Scientific Research Publishing

International Journal of Astronomy and Astrophysics, 2025, 15(3), 282-292
https://www.scirp.org/journal/ijaa
ISSN Online: 2161-4725
ISSN Print: 2161-4717

Relativistic Effects on the Existence of Libration Points in the Sitnikov Problem

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How to cite this paper: Sultana, S., Hassan, M.R., Kumar, L., Hassan, M.A., Kumar, P., Shubhra and Alam, M. (2025) Relativistic Effects on the Existence of Libration Points in the Sitnikov Problem. *International Journal of Astronomy and Astrophysics*, **15**, 282-292.
https://doi.org/10.4236/ijaa.2025.153018

Received: May 11, 2025
Accepted: September 9, 2025
Published: September 12, 2025

Abstract

In the present manuscript, we investigated the relativistic effects on the existence of libration points in the Sitnikov Restricted Three-Body Problem (SRTBP). The Brumberg relativistic model is employed to incorporate post-Newtonian effects into the classical framework. The analysis reveals that, due to relativistic contributions, libration points emerge along the Z-axis at equal distances on either side of the origin; an outcome absent in the classical Newtonian formulation.

Keywords

Restricted Three-Body Problem (RTBP), Sidereal Frame, Synodic Frame, Sitnikov Problem, Relativistic Sitnikov Problem, Libration Points

1. Introduction

The Sitnikov problem is a special case of the restricted three-body problem in which two primary bodies of equal masses $m_1 = m_2 = 1/2$ revolve around their common centre of mass under Newtonian gravitational attraction, following either circular or elliptical orbits. An infinitesimal mass m (infinitesimally small compared to the primaries) moves along a line perpendicular to the orbital plane of the primaries, passing through their centre of mass. Brumberg [1] extensively studied the motion of the Moon, a subject that has held a central place in celestial mechanics. Many prominent astronomers and mathematicians have contributed significantly to lunar theory. By the late 19th and early 20th centuries, several high-precision models describing lunar motion had been developed. Among these, the Hill-Brown theory emerged as the most accurate and remains in use today.

Faruque [2] has studied the axial oscillation of a planetoid in the circular restricted three-body problem by applying the iteration process of Green's function. Further, Faruque [3] has established a new analytical expression for the position of the infinitesimal mass in the elliptic Sitnikov problem. Sidorenko [4] has studied the circular Sitnikov problem with the alternation of stability and instability in the family of vertical motions. Douskos *et al.* [5] have discussed the Sitnikov-like motions generating new kinds of 3d periodic orbits in the restricted three-body problem with prolate primaries. Kovacs *et al.* [6] discussed the relativistic effects in the Chaotic Sitnikov problem. The Sitnikov problem is one of the simplest dynamical systems in celestial mechanics that provides all kinds of chaotic behaviours.

The configuration of the system is defined by two point-like bodies of equal masses (called primaries) orbiting around their common centre of mass due to their mutual gravitational forces and a third body of negligible mass moving along a line, perpendicular to the orbital plane of the primaries, passing through their barycenter. For the circular motion of the primaries, the problem is integrable, and MacMillan [7] gave a closed-form analytical solution with elliptic integrals. Wodnar [8] introduced a new formulation for the equation of motion by using the true anomaly of the primaries as an independent variable. Hagel *et al.* [9] extended the analytical approximations up to very high orders by using extensive computer algebra. Dvorak [10] showed by numerical computations that invariant curves exist for small oscillations around the barycentre.

An analytical expression for the position of the infinitesimal body in the elliptic Sitnikov restricted three-body problem was provided by Suraj and Hassan [11]. This solution is valid for small-bounded oscillations in the case of moderate eccentricity of primaries. They have linearized the equation of motion to obtain the Hill-type equation. Using the Courant and Snyder transformation, Hill's equations were transformed into a Harmonic oscillator-type equation. They used the Lindstedt-Poincare perturbation method, and again they applied the Courant and Snyder transformation to obtain the final result. They investigated the phase-space structure of the relativistic Sitnikov problem in the first post-Newtonian approximations. The phase-space portraits show a strong dependence on the gravitational radius, which describes the strength of the relativistic pericenter advance. Ullah *et al.* [12] developed the series solutions of the Sitnikov kite configuration by the methods given by Lindstedt-Poincare and MacMillan. They have developed an averaged equation of motion by applying the Van der Pol transformation and the averaging technique of Guckenheimer and Holmes.

No previous author has worked on the relativistic effects on the existence of libration points of the Sitnikov restricted three-body problem, so presently we propose to study the same.

2. Equations of Motion of the Infinitesimal Mass in the RTBP

Considering an inertial frame (Sidereal frame) $(O, X_0 Y_0 Z)$ and a rotating frame

(Synodic frame) (O, XYZ) to discuss the existence of libration points in the Sitnikov problem. Before discussing the existence of libration points with relativistic effects, let us introduce the existence of libration points in the classical case. Let the synodic frame rotate relative to the sidereal frame with a constant angular velocity $\vec{\omega} = n\hat{k}$ ($|\vec{\omega}| = n$) about the vertical Z-axis OZ, common to both frames. Let at any time t, $P(x, y, z)$ be the position of the infinitesimal mass moving in the gravitational field of two-point masses (spherical in shape) situated at $P_s(s = 1, 2)$. It is to be noted that the infinitesimal mass has no influence of attraction on the point masses, but it is being influenced by them.

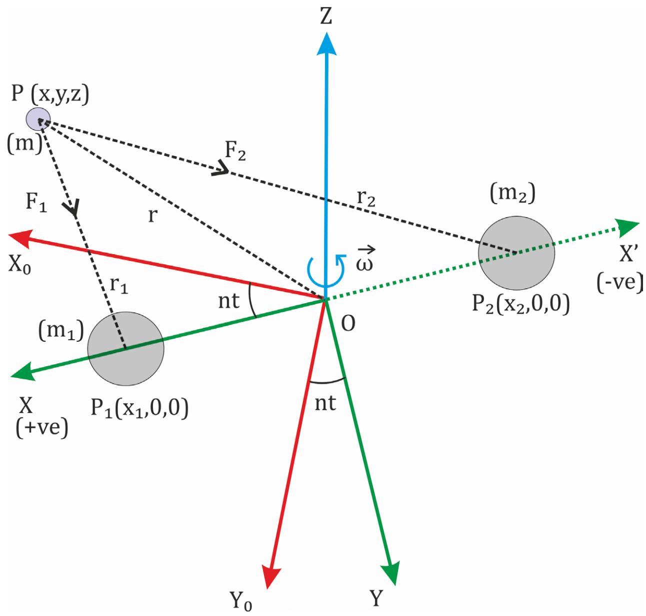

Figure 1. Representation of the restricted three-body problem.

In Figure 1, $\overrightarrow{OP} = \vec{r}$ then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\overrightarrow{P_sP} = \vec{r_s} = (x - x_s)\hat{i} + y\hat{j} + z\hat{k},$$

$$\vec{r_1} = (x - x_1)\hat{i} + y\hat{j} + z\hat{k}, \quad \vec{r_2} = (x - x_2)\hat{i} + y\hat{j} + z\hat{k}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2, \quad r_1^2 = (x - x_1)^2 + y^2 + z^2, \quad r_2^2 = (x - x_2)^2 + y^2 + z^2.$$

Let $\vec{F_s}$ be the forces of attraction of the point masses on the infinitesimal mass at $P(x, y, z)$, then

$$\vec{F_s} = -\frac{Gmm_s}{r_s^2}\hat{r_s} = -\frac{Gmm_s}{r_s^2}\frac{\vec{r_s}}{r_s} = -\frac{Gmm_s}{r_s^3}\vec{r_s},$$ $\hat{r_s}$ is the unit vector along $\overrightarrow{P_sP}$.

Therefore, the total force of attraction on the infinitesimal mass due to two-point masses is given by

$$\vec{F} = \sum_{s=1}^{2}\vec{F_s} = -\sum_{s=1}^{2}\frac{Gmm_s}{r_s^3}\vec{r_s} = -Gm\left[\frac{m_1}{r_1^3}\vec{r_1} + \frac{m_2}{r_2^3}\vec{r_2}\right]$$

$$= -Gm\left[\frac{m_1}{r_1^3}\left\{(x - x_1)\hat{i} + y\hat{j} + z\hat{k}\right\} + \frac{m_2}{r_2^3}\left\{(x - x_2)\hat{i} + y\hat{j} + z\hat{k}\right\}\right]$$

$$\Rightarrow \vec{F} = -Gm\left[\left\{\frac{m_1(x-x_1)}{r_1^3}+\frac{m_2(x-x_2)}{r_2^3}\right\}\hat{i}+\left\{\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}\right\}y\hat{j}+\left\{\frac{m_1}{r_1^3}+\frac{m_2}{r_2^3}\right\}z\hat{k}\right]. \quad (1)$$

Also,

$$\vec{F} = m\left[\frac{\partial^2\vec{r}}{\partial t^2}+2\vec{\omega}\times\frac{\partial\vec{r}}{\partial t}+\vec{\omega}\times(\vec{\omega}\times\vec{r})\right] \quad (2)$$

where, $r = x\hat{i}+y\hat{j}+z\hat{k}$ and $\vec{\omega} = n\hat{k}$.

Thus,

$$\frac{\partial^2\vec{r}}{\partial t^2} = \frac{\partial^2 x}{\partial t^2}\hat{i}+\frac{\partial^2 y}{\partial t^2}\hat{j}+\frac{\partial^2 z}{\partial t^2}\hat{k} = \ddot{x}\hat{i}+\ddot{y}\hat{j}+\ddot{z}\hat{k}, \quad (3)$$

$$\vec{\omega}\times\frac{\partial\vec{r}}{\partial t} = n\hat{k}\times\left(\frac{\partial x}{\partial t}\hat{i}+\frac{\partial y}{\partial t}\hat{j}+\frac{\partial z}{\partial t}\hat{k}\right) = -n\dot{y}\hat{i}+n\dot{x}\hat{j} \quad (4)$$

and

$$\begin{aligned}\vec{\omega}\times(\vec{\omega}\times\vec{r}) &= (\vec{\omega}\cdot\vec{r})\vec{\omega}-(\vec{\omega}\cdot\vec{\omega})\vec{r}\\ &= n\hat{k}\left(x\hat{i}+y\hat{j}+z\hat{k}\right)n\hat{k}-n^2\left(x\hat{i}+y\hat{j}+z\hat{k}\right)\\ &= nz\left(n\hat{k}\right)-n^2x\hat{i}-n^2y\hat{j}-n^2z\hat{k},\end{aligned}$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -n^2x\hat{i}-n^2y\hat{j}. \quad (5)$$

Combining Equations (2), (3), (4) and (5), we get

$$\vec{F} = \ddot{x}\hat{i}+\ddot{y}\hat{j}+\ddot{z}\hat{k}+2\left(-n\dot{y}\hat{i}+n\dot{x}\hat{j}\right)-n^2x\hat{i}-n^2y\hat{j}. \quad (6)$$

Thus, from Equations (1) and (6), we have

$$\begin{aligned}&\left(\ddot{x}-2n\dot{y}-n^2x\right)\hat{i}+\left(\ddot{y}+2n\dot{x}-n^2y\right)\hat{j}+\ddot{z}\hat{k}\\ &= -G\left[\left\{\frac{m_1}{r_1^3}(x-x_1)+\frac{m_2}{r_2^3}(x-x_2)\right\}\hat{i}+\left(\frac{m_1}{r_2^3}y+\frac{m_2}{r_2^3}y\right)\hat{j}+\left(\frac{m_1}{r_1^3}z+\frac{m_2}{r_2^3}z\right)\hat{k}\right].\end{aligned}$$

Choosing the unit of force in such a way that $G = 1$ and equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$ from both sides, we get

$$\left.\begin{aligned}\ddot{x}-2n\dot{y} &= n^2x-\frac{m_1}{r_1^3}(x-x_1)-\frac{m_2}{r_1^3}(x-x_2),\\ \ddot{y}+2n\dot{x} &= n^2y-\frac{m_1}{r_1^3}y-\frac{m_2}{r_2^3}y,\\ \ddot{z} &= -\frac{m_1}{r_1^3}z-\frac{m_2}{r_2^3}z.\end{aligned}\right\} \quad (7)$$

The system (7) represents the equations of motion of an infinitesimal mass moving under the gravitational influence of two primaries at P_1 and P_2.

By multiplying the equations of system (7) by $2\dot{x}, 2\dot{y}, 2\dot{z}$ respectively, then adding and integrating, we get

$$\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right) = \frac{n^2}{2}\left(x^2+y^2\right)+\frac{m_1}{r_1}+\frac{m_2}{r_2}-\frac{C}{2}. \quad (8)$$

This is known as Jacobi's integral or energy integral of the infinitesimal mass in the RTBP and C is the Jacobi Constant.

If v be the linear velocity of the infinitesimal mass at $P(x,y,z)$ then $v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2$.

Let us define a function of x, y and z as

$$\Omega = \frac{n^2}{2}\left(x^2 + y^2\right) + \frac{m_1}{r_1} + \frac{m_2}{r_2}, \text{ then} \tag{9}$$

$$v^2 = 2\Omega - C \tag{10}$$

where Ω is the kinetic potential or potential function of the infinitesimal mass.

The partial derivatives of Ω with respect to x, y, z are given by

$$\left.\begin{aligned} \frac{\partial \Omega}{\partial x} &= n^2 x - \frac{m_1(x-x_1)}{r_1^3} - \frac{m_2(x-x_2)}{r_2^3}, \\ \frac{\partial \Omega}{\partial y} &= n^2 y - \frac{m_1 y}{r_1^3} - \frac{m_2 y}{r_2^3}, \\ \frac{\partial \Omega}{\partial z} &= -\frac{m_1 z}{r_1^3} - \frac{m_2 z}{r_2^3}. \end{aligned}\right\} \tag{11}$$

Comparing Equation (7) with Equation (11), we get

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial \Omega}{\partial x} \\ \ddot{y} + 2n\dot{x} &= \frac{\partial \Omega}{\partial y} \\ \ddot{z} &= \frac{\partial \Omega}{\partial z} \end{aligned}\right\} \tag{12}$$

These are the equations of motion of an infinitesimal mass in the restricted three-body problem in a three-dimensional coordinate system in terms of velocities, accelerations, and partial derivatives of the potential function.

3. Sitnikov Problem

In Sitnikov motion

$$\begin{aligned} m_1 = m_2 &= 1/2, \qquad (\text{as } m_1 + m_2 = 1) \\ \text{and} \quad r_1 = r_2 &= \sqrt{z^2 + 1/4} \quad (\text{as } x = y = 0). \end{aligned} \tag{13}$$

Thus, the System of Equations (12) is reduced to a single equation

$$\ddot{z} = \frac{\partial \Omega}{\partial z}. \tag{14}$$

This is known as the equations of motion of an infinitesimal mass in the Sitnikov problem, which is one-dimensional along the vertical Z-axis about the origin.

Using Equation (13) in the last equation of System (11), we have

$$\frac{\partial \Omega}{\partial z} = -\frac{m_1 z}{r_1^3} - \frac{m_2 z}{r_2^3} = -\frac{m_1 z}{r_1^3} - \frac{m_1 z}{r_1^3} = -\frac{2m_1 z}{r_1^3} = -\frac{z}{r_1^3} = f(z)$$

$$\Rightarrow \frac{\partial \Omega}{\partial z} = -\frac{z}{\left(z^2 + 1/4\right)^{3/2}} = f(z) \tag{15}$$

Thus, only one libration point exists at $z = 0$ (origin) in the unperturbed case of the Sitnikov problem, while taking $\frac{\partial \Omega}{\partial z} = 0$.

Figure 2 depicts the graph of the function z vs $f(z) = \partial\Omega/\partial z$ passing through the origin, indicating the location of the libration point at the origin in the classical case. **Figure 3** presents a contour plot of the same function $f(z)$ for a specified domain *i.e.*, $-2 \le z \le 2$. The two white regions highlight the neighbourhood of the singularities located near $(0, -1/2)$ and $(0, 1/2)$. The surrounding elliptic trajectories illustrate periodic orbits of the infinitesimal mass around these singularities.

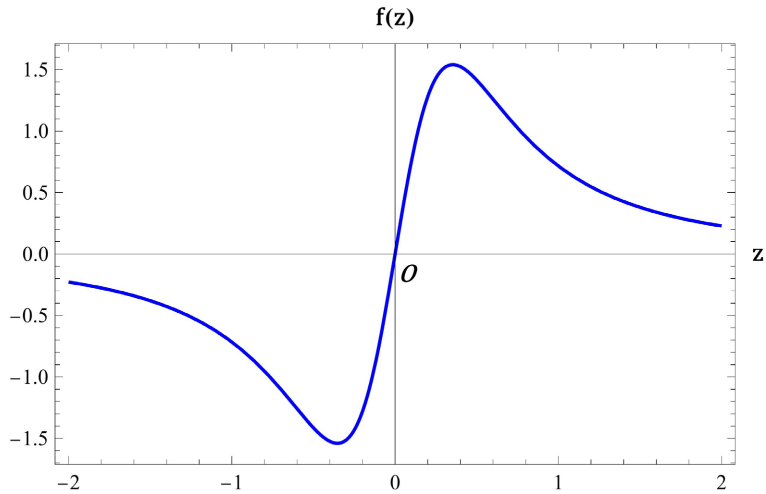

Figure 2. Graph of z vs. $f(z)$ (Classical case).

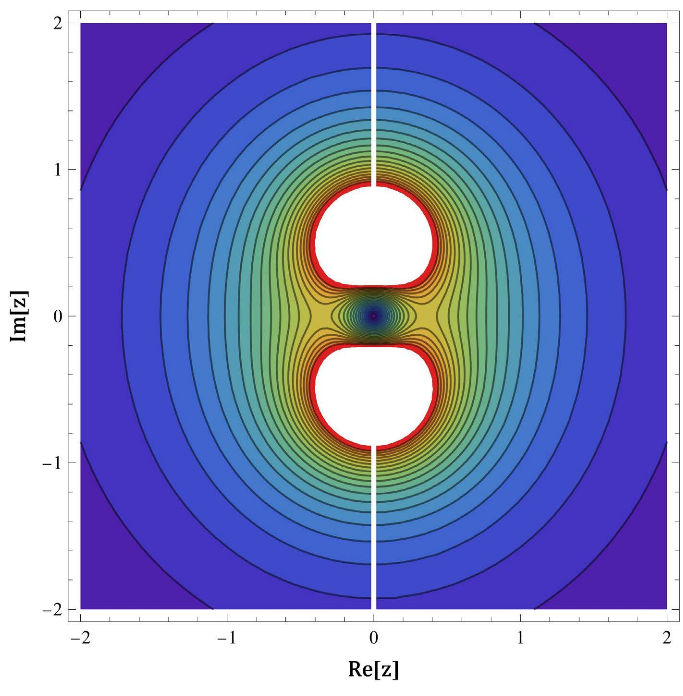

Figure 3. Contour plot of $|f(z)|$ for $-2 \le z \le 2$.

4. Equations of Motion in Relativistic Mechanics

The equations of motion of an infinitesimal mass in the relativistic field of two spherical bodies are given by Brumberg [1] as

$$\left.\begin{aligned} \ddot{x}-2n\dot{y}&=\frac{\partial U}{\partial x}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{x}}\right),\\ \ddot{y}+2n\dot{x}&=\frac{\partial U}{\partial y}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{y}}\right),\\ \ddot{z}&=\frac{\partial U}{\partial z}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{z}}\right), \end{aligned}\right\} \tag{16}$$

where,

$$\begin{aligned} U=&\frac{1}{2}\left(x^2+y^2\right)n^2+G\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)+\frac{1}{8c^2}\left\{\dot{x}^2+\dot{y}^2+\dot{z}^2+2n\left(x\dot{y}-y\dot{x}\right)+\left(x^2+y^2\right)n^2\right\}^2\\ &+\frac{3}{2c^2}G\left(\frac{m_1}{r_1}+\frac{m_2}{r_2}\right)\left\{\dot{x}^2+\dot{y}^2+\dot{z}^2+2n\left(x\dot{y}-y\dot{x}\right)+\left(x^2+y^2\right)n^2\right\}\\ &-\frac{1}{2c^2}G^2\left(\frac{m_1^2}{r_1^2}+\frac{m_2^2}{r_2^2}\right)+\frac{Gm_1m_2}{Mc^2}\left\{nR\left(4\dot{y}+\frac{7}{2}nx\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{n^2R^2}{2M}y^2\left(\frac{m_2}{r_1^3}+\frac{m_1}{r_2^3}\right)\right.\\ &\left.+n^2R^2\left(-\frac{R}{r_1r_2}+\frac{m_2-2m_1}{2Mr_1}+\frac{m_1-2m_2}{2Mr_2}\right)\right\}, \end{aligned} \tag{17}$$

where, c = speed of light and $n=\left|\frac{\sqrt{GM}}{R^{\frac{3}{2}}}\left[1-\frac{3GM}{2c^2R}\left(1-\frac{m_1m_2}{3M^2}\right)\right]\right|$.

Here in the relativistic case, the infinitesimal mass does not influence the point masses at P_1 and P_2, but is influenced by them.

Now, choosing units of force, units of mass and units of distance in such a way that $G=1$, $R=P_1P_2=1$, $M=m_1+m_2=1$, $m_1=m_2=1/2$, then Equation (17) is reduced to

$$\begin{aligned} U=&\frac{1}{2}\left(x^2+y^2\right)n^2+\frac{1}{2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\\ &+\frac{1}{8c^2}\left\{\dot{x}^2+\dot{y}^2+\dot{z}^2+2n\left(x\dot{y}-y\dot{x}\right)+\left(x^2+y^2\right)n^2\right\}^2\\ &+\frac{3}{4c^2}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left\{\dot{x}^2+\dot{y}^2+\dot{z}^2+2n\left(x\dot{y}-y\dot{x}\right)+\left(x^2+y^2\right)n^2\right\}\\ &-\frac{1}{8c^2}\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}\right)+\frac{1}{4c^2}\left\{n\left(4\dot{y}+\frac{7}{2}nx\right)\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{n^2}{4}y^2\left(\frac{1}{r_1^3}+\frac{1}{r_2^3}\right)\right.\\ &\left.+n^2\left(-\frac{1}{r_1r_2}-\frac{1}{4r_1}-\frac{1}{4r_2}\right)\right\} \end{aligned} \tag{18}$$

where,

$$n=\left|1-\frac{3}{2c^2}\left(1-\frac{1/4}{3}\right)\right|=\left|1-\frac{11}{8c^2}\right|<1 \tag{19}$$

For Sitnikov's relativistic restricted three-body problem $m_1=m_2=1/2$,

$r_1 = r_2 = \sqrt{z^2 + 1/4}$ (as $x = y = 0$).

From Equation (18), we have

$$U = \frac{1}{\sqrt{z^2+1/4}} + \frac{\dot{z}^4}{8c^2} + \frac{3\dot{z}^2}{2c^2\sqrt{z^2+1/4}} - \frac{1}{4c^2\left(z^2+1/4\right)} - \frac{n^2}{4c^2\left(z^2+1/4\right)} - \frac{n^2}{8c^2\sqrt{z^2+1/4}}, \tag{20}$$

$$\Rightarrow \frac{\partial U}{\partial z} = -\frac{z}{\left(z^2+1/4\right)^{3/2}} + \frac{\dot{z}^2\ddot{z}}{2c^2} + \frac{3\ddot{z}}{c^2\left(z^2+1/4\right)^{1/2}} - \frac{3\dot{z}^2 z}{2c^2\left(z^2+1/4\right)^{3/2}} + \frac{z}{2c^2\left(z^2+1/4\right)^2} + \frac{n^2 z}{2c^2\left(z^2+1/4\right)^2} + \frac{n^2 z}{8c^2\left(z^2+1/4\right)^{3/2}}, \tag{21}$$

and

$$\frac{\partial U}{\partial \dot{z}} = \frac{4\dot{z}^3}{8c^2} + \frac{6\dot{z}}{2c^2\sqrt{z^2+1/4}} = \frac{\dot{z}^3}{2c^2} + \frac{3\dot{z}}{c^2\sqrt{z^2+1/4}}, \tag{22}$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{z}}\right) = \frac{1}{2c^2}\cdot 3\dot{z}^2\cdot\ddot{z} + \frac{3}{c^2\sqrt{z^2+1/4}}\ddot{z} = \frac{3\dot{z}^2\ddot{z}}{2c^2} + \frac{3\ddot{z}}{c^2\sqrt{z^2+1/4}}. \tag{23}$$

Thus, the equations of motion (16) of an infinitesimal mass in the relativistic Sitnikov restricted three-body problem is reduced to a single equation as

$$\ddot{z} = -\frac{z}{\left(z^2+1/4\right)^{3/2}} + \frac{\dot{z}^2\ddot{z}}{2c^2} + \frac{3\ddot{z}}{c^2\left(z^2+1/4\right)^{1/2}} - \frac{3z\dot{z}^2}{2c^2\left(z^2+1/4\right)^{3/2}} + \frac{z}{2c^2\left(z^2+1/4\right)^2} + \frac{n^2 z}{2c^2\left(z^2+1/4\right)^2} + \frac{n^2 z}{8c^2\left(z^2+1/4\right)^{3/2}} - \frac{3\dot{z}^2\ddot{z}}{2c^2} - \frac{3\ddot{z}}{c^2\sqrt{z^2+1/4}},$$

$$\Rightarrow \ddot{z}\left(1+\frac{2\dot{z}^2}{2c^2}\right) = -\frac{z}{\left(z^2+1/4\right)^{3/2}} - \frac{3\dot{z}^2 z}{2c^2\left(z^2+1/4\right)^{3/2}} + \frac{z}{2c^2\left(z^2+1/4\right)^2} + \frac{n^2 z}{2c^2\left(z^2+1/4\right)^2} + \frac{n^2 z}{8c^2\left(z^2+1/4\right)^{3/2}}. \tag{24}$$

This represents one-dimensional oscillatory motions of an infinitesimal mass along the Z-axis in the relativistic case.

5. Libration Points in Relativistic Mechanics

For libration points along the Z-axis, $\ddot{z} = \dot{z} = 0$ then from Equation (24)

$$\Rightarrow z\left[-\frac{1}{\left(z^2+1/4\right)^{3/2}} + \frac{n^2+1}{2c^2\left(z^2+1/4\right)^2} + \frac{n^2}{8c^2\left(z^2+1/4\right)^{3/2}}\right] = 0. \tag{25}$$

Taking the dimensionless value of c in Planck units (*i.e.*, $c = 1$), then from Equation (19), we get $n = 3/8$ (Refer to Robert [13]).

Thus, from Equation (25), we have

$$F(z) = z\left[\frac{73}{128\left(z^2+1/4\right)^2} - \frac{503}{512\left(z^2+1/4\right)^{3/2}}\right] = 0. \tag{26}$$

$$\Rightarrow z = 0, \quad z = \pm 1.648.$$

Figure 4 illustrates the graphical representation of libration points in the z vs. $F(z)$ at $z = 0$, $z = \pm 1.648$ for the relativistic case in the specified domain $-2 \le z \le 2$. Figure 5 presents contour plot of $F(z)$ for $-2 \le z \le 2$, corresponding to relativistic conditions. Unlike the classical case, the function $F(z)$, as defined in Equation (26), exhibits no singularities in the relativistic framework. Figure 6 compares the classical and relativistic scenarios: the blue curve represents the classical case, which has a single libration point at the origin, while the red curve represents the relativistic case, showing three distinct libration points located symmetrically along the vertical axis.

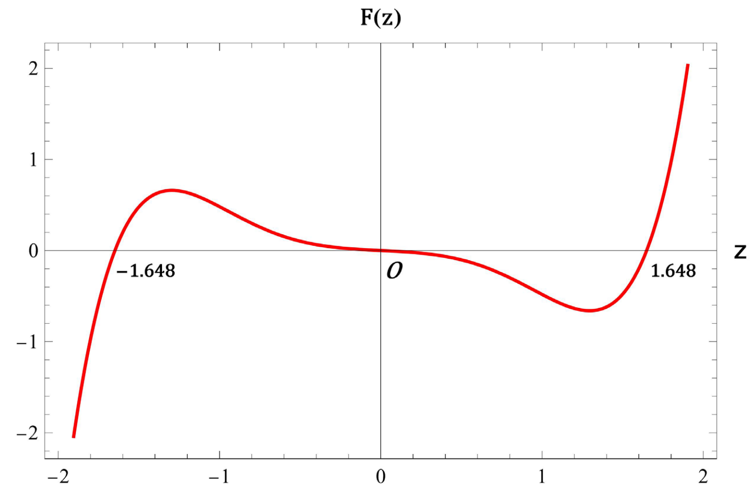

Figure 4. Graph of z vs. $F(z)$ (Relativistic case) for $-2 \le z \le 2$.

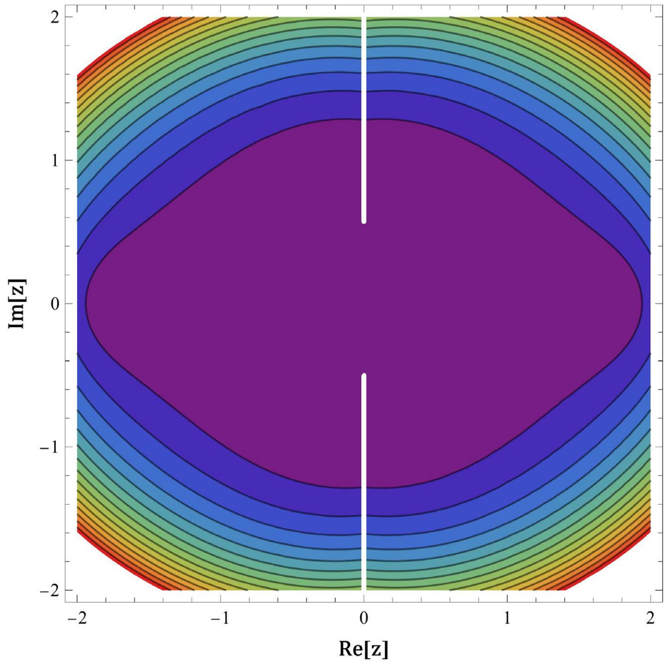

Figure 5. Contour plot of $|F(z)|$ for $-2 \le z \le 2$.

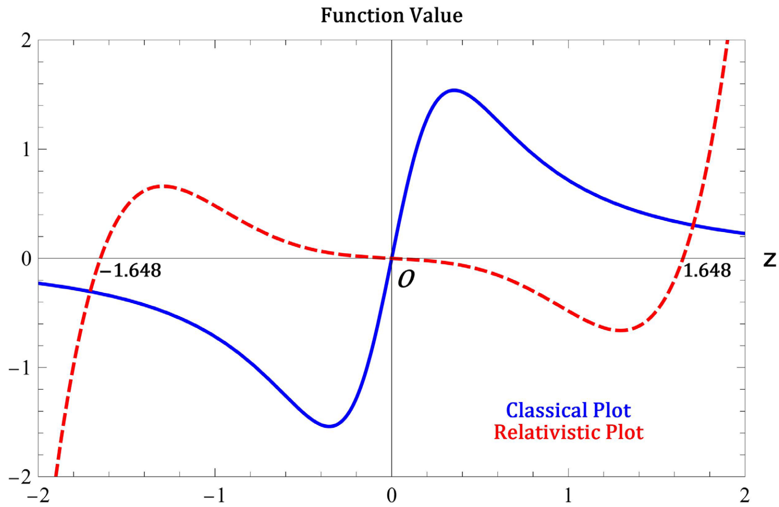

Figure 6. Comparison graph of classical $f(z)$ and relativistic $F(z)$.

6. Conclusion

This manuscript presents a study on the existence of libration points in the Sitnikov Restricted Three-Body Problem (SRTBP), along with an analysis of the effects of relativity on these points, organized across several sections. In Section 1, we have outlined the contributions of previous researchers related to the problem. In Section 2, the equations governing the restricted three-body problem are derived in a three-dimensional coordinate system, the gravitational field generated by two spherical primaries. In Section 3, these equations are reduced to formulate the motion specific to the Sitnikov restricted three-body problem. The resulting equation of motion demonstrates that the infinitesimal mass moves along the Z-axis. Thus, the motion of the infinitesimal mass is one-dimensional oscillating about the origin. In Section 4, we derive the equation of motion for the relativistic restricted three-body Sitnikov problem. The relativistic formulation yields three real libration points $z = 0, \pm 1.648$, as illustrated in Figure 6. It is evident that the two additional libration points $z = \pm 1.648$, arises purely due to the relativistic effects in the Sitnikov restricted three-body problem.

Acknowledgements

We express our heartfelt gratitude to the "Variant Research Centre", Bhagalpur, Bihar, India, for extending their generous support and excellent research facilities.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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