, 1.674$ are invalid, so the cyclic straight-line configuration lying on the Y-axis is rotating with the mean motion $n = 1$. Thus for $\mu = 0$, $\mu = 1/2$, the kite configuration of the second kind does not exist, so the exact domain of μ is $\mu \in \left]0, 1/2\right[$.

5. The Equations of Motion of the Infinitesimal Mass

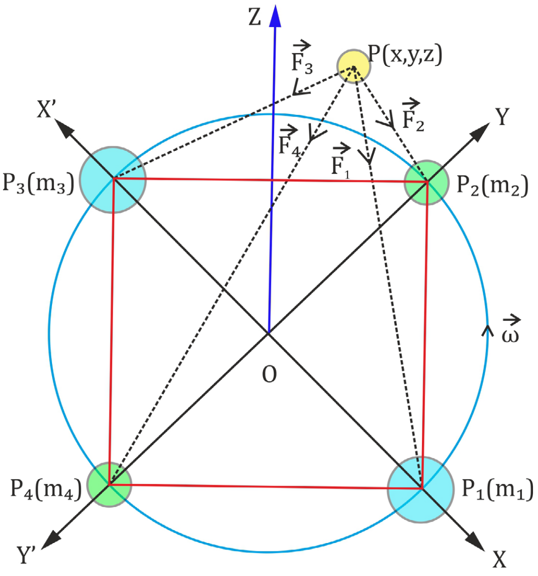

Figure 5. Position of an Infinitesimal mass in the kite configuration of second kind.

As shown in **Figure 5**, let at any time t, $P(x, y, z)$ be the position of an infinitesimal mass moving in the gravitational field of the four-point masses. It is to be noted that an infinitesimal mass has no influence of attraction on the point masses, but influenced by them.

Let $\overrightarrow{OP} = \vec{\rho}$, $\overrightarrow{P_k P} = \vec{\rho}_k, \left(k = 1,2,3,4\right)$ then,

$$\vec{\rho} = x\hat{i} + y\hat{j} + z\hat{k}, \vec{\rho}_k = \left(x - x_k\right)\hat{i} + \left(y - y_k\right)\hat{j} + \left(z - z_k\right)\hat{k}$$

$$\Rightarrow \rho^2 = x^2 + y^2 + z^2 \text{ and } \rho_k^2 = \left(x - x_k\right)^2 + \left(y - y_k\right)^2 + \left(z - z_k\right)^2$$

$$\rho_1^2 = \left(x - 1/2\right)^2 + y^2 + z^2, \quad \rho_2^2 = x^2 + \left(y - 1/2\right)^2 + z^2,$$

$$\rho_3^2 = \left(x + 1/2\right)^2 + y^2 + z^2, \quad \rho_4^2 = x^2 + \left(y + 1/2\right)^2 + z^2.$$

The force of attraction on the infinitesimal mass m due to the masses m_k, are

$$F_k = \frac{Gmm_k}{\rho_k^2} \Rightarrow \vec{F}_k = -\frac{Gmm_k}{\rho_k^2}\hat{\rho}_k = -\frac{Gmm_k\vec{\rho}_k}{\rho_k^3},$$

where $\hat{\rho}_k$ is the unit vector along $\overrightarrow{P_k P} = \vec{\rho}_k$.

Therefore, the total force of attraction on the infinitesimal mass due to four-point masses at the vertices of a cyclic kite configuration is given by

$$\vec{F} = \sum_{k=1}^{4} \vec{F}_k = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = -Gm\left[\frac{m_1\vec{\rho}_1}{\rho_1^3} + \frac{m_2\vec{\rho}_2}{\rho_2^3} + \frac{m_3\vec{\rho}_3}{\rho_3^3} + \frac{m_4\vec{\rho}_4}{\rho_4^3}\right],$$

$$= -Gm\left[\frac{\left(1-2\mu\right)\vec{\rho}_1}{2\rho_1^3} + \frac{\mu\vec{\rho}_2}{\rho_2^3} + \frac{\left(1-2\mu\right)\vec{\rho}_3}{2\rho_3^3} + \frac{\vec{\mu}\rho_4}{\rho_4^3}\right]. \tag{34}$$

But

$$\vec{F} = m\underbrace{\left[\overbrace{\frac{\partial^2\vec{\rho}}{\partial t^2}}^{\text{Relative}} + \overbrace{2\vec{\omega}\times\frac{\partial\vec{\rho}}{\partial t}}^{\text{Coriolis}} + \overbrace{\frac{\partial\vec{\omega}}{\partial t}\times\vec{\rho}}^{\text{Euler}} + \overbrace{\vec{\omega}\times\left(\vec{\omega}\times\vec{\rho}\right)}^{\text{Centrifugal}}\right]}_{\text{Acceleration}} \tag{35}$$

where $\vec{\omega} = n\hat{k}$ is the angular velocity of the plane of motion about the Z-axis, and $\vec{\rho} = x\hat{i} + y\hat{j} + z\hat{k}$ so $\frac{\partial^2\vec{\rho}}{\partial t^2} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}$,

$$\vec{\omega}\times\frac{\partial\vec{\rho}}{\partial t} = -n\dot{y}\hat{i} + n\dot{x}\hat{j}, \quad \frac{\partial\vec{\omega}}{\partial t} = 0, \quad \vec{\omega}\times\left(\vec{\omega}\times\vec{\rho}\right) = -n^2x\hat{i} - n^2y\hat{j}.$$

Thus, from Equation (35)

$$\vec{F} = m\left[\ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k} - 2n\dot{y}\hat{i} + 2n\dot{x}\hat{j} - n^2x\hat{i} - n^2y\hat{j}\right] \tag{36}$$

Comparing Equations (34) and (36), we get

$$\begin{aligned}
&\left(\ddot{x} - 2n\dot{y} - n^2x\right)\hat{i} + \left(\ddot{y} + 2n\dot{x} - n^2y\right)\hat{j} + \ddot{z}\hat{k} \\
&= -G\Bigg[\left\{\frac{\left(1-2\mu\right)\left(x-1/2\right)}{2\rho_1^3} + \frac{\mu x}{\rho_2^3} + \frac{\left(1-2\mu\right)\left(x+1/2\right)}{2\rho_3^3} + \frac{\mu x}{\rho_4^3}\right\}\hat{i} \\
&\quad + \left\{\frac{\left(1-2\mu\right)y}{2\rho_1^3} + \frac{\mu\left(y-1/2\right)}{\rho_2^3} + \frac{\left(1-2\mu\right)y}{2\rho_3^3} + \frac{\mu\left(y+1/2\right)}{\rho_4^3}\right\}\hat{j} \\
&\quad + \left\{\frac{\left(1-2\mu\right)z}{2\rho_1^3} + \frac{\mu z}{\rho_2^3} + \frac{\left(1-2\mu\right)z}{2\rho_3^3} + \frac{\mu z}{\rho_4^3}\right\}\hat{k}\Bigg].
\end{aligned}$$

Now choosing units of force so that $G=1$ and taking scalar products of $\hat{i},\hat{j},\hat{k}$ we get

$$\left.\begin{aligned}\ddot{x}-2n\dot{y}&=n^2x-\frac{(1-2\mu)(x-1/2)}{2\rho_1^3}-\frac{\mu x}{\rho_2^3}-\frac{(1-2\mu)(x+1/2)}{2\rho_3^3}-\frac{\mu x}{\rho_4^3},\\ \ddot{y}+2n\dot{x}&=n^2y-\frac{(1-2\mu)y}{2\rho_1^3}-\frac{\mu(y-1/2)}{\rho_2^3}-\frac{(1-2\mu)y}{2\rho_3^3}-\frac{\mu(y+1/2)}{\rho_4^3},\\ \ddot{z}&=-\frac{(1-2\mu)z}{2\rho_1^3}-\frac{\mu z}{\rho_2^3}-\frac{(1-2\mu)z}{2\rho_3^3}-\frac{\mu z}{\rho_4^3}.\end{aligned}\right\}\tag{37}$$

The system of Equation (37) represents equations of motion of the infinitesimal mass in the gravitational field of the kite of the second kind.

Multiplying respective equations of Equation (37) by $2x,2y$ and $2z$ and adding, then integrating with respect to t, we get

$$\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)=\frac{n^2}{2}\left(x^2+y^2\right)+\frac{1-2\mu}{2\rho_1}+\frac{\mu}{\rho_2}+\frac{1-2\mu}{2\rho_3}+\frac{\mu}{\rho_4}-\frac{C}{2}\tag{38}$$

Equation (38) is known as the energy integral of the infinitesimal mass, and C is called the Jacobi constant, depending upon the initial position and velocity of the infinitesimal mass. Defining a function Ω of position vectors of an infinitesimal mass as

$$\Omega=\frac{n^2}{2}\left(x^2+y^2\right)+\frac{1-2\mu}{2\rho_1}+\frac{\mu}{\rho_2}+\frac{1-2\mu}{2\rho_3}+\frac{\mu}{\rho_4}.\tag{39}$$

If $\vec{v}=\dot{x}\hat{i}+\dot{y}\hat{j}+\dot{z}\hat{k}$ is the linear velocity vector of an infinitesimal mass at $P(x,y,z)$, then from Equation (38), we get $v^2=2\Omega-C$. The function Ω is generally known as the kinetic potential of an infinitesimal mass. There will be curves of zero velocity given by $v=0$, *i.e.*, $2\Omega=C$.

$$\Rightarrow n^2\left(x^2+y^2\right)+\frac{1-2\mu}{\rho_1}+\frac{2\mu}{\rho_2}+\frac{1-2\mu}{\rho_3}+\frac{2\mu}{\rho_4}=C\tag{40}$$

represents the surface of zero velocity.

The motion of an infinitesimal mass can occur only in those regions for which $2\Omega-C>0$ *i.e.*, $2\Omega>C$.

Differentiating Ω partially with respect to x,y and z, we get

$$\left.\begin{aligned}\frac{\partial\Omega}{\partial x}&=n^2x-\frac{(1-2\mu)(x-1/2)}{2\rho_1^3}-\frac{\mu x}{\rho_2^3}-\frac{(1-2\mu)(x+1/2)}{2\rho_3^3}-\frac{\mu x}{\rho_4^3},\\ \frac{\partial\Omega}{\partial y}&=n^2y-\frac{(1-2\mu)y}{2\rho_1^3}-\frac{\mu(y-1/2)}{\rho_2^3}-\frac{(1-2\mu)y}{2\rho_3^3}-\frac{\mu(y+1/2)}{\rho_4^3},\\ \frac{\partial\Omega}{\partial z}&=-\frac{(1-2\mu)z}{2\rho_1^3}-\frac{\mu z}{\rho_2^3}-\frac{(1-2\mu)z}{2\rho_3^3}-\frac{\mu z}{\rho_4^3}.\end{aligned}\right\}\tag{41}$$

Comparing Equations of (37) and (41), we get

$$\left.\begin{aligned}\ddot{x}-2n\dot{y}&=\frac{\partial\Omega}{\partial x},\\ \ddot{y}+2n\dot{x}&=\frac{\partial\Omega}{\partial y},\\ \ddot{z}&=\frac{\partial\Omega}{\partial z}.\end{aligned}\right\}\tag{42}$$

This system of equations represents the equations of motion of an infinitesimal mass at $P(x,y,z)$, in the gravitational field of the cyclic kite configuration of the second kind, in terms of kinetic potential Ω.

6. Application of the Equations of Motion

If Ω is independent of x and y, then $\frac{\partial\Omega}{\partial x}=\frac{\partial\Omega}{\partial y}=0$ and hence, the equation of an infinitesimal mass is reduced to a single equation as

$$\ddot{z}=-\frac{(1-2\mu)z}{2\rho_1^3}-\frac{\mu z}{\rho_2^3}-\frac{(1-2\mu)z}{2\rho_3^3}-\frac{\mu z}{\rho_4^3} \tag{43}$$

with $\rho_1^2=\rho_2^2=\rho_3^2=\rho_4^2=z^2+1/4$.

Thus, the equation of motion of an infinitesimal mass of the convex central kite configuration along the Z-axis is

$$\ddot{z}=-\frac{z}{\left(z^2+1/4\right)^{3/2}}=f(z) \tag{44}$$

i.e., force per unit mass towards the origin is a function of *z* as shown in **Figure 6**.

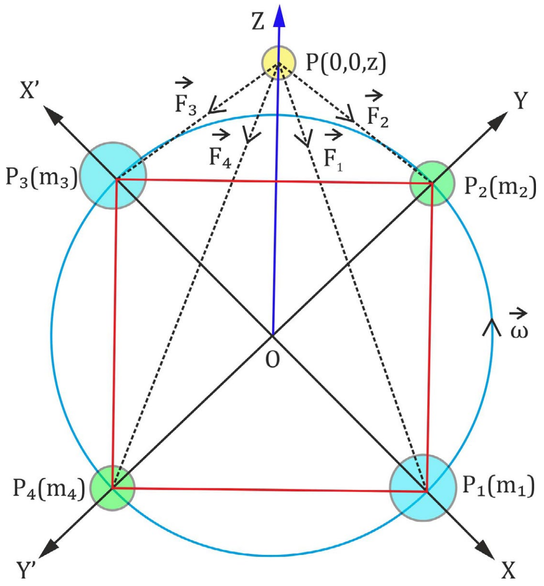

Figure 6. Position of an infinitesimal mass in sitnikov form in kite configuration.

Equation (44) is nothing but the Sitnikov equation of motion of an infinitesimal mass in the gravitational field of the four-body problem with the masses $\mu, 1/2-\mu, \mu, 1/2-\mu$. The classical Sitnikov problem is the particular case of the restricted three-body problem with the primaries of equal masses, and similarly, in our case Sitnikov problem is the specific case of a cyclic kite configuration; the same equation has been established with the two pairs of equal masses at the opposite vertices of the cyclic kite configuration of the second kind. The Sitnikov

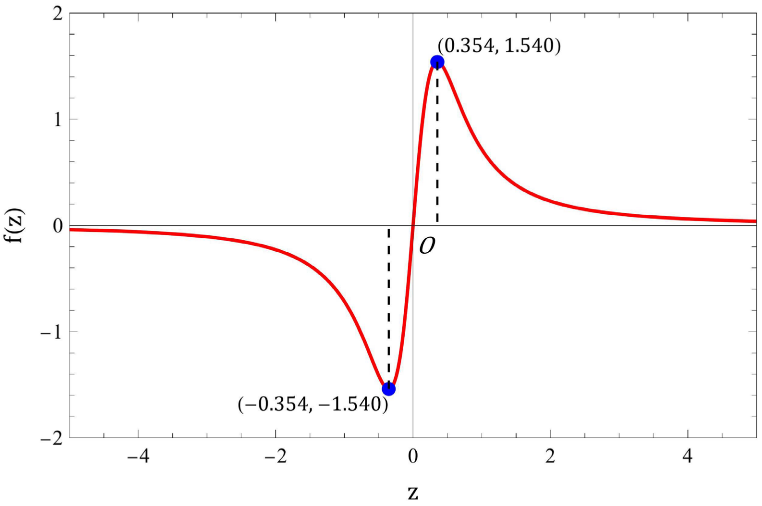

Figure 7. The graph of displacement z versus acceleration $\ddot{z} = f(z)$.

motion of an infinitesimal mass is one-dimensional, *i.e.*, oscillates on the Z-axis about the origin, whose maximum and minimum acceleration are shown in **Figure 7**. Equation (43) is not easily tractable; however, Equation (44) is comparatively simpler to solve, as it reduces to a one-dimensional form. The series solution of Equation (44) cannot be obtained through the Lindstedt-Poincare method or other conventional techniques. Therefore, it is solved exclusively by employing the method of iteration of Green's function, as outlined below:

From Equation (44), we have

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -\frac{z}{\left(z^2 + 1/4\right)^{3/2}} = -8z + 48z^3 - 240z^5 + \cdots.$$

In comparison of units of distance (*i.e.*, $2R = 1$), z is very-very small, so $z \ll 1$. Neglecting higher order terms of z above the third order and hence, we get

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} = -8z + 48z^3 \tag{45}$$

This equation cannot be solved by the Lindstedt-Poincare method as the coefficient of $z^3 = 48 > 1$.

Let us solve the equation by the iteration of Green's function. From Equation (45), we have

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + 8z = 48z^3 \quad \Rightarrow \frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \eta_0^2 z = \left(\varepsilon z^2\right) z \quad \text{where } \eta_0^2 = 8 \text{ and } \varepsilon = 48 > 1.$$

Let us write

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \eta_0^2 z = \left(\varepsilon z^2\right) z = -f\left(z(t)\right). \tag{46}$$

The general solution of the equation

$$\frac{\mathrm{d}^2 z}{\mathrm{d}t^2} + \eta_0^2 z = 0, \tag{47}$$

can be written as $z = c\exp\left(i\eta_0 t\right)$, where c is the initial value of z at $t = 0$.

Let the solution of Equation (46) be

$$P(x,y,z), z = c\exp(i\eta_0 t) + z^*(t). \tag{48}$$

Since the first term of the right-hand side of Equation (45) satisfies the homogeneous Equation (47), hence

$$\frac{\mathrm{d}^2 z^*}{\mathrm{d}t^2} + \eta_0^2 z^* = (\varepsilon z^2) z = -f(z(t)), \tag{49}$$

where z^* is the solution of the integral equation

$$z^*(t) = \int G(t,\xi') f(\xi')\,\mathrm{d}\xi' \tag{50}$$

where the Green's function G is defined by the particular integral of the differential equation

$$\frac{\mathrm{d}^2 G}{\mathrm{d}t^2} + \eta_0^2 G = -\delta(t-\xi') \tag{51}$$

where the Dirac delta function $\delta(t-\xi')$ is defined as

$$\delta(t-\xi') = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{is(t-\xi')\}\,\mathrm{d}s. \tag{52}$$

Now, the Green's function $G(t,\xi')$ is the particular integral of Equation (51), *i.e.*,

$$P.I. = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{s^2-\eta_0^2}\exp\{is(t-\xi')\}\,\mathrm{d}s \tag{53}$$

On the right-hand side of Equation (53), $s = \pm\eta_0$ are the poles of order one, so by the residue theorem, we find the solution to Equation (53) by an iteration process using the term ($\in z^2$) as the iteration and using

$$f(t) = \int \delta(t-\xi') f(\xi')\,\mathrm{d}\xi',$$

The first iteration is $z^*(\xi') = -\varepsilon\int G(t,\xi')(z(\xi'))^2 z(\xi')\,\mathrm{d}\xi'$ and;

The second iteration is $z^*(\xi'') = -\varepsilon\int G(\xi',\xi'')(z(\xi''))^2 z(\xi'')\,\mathrm{d}\xi''$.

Hence, from Equation (50), we have

$$\begin{aligned} z^*(t) &= -\varepsilon\int G(t,\xi')(z(\xi'))^2 z(\xi')\,\mathrm{d}\xi' \\ &\quad + \varepsilon^2\iint G(t,\xi')(z(\xi'))^2 G(\xi',\xi'')(z(\xi''))^2 z(\xi'')\,\mathrm{d}\xi''\mathrm{d}\xi'. \end{aligned}$$

Finally, from Equation (48), we have

$$z = c\exp(t,\xi') - \varepsilon\int G(t,\xi')(z(\xi'))^2 z(\xi')\,\mathrm{d}\xi' \tag{54}$$

Here in Equation (54), the iteration giving the position of an infinitesimal mass is given by the first approximation of the simple harmonic term of Equation (49) from $z(\xi'')$ to $z(\xi')$ and then from $z(\xi')$ to $z(t)$ at the time ξ' and ξ''. The iterations are $\varepsilon z^2(\xi'')$ and $\varepsilon z^2(\xi')$ with the respective propagators $G(\xi',\xi'')$ and $G(t,\xi')$.

To the first approximation, we use Equation (48) as $z(\xi') = c\exp(i\eta_0\xi')$ and hence, Equation (54) reduces to

$$z = c\exp\left(i\eta_0 t\right) + z_1 + z_2 \tag{55}$$

where $z_1 = -\varepsilon\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 z\left(\xi'\right)\mathrm{d}\xi'$,

$$z_2 = \varepsilon^2\int\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 G\left(\xi',\xi''\right)\left(z\left(\xi''\right)\right)^2 z\left(\xi''\right)\mathrm{d}\xi''\mathrm{d}\xi'.$$

Now,

$$\left.\begin{aligned} z_1 &= -\varepsilon\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 z\left(\xi'\right)\mathrm{d}\xi' \\ &= \frac{\varepsilon^2 c^5}{16\eta_0^4}\left[\exp\left(i\eta_0 t\right) - 2\exp\left(i3\eta_0 t\right) + \exp\left(i5\eta_0 t\right)\right] \\ z_2 &= \varepsilon^2\int\int G\left(t,\xi'\right)\left(z\left(\xi'\right)\right)^2 G\left(\xi',\xi''\right)\left(z\left(\xi''\right)\right)^2 z\left(\xi''\right)\mathrm{d}\xi''\mathrm{d}\xi' \\ &= \frac{9c^5}{4}\left[\exp\left(i2\sqrt{2}t\right) - 2\exp\left(i6\sqrt{2}t\right) + \exp\left(i10\sqrt{2}t\right)\right]. \end{aligned}\right\} \tag{56}$$

Thus, finally the solution of Equation (55) can be written as

$$\begin{aligned} z &= c\exp\left(i2\sqrt{2}t\right) + z_1 + z_2 \\ &= c\exp\left(i2\sqrt{2}t\right) + \frac{3c^3}{2}\left[\exp\left(i2\sqrt{2}t\right) - \exp\left(i6\sqrt{2}t\right)\right] \\ &\quad + \frac{9c^5}{4}\left[\exp\left(i2\sqrt{2}t\right) - 2\exp\left(i6\sqrt{2}t\right) + \exp\left(i10\sqrt{2}t\right)\right]. \end{aligned}$$

$$\begin{aligned} \Rightarrow z &= c\left(1 + \frac{3c^2}{2} + \frac{9c^4}{4}\right)\exp\left(i2\sqrt{2}t\right) - \frac{3c^3}{2}\left(1 + 3c^2\right)\exp\left(i6\sqrt{2}t\right) \\ &\quad + \frac{9c^5}{4}\exp\left(i10\sqrt{2}t\right). \end{aligned} \tag{57}$$

This is the required periodic solution of the Sitnikov equation of motion of an infinitesimal mass moving in the gravitational field of the central kite configuration of the second kind. **Figures 8-12** represent the periodic solution of the infinitesimal mass's motion with different values of *c* varying from 0.1, 0.5, 0.9, 1 and 3. **Figures 13-16** represent the trajectory of the periodic solutions *z(t)* with different values of *c* varying from 0.3, 0.7, 1 and 2.

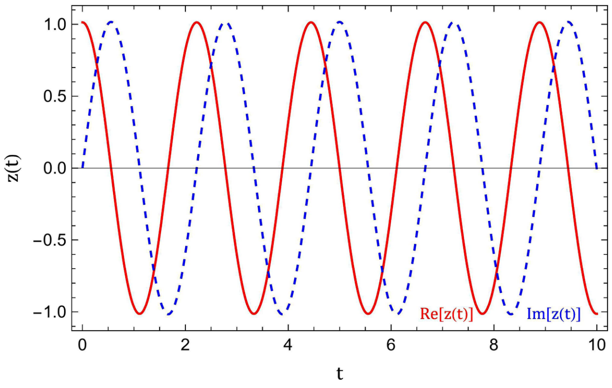

Figure 8. Periodic Solution for $c = 0.1$.

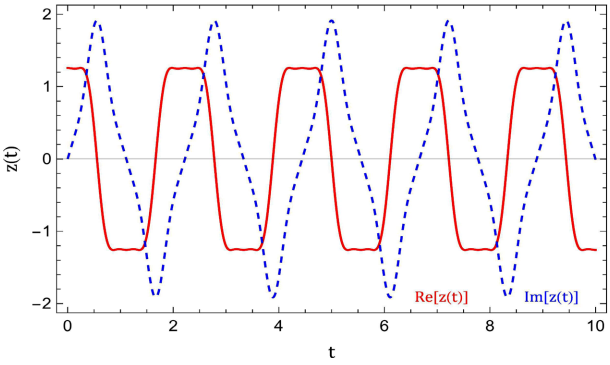

Figure 9. Periodic Solution for $c = 0.5$.

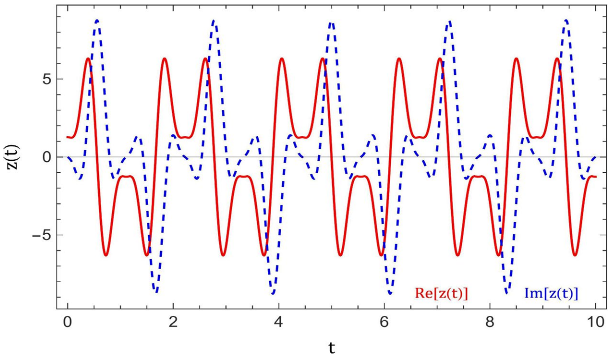

Figure 10. Periodic Solution for $c = 0.9$.

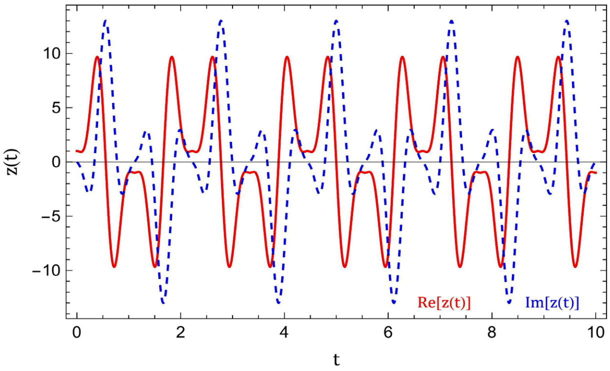

Figure 11. Periodic Solution for $c = 1$.

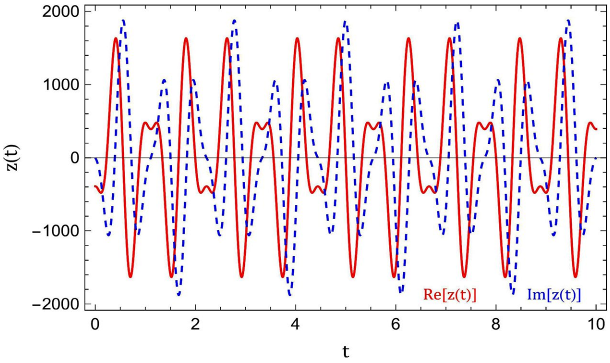

Figure 12. Periodic Solution for $c = 3$.

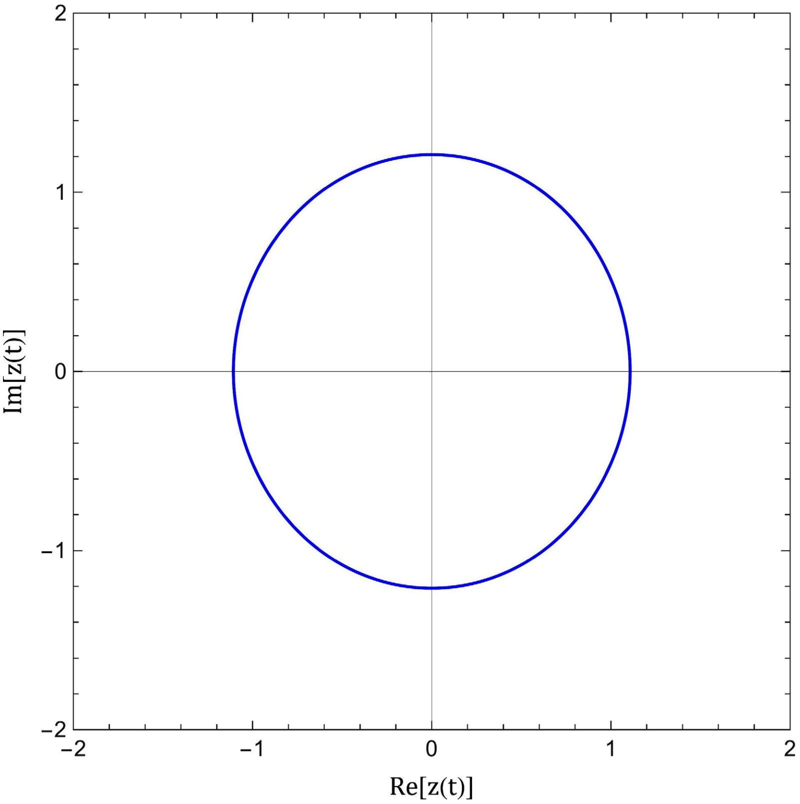

Figure 13. Trajectory of periodic solutions for $c = 0.3$ in the complex plane.

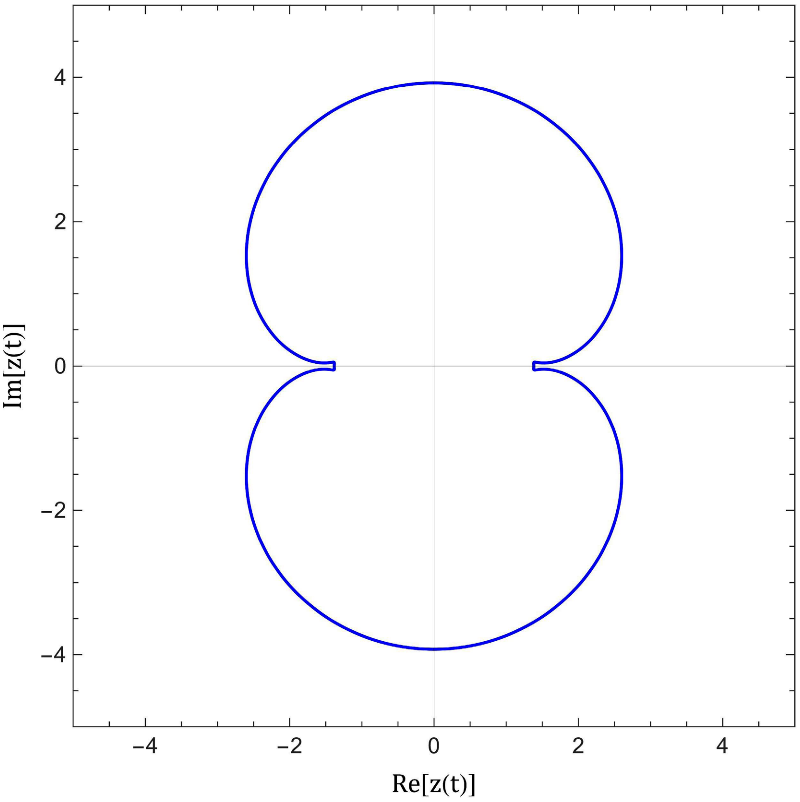

Figure 14. Trajectory of periodic solutions for $c = 0.7$ in the complex plane.

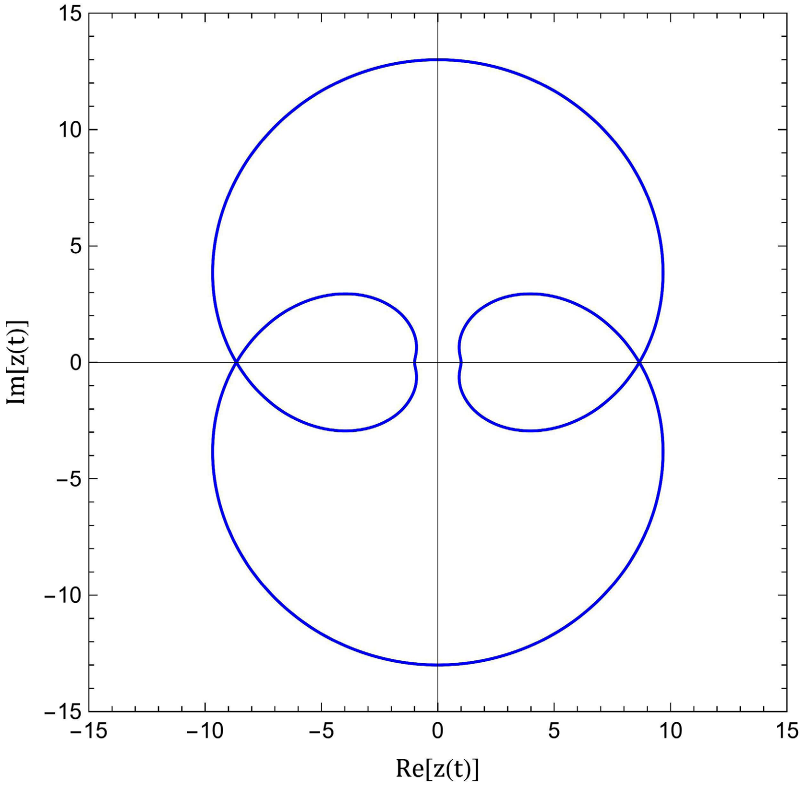

Figure 15. Trajectory of Periodic Solutions for $c = 1$ in the Complex Plane.

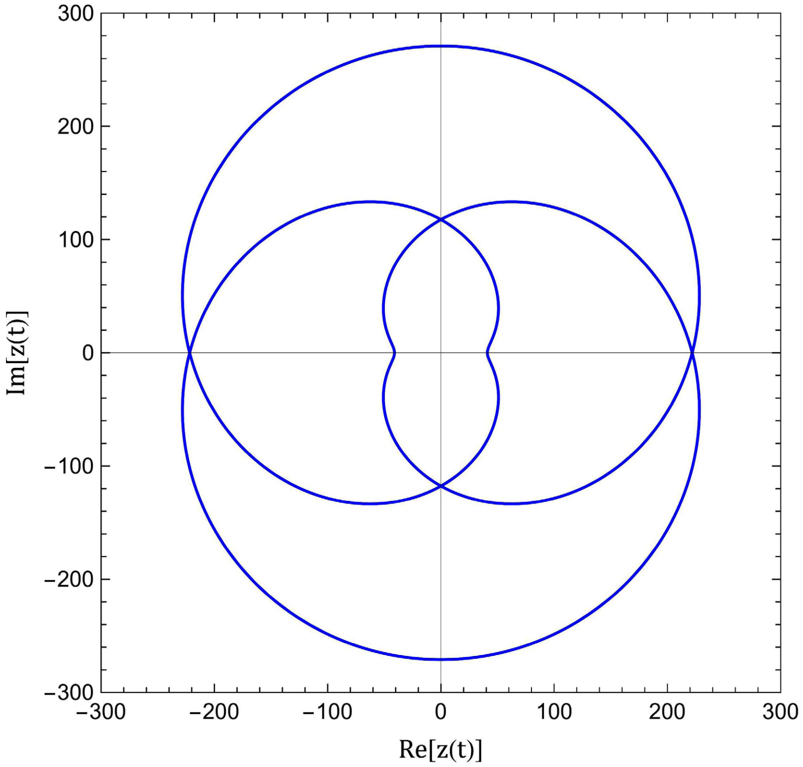

Figure 16. Trajectory of Periodic Solutions for $c = 2$ in the Complex Plane.

7. Conclusion

This manuscript presents a comprehensive study of the equations of motion of an infinitesimal mass moving in the gravitational field of a cyclic kite configuration of the second kind, with applications to the Sitnikov problem. Section 1 provides a review of previous works ranging from MacMillan *et al.* [1] to Alam *et al.* [14]. Section 2 discusses the classification of the kite configuration and establishes three theorems, along with the collapse condition of the kite configuration of the first kind and their justifications. In Section 3, the equations of motion of four bodies forming a kite, expressed in terms of position vectors, the mass parameter, and the mean motion, are derived. Section 4 establishes the admissible domain of the mass parameter. Section 5 formulates the equations of motion of the infinitesimal mass under the influence of the kite configuration. Finally, in Section 6, these results are applied to the Sitnikov problem, demonstrating that the motion of the infinitesimal mass in the kite configuration reduces to the same form as in the classical Sitnikov problem. The equations are then solved using iteration of the Green's function and periodic series solutions, with the corresponding trajectories illustrated graphically for different initial values of c.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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