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New Probability Distributions in Astrophysics: XV. Truncation of the Exponentiated Gamma Distribution

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Abstract

The gamma function is a good approximation to the luminosity function (LF) of galaxies, and an exponentiated gamma distribution would permit a more rigorous analysis. This paper examines the exponentiated gamma distribution and its double truncation. The new results are applied to five clusters of stars. The magnitude version of the truncated LF is derived in order to fit the observed LF for galaxies. The regular and truncated LFs are applied to the five bands of SDSS galaxies.

Keywords

Stars: Normal, Stars: Luminosity Function, Mass Function Stars: Statistics

1. Introduction

The analysis of the exponentiated distributions started with the Weibull family in 1993 [1], followed by the exponential distribution in 1998 [2]. The exploration of the exponentiated gamma distribution started in 2007 with [3], where an application to drought data from the State of Nebraska in the USA was analyzed.

Careful attention should be given to the number of adopted parameters for the exponentiated gamma. For example, some authors [4] used two parameters. Regarding the astrophysical applications of the exponentiated gamma, we considered three parameters: the scale, shape and exponent. The following conjecture is useful when going from the standard gamma distribution to the exponentiated distribution.

Conjecture 1 *The goodness of fit for the gamma distribution with the Kolmogorov-Smirnov test or the* χ^2 *test increases with the number of parameters adopted*. The usual standard distributions, such as the gamma, lognormal, and

Weibull, are defined between zero and infinity. But in astrophysics, the variables, e.g., the mass, are measured between a minimum and a maximum value. For example, the mass of a star varies between a minimum value of $0.01\mathcal{M}_\odot$ and a maximum value of $100\mathcal{M}_\odot$. The mass of a galaxy varies between a minimum value of $10^8\mathcal{M}_\odot$ and a maximum value of $30\times10^{12}\mathcal{M}_\odot$. The above two arguments show the importance of considering truncated distributions. In order to answer Conjecture 1, this paper reviews in Section 3 the gamma distribution with two parameters and the truncated gamma distribution with four parameters. Section 4 introduces the exponentiated gamma distribution and then Section 5 derives its truncated version. A sketch of this pattern is presented in **Figure 1**. Section 6 presents two luminosity functions for galaxies derived in the framework of the regular and the truncated exponentiated gamma distribution. The applications of the developed formulae to the mass distribution of the stars in the clusters are presented in Section 7. Section 8 contains the fit of the luminosity functions of the SDSS galaxies with the new theoretical luminosity functions.

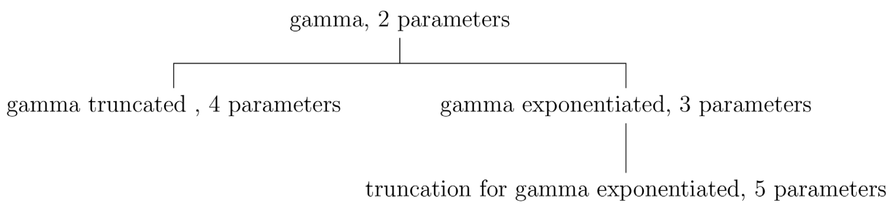

Figure 1. Trees diagram for gamma distributions.

2. The Lognormal Distribution

Let X be a random variable taking values x in the interval $[0,\infty]$; the first definition for the *lognormal* PDF, following [5] or formula (14.2) in [6], is

$$f(x:m,\sigma)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\frac{-\left[\ln(x/m)\right]^2}{2\sigma^2}. \tag{1}$$

Its average value, $E(m,\sigma)$, is

$$E(m,\sigma)=m\mathrm{e}^{\frac{\sigma^2}{2}}, \tag{2}$$

and its distribution function, $F(x:m,\sigma)$, is given by

$$F(x:m,\sigma)=\frac{1}{2}-\frac{\operatorname{erf}\left(\frac{\sqrt{2}\left(\ln(m)-\ln(x)\right)}{2\sigma}\right)}{2}. \tag{3}$$

The second definition is

$$f(x:\mu,\sigma)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\frac{-(\ln x-\mu)^2}{2\sigma^2}, \tag{4}$$

where $m=\exp\mu$ and $\mu=\ln m$. Its average value, $E(\mu,\sigma)$ is

$$E(\mu,\sigma)=\mathrm{e}^{\frac{\sigma^2}{2}+\mu}, \tag{5}$$

and its distribution function, $F(x:\mu,\sigma)$, is

$$F(x:\mu,\sigma)=\frac{1}{2}+\frac{\operatorname{erf}\left(\frac{\sqrt{2}(\ln(x)-\mu)}{2\sigma}\right)}{2}. \tag{6}$$

3. The Gamma Family

This section reviews the adopted statistics, the gamma distribution and the truncated gamma distribution.

3.1. Adopted Statistics

The χ^2 statistic is defined by

$$\chi^2=\sum_{i=1}^{n}\frac{(T_i-O_i)^2}{T_i}, \tag{7}$$

where n is the number of bins, T_i is the theoretical value, and O_i is the experimental value represented by the frequencies. The merit function χ^2_{red} is given by

$$\chi^2_{red}=\chi^2/NF, \tag{8}$$

where $NF=n-k$ is the number of degrees of freedom, n is the number of bins, and k is the number of parameters. The goodness of the fit can be expressed by the probability Q, see equation 15.2.12 in [7], which involves the number of degrees of freedom and χ^2. The Akaike information criterion (AIC), see [8], is defined by

$$\text{AIC}=2k-2\ln(L), \tag{9}$$

where L is the likelihood function and k the number of free parameters in the model. We assume a Gaussian distribution for the errors and the likelihood function can be derived from the χ^2 statistic $L\propto\exp\left(-\frac{\chi^2}{2}\right)$ where χ^2 has been computed by Equation (7), see [9] [10]. Now the AIC becomes

$$\text{AIC}=2k+\chi^2. \tag{10}$$

3.2. The Gamma Distribution

Let X be a random variable taking values x in the interval $[0,\infty]$; the *gamma* PDF is

$$f(x;b,c)=\frac{\left(\frac{x}{b}\right)^{c-1}\mathrm{e}^{-\frac{x}{b}}}{b\Gamma(c)} \tag{11}$$

where

$$\Gamma(z)=\int_0^\infty \mathrm{e}^{-t}t^{z-1}\mathrm{d}t, \tag{12}$$

is the gamma function, $b>0$ is the scale and $c>0$ is the shape, see formula

(17.23) in [6]. Its expected value is

$$E(b,c) = bc, \tag{13}$$

and its variance,

$$Var(b,c) = b^2 c. \tag{14}$$

The mode is at

$$m(b,c) = bc - b \quad \text{when } c > 1. \tag{15}$$

We now present three expressions for its distribution function (DF). The *first* expression is

$$DF(x;b,c) = \frac{x\, x^{-\frac{c}{2}-1} b^{\frac{c}{2}} \left(\mathrm{e}^{-\frac{x}{b}} x^{\frac{3c}{2}} b^{-\frac{3c}{2}} c + \mathrm{e}^{-\frac{x}{2b}} x^{c} b^{-c} M_{\frac{c}{2},\frac{c}{2}+\frac{1}{2}}\left(\frac{x}{b}\right) + \mathrm{e}^{-\frac{x}{b}} x^{\frac{3c}{2}} b^{-\frac{3c}{2}} \right)}{\Gamma(c+2)}, \tag{16}$$

where $M_{\mu,\nu}(z)$ is the Whittaker M function, see [11]. The *second* expression is

$$DF(x;b,c) = 1 - \frac{\Gamma\left(c, \frac{x}{b}\right)}{\Gamma(c)}, \tag{17}$$

where $\Gamma(a,z)$ is the incomplete Gamma function defined as

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1} \mathrm{e}^{-t} \mathrm{d}t, \tag{18}$$

see [11]. The *third* expression is

$$DF(x;b,c) = \frac{\gamma\left(c, \frac{x}{b}\right)}{\Gamma(c)}, \tag{19}$$

where

$$\gamma(a,z) = \int_0^{z} t^{a-1} \mathrm{e}^{-t} \mathrm{d}t, \tag{20}$$

is the lower incomplete gamma function, see [11] [12]. One method to derive the parameters is to estimate them by matching the moments (MOM):

$$b = \frac{s^2}{\overline{x}} \tag{21}$$

$$c = \left(\frac{\overline{x}}{s}\right)^2, \tag{22}$$

where s^2 and $\overline{x}$ are the sample variance and the sample mean. More details can be found in [5]. Another method is maximum likelihood (MLE), which maximizes

$$\Lambda = -nc\ln(b) - n\ln(\Gamma(c)) + \sum_{i=1}^{n} \left(c\ln(x_i) - \ln(x_i) - \frac{x_i}{b} \right), \tag{23}$$

where n is the number of elements in the sample x_j. The derivatives of Λ with respect to b, and c form a system of two nonlinear equations:

$$\frac{\partial b}{\partial b} = \frac{-ncb + \sum_{i=1}^{n} x_i}{b^2} = 0, \tag{24}$$

$$\frac{\partial \Lambda}{\partial c} = -n\ln(b) - n\Psi(c) + \sum_{i=1}^{n} \ln(x_i) = 0, \tag{25}$$

where $\Psi(x)$ is the digamma function.

3.3. The Truncated Gamma Distribution

Let X be a random variable taking values x in the interval $[x_l, x_u]$; the truncated gamma (TG) PDF, as in [13], is

$$f(x;b,c,x_l,x_u) = k\left(\frac{x}{b}\right)^{c-1} \mathrm{e}^{-\frac{x}{b}}, \tag{26}$$

where the constant k is

$$k = \frac{c}{b\Gamma\left(1+c,\frac{x_l}{b}\right) - b\Gamma\left(1+c,\frac{x_u}{b}\right) + \mathrm{e}^{-\frac{x_u}{b}} b^{-c+1} x_u^c - \mathrm{e}^{-\frac{x_l}{b}} b^{-c+1} x_l^c}, \tag{27}$$

where

$$\Gamma(a,z) = \int_z^{\infty} t^{a-1}\mathrm{e}^{-t}\mathrm{d}t, \tag{28}$$

is the upper incomplete gamma function, see [11] [12]. Its expected value is

$$E(b,c,x_l,x_u) = -b^2 k\left(-\Gamma\left(1+c,\frac{x_l}{b}\right) + \Gamma\left(1+c,\frac{x_u}{b}\right)\right), \tag{29}$$

and its rth moment about the origin, $\mu'_r(b,c)$, is

$$\mu'_r = \frac{kb\left(x_u^{c+r} b^{-c} \mathrm{e}^{-\frac{x_u}{b}} - x_l^{c+r} b^{-c} \mathrm{e}^{-\frac{x_l}{b}} - b^r\Gamma\left(1+c+r,\frac{x_u}{b}\right) + b^r\Gamma\left(1+c+r,\frac{x_l}{b}\right)\right)}{c+r}. \tag{30}$$

The mode is at

$$m(b,c,x_l,x_u) = bc - b \quad \text{when } c > 1, \tag{31}$$

but in order for it to exist, the inequality $x_l < m < x_u$ should be satisfied. Its distribution function is

$$\begin{aligned} &DF(x;b,c,x_l,x_u) \\ &= k\left(b\Gamma\left(1+c,\frac{x_l}{b}\right) - b\Gamma\left(1+c,\frac{x}{b}\right) + \mathrm{e}^{-\frac{x}{b}} b^{-c+1} x^c - \mathrm{e}^{-\frac{x_l}{b}} b^{-c+1} x_l^c\right). \end{aligned} \tag{32}$$

A random number generation can be implemented by solving the following nonlinear equation for x

$$DF(x;b,c,x_l,x_u) - \mathbf{R} = 0, \tag{33}$$

where we have a pseudorandom number generator generating $\mathbf{R}$ between zero and one, see [14]. The lower and upper boundaries are derived according to the following recipe:

$$x_l = \text{minimum of sample}, \quad x_u = \text{maximum of sample}. \tag{34}$$

One method to determine the parameters is to introduce the moments of the experimental sample

$$\overline{x}_r = \frac{1}{n}\sum_i^n x_i^r . \tag{35}$$

As a consequence, the two parameters can be found by solving the following two non-linear equations, which constitutes the method of moments (MOM)

$$\overline{x}_1 = \mu_1'(b,c), \tag{36}$$

$$\overline{x}_2 = \mu_2'(b,c). \tag{37}$$

4. The Exponentiated Gamma Distribution

Let X be a random variable taking values x in the interval $[0,\infty]$; the *exponentiated gamma* DF is here obtained from the second definition of the gamma DF, Equation (17),

$$F(x;b,c,d) = \left(1 - \frac{\Gamma\left(c,\frac{x}{b}\right)}{\Gamma(c)}\right)^d , \tag{38}$$

where $b>0$, $c>0$ and $d>0$. Its PDF is

$$f(x;b,c,d) = \frac{\left(1 - \frac{\Gamma\left(c,\frac{x}{b}\right)}{\Gamma(c)}\right)^d d\left(\frac{x}{b}\right)^{c-1} \mathrm{e}^{-\frac{x}{b}}}{b\Gamma(c)\left(1 - \frac{\Gamma\left(c,\frac{x}{b}\right)}{\Gamma(c)}\right)}. \tag{39}$$

An analytical expression for the moments can be derived only when d is an integer. As an example, when $d=2$, the expected value is

$$E(b,c,2) = \frac{4b\Gamma(2c)\,{}_2F_1(c,1+2c;c+1;-1)}{\Gamma(c)^2}, \tag{40}$$

the second moment about the origin, $\mu_2'(b,c,2)$, is

$$\mu_2'(b,c,2) = \frac{2b^2\Gamma(2+2c)\,{}_2F_1(c,2+2c;c+1;-1)}{c\Gamma(c)^2}, \tag{41}$$

and the variance is

$$Var(x;b,c,2) = b^2\left(-c^2 + \frac{4(1+2c)\,{}_2F_1(c,1+2c;c+1;-1)\Gamma(2c)}{\Gamma(c)^2} - \frac{16\,{}_2F_1(c,1+2c;c+1;-1)^2\,\Gamma(2c)^2}{\Gamma(c)^4}\right), \tag{42}$$

where ${}_2F_1(a,b;c;z)$ is the regularized hypergeometric function. A series representation for the PDF is

$$f(x;b,c,d)=\sum_{n=0}^{\infty}\left(-\frac{x^{c-1}\mathrm{e}^{-\frac{x}{b}}b^{-c}\left(\frac{\Gamma\left(c,\frac{x}{b}\right)}{\Gamma(c)}\right)^{n}\Gamma(n+1-d)}{\Gamma(-d)\Gamma(c)\Gamma(n+1)}\right). \quad (43)$$

As an example, an approximation for the expected value E (2, 2, 0.8), up to order 5, has a percentage error of 0.04%. The three parameters can be found through the MLE:

$$\Lambda=-n\ln(b)c+n\ln(d)+\sum_{j=1}^{n}\left(\ln(x_j)c-\ln(x_j)+\ln\left(\frac{\left(\frac{\Gamma(c)-\Gamma\left(c,\frac{x_j}{b}\right)}{\Gamma(c)}\right)^{d}}{\Gamma(c)-\Gamma\left(c,\frac{x_j}{b}\right)}\right)-\frac{x_j}{b}\right), \quad (44)$$

The derivatives of Λ with respect to b, c and d form a system of three nonlinear equations:

$$\frac{\partial b}{\partial b}=\frac{-ncb-\sum_{j=1}^{n}\frac{-x_j^{c}b^{-c+1}(-1+d)\mathrm{e}^{-\frac{x_j}{b}}-x_j\left(-\Gamma(c)+\Gamma\left(c,\frac{x_j}{b}\right)\right)}{-\Gamma(c)+\Gamma\left(c,\frac{x_j}{b}\right)}}{b^2}=0, \quad (45)$$

$$\frac{\partial\Lambda}{\partial c}=\frac{-n\ln(b)c^2}{c^2}+\frac{\sum_{j=1}^{n}\frac{x_j^{c}b^{-c}(-1+d)\,{}_2F_2\left(c,c;c+1,c+1;-\frac{x_j}{b}\right)+\left((1-d)\ln(b)+d\left(\ln(x_j)-\Psi(c)\right)\right)c^2\left(-\Gamma(c)+\Gamma\left(c,\frac{x_j}{b}\right)\right)}{-\Gamma(c)+\Gamma\left(c,\frac{x_j}{b}\right)}}{c^2}=0, \quad (46)$$

$$\frac{\partial\Lambda}{\partial d}=\frac{n}{d}+\sum_{j=1}^{n}\left(\frac{\Gamma(c)-\Gamma\left(c,\frac{x_j}{b}\right)}{\Gamma(c)}\right)=0, \quad (47)$$

where $\Psi(x)$ is the digamma function.

We now evaluate the bias of the expected value, $\mathcal{B}(\hat{E})$, which is

$$\mathcal{B}(\hat{E}) = \overline{x} - E \tag{48}$$

where $\overline{x}$ is the average of the sample and E the true value of the average of the distribution. In order to evaluate the bias for $E(0.173, 0.53, 12.26)$, which is the case of γ Velorum cluster of stars, we perform a simulation, see Figure 2.

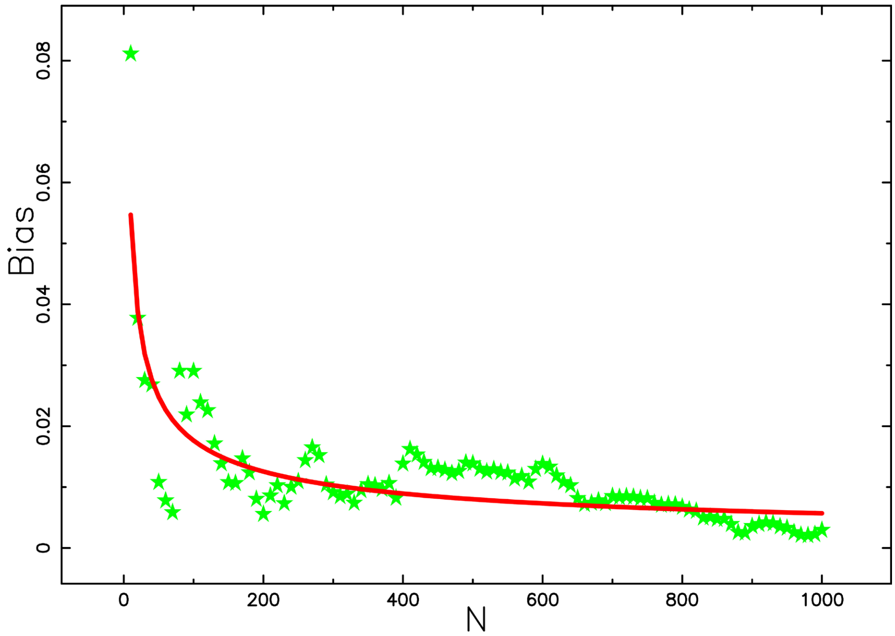

Figure 2. Empirical bias, green stars, versus fitted bias, red line, of the MLE estimator for E as a function of N.

The red line of the above fit is parametrized by

$$\mathcal{B}(\hat{E}) = 0.169 \times N^{-0.491}, \tag{49}$$

which means that $\mathcal{B}(\hat{E}) = 0.015$ when $N = 237$, which is the size of the sample for the γ Velorum cluster of stars.

5. Truncation of the Exponentiated Gamma Distribution

The DF of the truncated exponentiated gamma distribution, $F_T(x)$, is

$$F_T(x; b, c, d, x_l, x_u) = \frac{-\left(\frac{\Gamma(c) - \Gamma\left(c, \frac{x}{b}\right)}{\Gamma(c)}\right)^d + \left(\frac{\Gamma(c) - \Gamma\left(c, \frac{x_l}{b}\right)}{\Gamma(c)}\right)^d}{-\left(\frac{\Gamma(c) - \Gamma\left(c, \frac{x_u}{b}\right)}{\Gamma(c)}\right)^d + \left(\frac{\Gamma(c) - \Gamma\left(c, \frac{x_l}{b}\right)}{\Gamma(c)}\right)^d}, \tag{50}$$

and its PDF, $f_T(x)$, is

$$f_T(x;b,c,d,x_l,x_u)=\frac{-\left(\frac{\Gamma(c)-\Gamma\left(c,\frac{x}{b}\right)}{\Gamma(c)}\right)^d+\left(\frac{\Gamma(c)-\Gamma\left(c,\frac{x_l}{b}\right)}{\Gamma(c)}\right)^d}{\left(-\Gamma(c)+\Gamma\left(c,\frac{x_u}{b}\right)\right)^d-\left(-\Gamma(c)+\Gamma\left(c,\frac{x_l}{b}\right)\right)^d}, \tag{51}$$

with x taking values in $[x_l,x_u]$. Its expected value when $d=1$ is

$$E_T(b,c,x_l,x_u)=\frac{b\left(\Gamma\left(c+1,\frac{x_u}{b}\right)-\Gamma\left(c+1,\frac{x_l}{b}\right)\right)}{\Gamma\left(c,\frac{x_u}{b}\right)-\Gamma\left(c,\frac{x_l}{b}\right)}. \tag{52}$$

We now outline how to obtain the parameters. Consider a sample $\mathcal{X}=x_1,x_2,\cdots,x_n$ and let $x_{(1)}\ge x_{(2)}\ge\cdots\ge x_{(n)}$ denote their order statistics, so that $x_{(1)}=\max(x_1,x_2,\cdots,x_n)$, $x_{(n)}=\min(x_1,x_2,\cdots,x_n)$. The two parameters x_l and x_u are

$$x_l=x_{(n)};\quad x_u=x_{(1)}. \tag{53}$$

The remaining parameters are determined by the MLE, where the derivatives are now evaluated numerically.

6. Luminosity Function for Galaxies

6.1. Schechter Luminosity

The Schechter function Φ, introduced by [15] in 1976, provides a useful reference for the LF of galaxies:

$$\Phi(L;\alpha,L^*,\Phi^*)\mathrm{d}L=\frac{\Phi^*}{L^*}\left(\frac{L}{L^*}\right)^\alpha\exp\left(-\frac{L}{L^*}\right)\mathrm{d}L. \tag{54}$$

Here, α sets the slope for low values of L, L^* is the characteristic luminosity and Φ^* is the normalization. At the moment of writing, that paper has 3175 citations and was cited for the first time in 1976 in the framework of the generalization of Schmidt's estimator, see [16]. The equivalent distribution in absolute magnitude is

$$\Phi(M)\mathrm{d}M=0.921\Phi^*10^{0.4(\alpha+1)(M^*-M)}\exp\left(-10^{0.4(M^*-M)}\right)\mathrm{d}M, \tag{55}$$

where M^* is the characteristic magnitude as derived from the data. We now introduce the parameter h, which is $H_0/100$, where H_0 is the Hubble constant. The scaling with h is $M^*-5\log_{10}h$ and Φ^*h^3 [Mpc^{-3}].

6.2. The Exponentiated Gamma Luminosity Function

In order to derive the exponentiated gamma LF, we start from the PDF as given by Equation (39) and we substitute b with L^* and x with L:

$$\Psi\left(L;L^*,c,d,\Psi^*\right)\mathrm{d}L = \Psi^* \frac{\left(1-\frac{\Gamma\left(c,\frac{L}{L^*}\right)}{\Gamma(c)}\right)^d d\left(\frac{L}{L^*}\right)^{c-1} \mathrm{e}^{-\frac{L}{L^*}}\Gamma(c)}{L^*\Gamma(c)\left(\Gamma(c)-\Gamma\left(c,\frac{L}{L^*}\right)\right)}\mathrm{d}L, \tag{56}$$

where L is the luminosity defined in $[0,\infty]$, L^* is the characteristic luminosity and Ψ^* is a normalization, *i.e.* the number of galaxies in a cubic Mpc. We now introduce the following useful formulae relating the absolute magnitude and luminosity:

$$\frac{L}{L_\odot} = 10^{0.4(M_\odot - M)}, \quad \frac{L^*}{L_\odot} = 10^{0.4\left(M_\odot - M^*\right)} \tag{57}$$

where $L_\odot$ and $M_\odot$ are the luminosity and absolute magnitude of the sun in the considered band. The LF in absolute magnitude is therefore

$$\begin{aligned}&\Psi\left(M;M^*,c,d,\Psi^*\right)\mathrm{d}M \\ &= \Psi^* 0.4\left(\Gamma(c)-\Gamma\left(c,10^{-0.4M+0.4M^*}\right)\right)^{-1+d} d10^{c\left(-0.4M+0.4M^*\right)}\Gamma(c)^{-d}\,\mathrm{e}^{-10^{-0.4M+0.4M^*}}\ln(10)\,\mathrm{d}M.\end{aligned} \tag{58}$$

6.3. Exponentiated Gamma Luminosity Function with Truncation

The truncated exponentiated gamma LF for galaxies according to Equation (51) is

$$\begin{aligned}&\Psi\left(L;L^*,\Psi^*,c,d,L_l,L_u\right)\mathrm{d}L \\ &= \Psi^* \frac{\left(1-\frac{\Gamma\left(c,\frac{L}{L^*}\right)}{\Gamma(c)}\right)^d d\left(\frac{L}{L^*}\right)^{c-1}\mathrm{e}^{-\frac{L}{L^*}}\Gamma(c)}{L^*\Gamma(c)\left(\Gamma(c)-\Gamma\left(c,\frac{L}{L^*}\right)\right)\left(\left(1-\frac{\Gamma\left(c,\frac{L_u}{L^*}\right)}{\Gamma(c)}\right)^d-\left(1-\frac{\Gamma\left(c,\frac{L_l}{L^*}\right)}{\Gamma(c)}\right)^d\right)}\mathrm{d}L,\end{aligned} \tag{59}$$

where the random variable L is defined in $[L_l, L_u]$, L_l is the lower bound on the luminosity, L_u is the upper bound on the luminosity, L^* is the characteristic luminosity and Ψ^* is the normalization. The absolute magnitude version is

$$\begin{aligned}&\Psi\left(M;M^*,\Psi^*,c,d,L_l,L_u\right)\mathrm{d}M \\ &= \Psi^* \frac{0.4\times 10^{0.4c\left(M^*-M\right)} d\,\mathrm{e}^{-10^{0.4M^*-0.4M}}\left(\Gamma(c)-\Gamma\left(c,10^{0.4M^*-0.4M}\right)\right)^{-1+d}\left(\ln(2)+\ln(5)\right)}{\left(\Gamma(c)-\Gamma\left(c,10^{-0.4M_l+0.4M^*}\right)\right)^d-\left(\Gamma(c)-\Gamma\left(c,10^{-0.4M_u+0.4M^*}\right)\right)^d}\mathrm{d}M,\end{aligned} \tag{60}$$

where M is the absolute magnitude, M^* is the characteristic magnitude, M_l is the lower bound on the magnitude and M_u is the upper bound on the magnitude. The two luminosities L_l and L_u are connected with the absolute magnitudes M_l and M_u through the following relation:

$$\frac{L_l}{L_\odot} = 10^{0.4(M_\odot - M_u)}, \quad \frac{L_u}{L_\odot} = 10^{0.4(M_\odot - M_l)} \tag{61}$$

where the indices u and l are inverted in the transformation from luminosity to absolute magnitude. The mean theoretical absolute magnitude, $\langle M \rangle$, can be evaluated as

$$\langle M \rangle = \frac{\int_{M_l}^{M_u} M \times \Psi\left(M; a, c, M^*, \Psi^*, M_l, M_u\right) \mathrm{d}M}{\int_{M_l}^{M_u} \Psi\left(a, M; c, M^*, \Psi^*, M_l, M_u\right) \mathrm{d}M}. \tag{62}$$

7. Applications to the Stars

The first test is performed on NGC 2362, where the 271 stars have a range of $1.47 M_\odot \geq M \geq 0.11 M_\odot$, see [17] and CDS catalog J/MNRAS/384/675/Table 1. According to [18], the distance of NGC 2362 is 1480 pc. The second test is performed on the low-mass stars in the young cluster NGC 6611, see [19] and CDS catalog J/MNRAS/392/1034. This massive cluster has an age of 2 - 3 Myr and contains masses from $1.5 M_\odot \geq M \geq 0.02 M_\odot$. Therefore, the brown dwarfs (BD) region, $\approx 0.2 \mathcal{M}_\odot$, is covered. The third test is performed on the γ Velorum cluster, where the 237 stars have a range of $1.31 M_\odot \geq M \geq 0.15 M_\odot$, see [20] and CDS catalog J/A+A/589/A70/Table 5. The fourth test is performed on the young cluster Berkeley 59, where the 420 stars have a range of $2.24 M_\odot \geq M \geq 0.15 M_\odot$, see [21] and CDS catalog J/AJ/155/44/Table 3. The fifth test is performed on the Hyades, where the 602 stars have a range of $2.20 M_\odot \geq M \geq 0.11 M_\odot$, see [22] and CDS catalog J/AJ/165/108/Table 1.

The results are presented in **Table 1** for the Gamma distribution and in **Table 2** for the truncated Gamma distribution. **Table 3** and **Table 4** present the results for the standard and truncated exponentiated gamma distributions, respectively.

Table 1. Numerical values of D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K–S test for the gamma distribution, see Equation (11), for different mass distributions. The last column shows whether the results of the K–S test are better when compared to the lognormal distribution (Y) or worse (N).

Cluster	parameters	D	P_{KS}	Y/N
NGC 2362	b = 0.143, c = 4.48	0.05	0.478	Y
NGC 6611	b = 0.3, c = 1.37	0.07	0.158	Y
γ Velorum	b = 0.1, c = 3.68	0.127	7.8×10^{-4}	N
Berkeley 59	b = 0.105, c = 3.68	0.127	7.8×10^{-4}	Y
Hyades	b = 0.239, c = 2.14	0.094	3.8×10^{-5}	N

As an example, **Figure 3** and **Figure 4** present the PDFs of the gamma exponentiated with truncation for NGC 2362 and NGC 6611 respectively.

Table 2. Numerical values of D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K–S test for the truncated gamma distribution, see Equation (26), for different mass distributions. The last column shows whether the results of the K–S test are better when compared to the lognormal distribution (Y) or worse (N).

Cluster	parameters	D	P_{KS}	Y/N
NGC 2362	b= 0.164, c =4.05	0.042	0.65	Y
NGC 6611	b= 0.36, c =1.126	0.084	0.1	Y
γ Velorum	b= 9.45, c = 1.39	0.105	9.28×10^{-16}	N
Berkeley 59	b= 0.309, c =0.966	0.14	7.8×10^{-4}	Y
Hyades	b= 0.239, c =2.14	0.094	1.0×10^{-7}	N

Table 3. Numerical values of D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K–S test for the exponentiated gamma distribution, see Equation (39), for different mass distributions. The last column shows whether the results of the K–S test are better when compared to the lognormal distribution (Y) or worse (N).

Cluster	parameters	D	P_{KS}	Y/N
NGC 2362	b = 0.141, c = 4.69, d = 0.926	0.049	0.515	Y
NGC 6611	b = 0.2, c = 5, d = 0.2	0.057	0.483	Y
γ Velorum	b = 0.173, c = 0.53, d = 12.26	0.103	1.2×10^{-2}	Y
Berkeley 59	b = 0.186, c = 0.927, d = 6.59	0.125	3.3×10^{-6}	N
Hyades	b = 0.369, c = 0.324, d = 8.73	0.086	2.11×10^{-4}	N

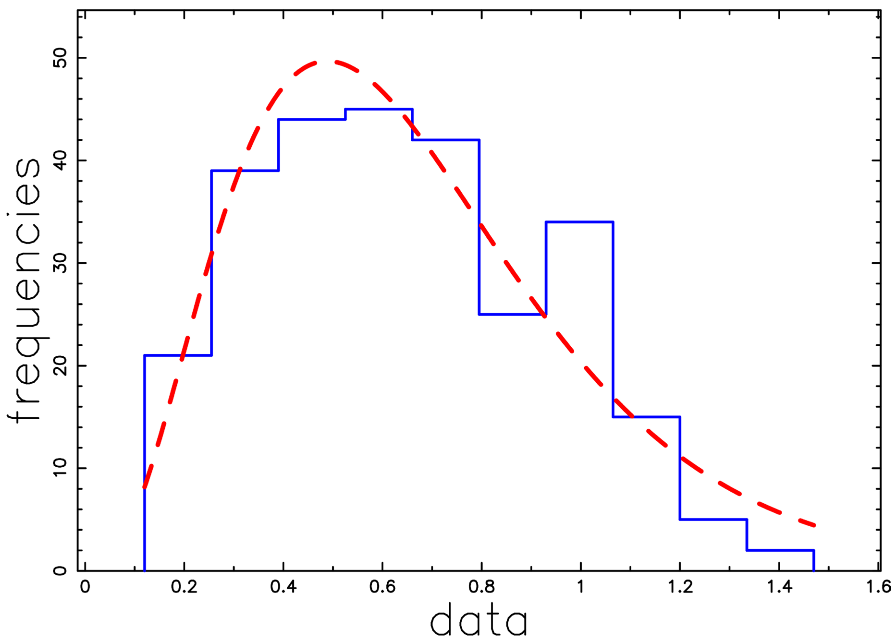

Figure 3. Empirical histogram for the mass distribution of NGC 2362 (blue) with a superposition of the PDF for the exponentiated gamma distribution with truncation, (red dashed line). Theoretical parameters as in **Table 4**.

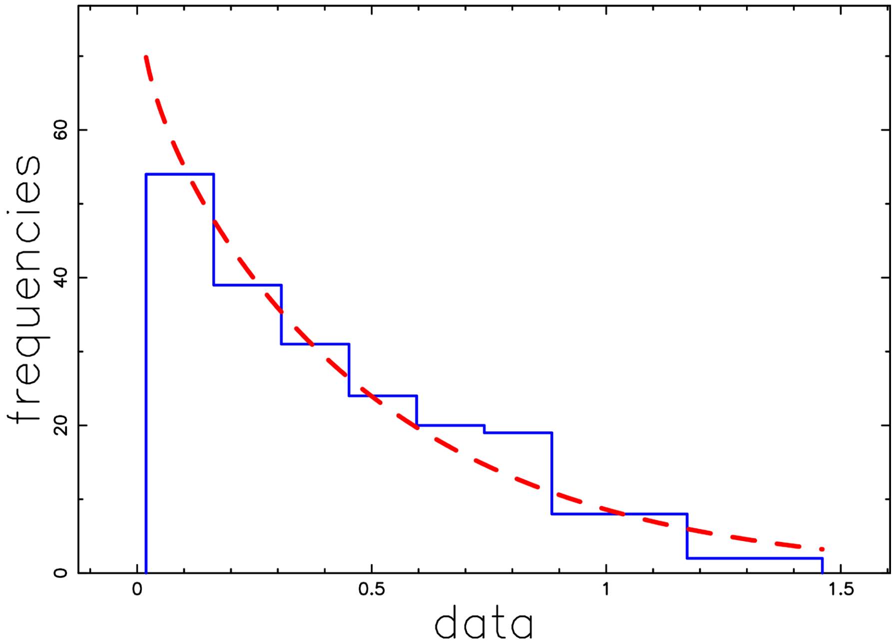

Figure 4. Empirical histogram for the mass distribution of NGC 6611 (blue) with a superposition of the PDF for the exponentiated gamma distribution with truncation, (red dashed line). Theoretical parameters as in Table 4.

Table 4. Numerical values of D, the maximum distance between theoretical and observed DF, and P_{KS}, significance level, in the K–S test for the truncated and exponentiated gamma distribution, see Equation (51), for different mass distributions. The last column shows whether the results of the K–S test are better when compared to the lognormal distribution (Y) or worse (N).

Cluster	parameters	D	P_{KS}	Y/N
NGC 2362	b = 0.198, c = 3.144, d = 1.2	0.036	0.856	Y
NGC 6611	b = 0.37, c = 1.921, d = 0.491	0.054	0.558	Y
γ Velorum	b = 0.394, c = 0.394, d = 1.583	0.068	0.205	Y
Berkeley 59	b = 0.43, c = 0.576, d = 3.01	0.147	1.89×10^{-8}	N
Hyades	b = 0.499, c = 0.576, d = 1.58	0.045	0.169	Y

8. Applications to the LF for Galaxies

In order to perform a test we selected the data of the Sloan Digital Sky Survey (SDSS), which has five bands: u^* ($\lambda = 3550$ Å), g^* ($\lambda = 4770$ Å), r^* ($\lambda = 6230$ Å), i^* ($\lambda = 7620$ Å) and z^* ($\lambda = 9130$ Å), with λ denoting the wavelength of the CCD camera, see [23]. The data of the astronomical LF are reported in [24] and are available at https://cosmo.nyu.edu/blanton/lf.html. The numerical values of the four parameters M^*, c, d and Ψ^* are given in Table 5.

The numerical values of the six parameters M^*, M_l, M_u, c, d and Ψ^* are given in Table 6.

Table 5. Four parameters of the gamma exponentiated LF as represented by formula (58).

parameter	u^*	g^*	r^*	i^*	z^*
$M^* - 5\log_{10} h$	−17.95	−19.35	−20.39	−20.81	−21.12
Ψ^* [h^3 Mpc^{-3}]	0.187	0.298	24.87	0.756	24.094
c	1.014	0.216	8.6410^{-4}	8.4510^{-3}	7.6910^{-4}
d	1.0110^{-2}	0.302	0.739	13.56	0.794
χ^2	317	736	2460	2269	3687
χ^2_{red}	0.662	1.237	3.672	3.218	5.01
$AIC\ k = 4$	325.22	744.58	2468	2277	3695
χ^2 *Schechter LF*	330.73	753.3	2260	2282	3245
χ^2_{red} − *Schechter LF*	0.689	1.263	3.368	3.232	4.403

Table 6. Parameters of the gamma exponentiated LF with truncation as represented by formula (60).

parameter	u^*	g^*	r^*	i^*	z^*
$M_l - 5\log_{10} h$	−20.65	−22.09	−22.94	−23.42	−23.73
$M_u - 5\log_{10} h$	−15.78	−16.32	−16.30	−17.21	−17.48
$M^* - 5\log_{10} h$	−17.94	−19.35	−20.42	−20.81	−21.12
Ψ^* [h^3 Mpc^{-3}]	4.310^{-2}	4.3810^{-2}	5.2910^{-2}	4.0310^{-2}	4.1210^{-2}
c	1.6310^{-2}	2.1710^{-1}	2.2910^{-2}	3.0210^{-3}	6.3110^{-5}
d	9.92	0.30	2	5.95	1.8
χ^2	318	736	2273	2271	3682
χ^2_{red}	0.666	1.242	3.403	3.231	5.016
$AIC\ k = 6$	330.031	748.58	2285.67	2283.69	3694
χ^2 *Schechter*	330.73	753.3	2260	2282	3245
χ^2_{red} − *Schechter*	0.689	1.263	3.368	3.232	4.403

As a visual example the Schechter function, the new four-parameter LF as represented by formula (58) and the data are presented in **Figure 5**, and **Figure 6**, where the two bands u^* and g^* are considered.

9. Conclusions

The exponentiated gamma distribution We derived the PDF and the DF of the exponentiated gamma distribution. An analytical expression for the average value was derived for $d = 2$, which is the parameter which regulates the power of the DF. Three equations which provide the three parameters b, c and d through the MLE were derived. A synoptic table of reference for the different gamma distributions here used with the number equation is presented in **Table 7**.

Truncation for the exponentiated gamma distribution

We derived the PDF and the DF of the truncated exponentiated gamma distribution. An expression for its average value was derived when $d = 1$.

Mass of the stars

The increase from two to three for the numbers of parameters in going from the gamma distribution to the exponentiated gamma distribution produces an increase of the P_{KS}, significance level in the K-S test, in four cases out of five of the clusters of stars here analyzed, see Table 1 and Table 3.

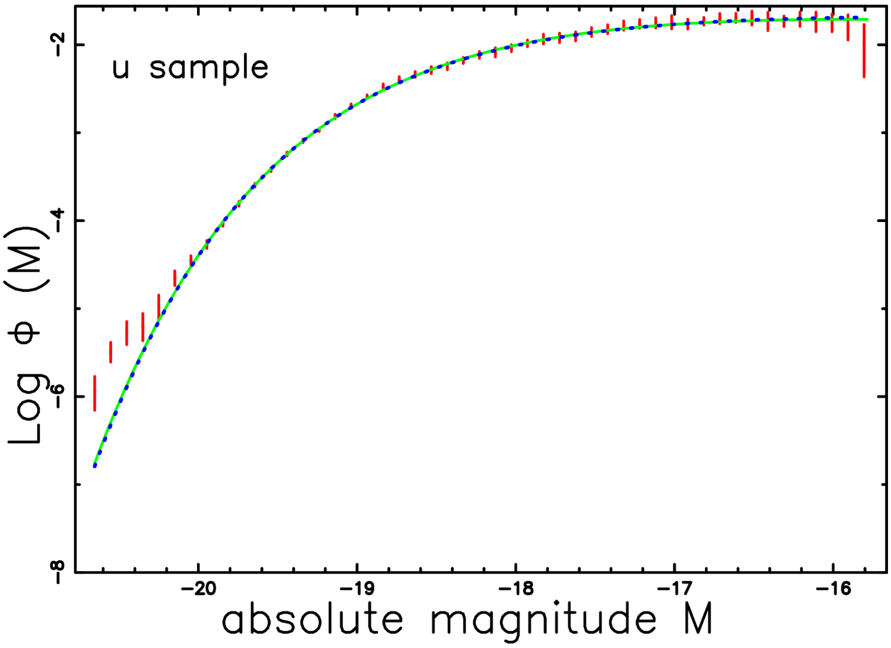

Figure 5. The luminosity function data of SDSS (u^*) are represented with error bars (red). The continuous line fit (green) represents the four-parameter gamma exponentiated LF (54) and the dotted line represents the Schechter function (blue).

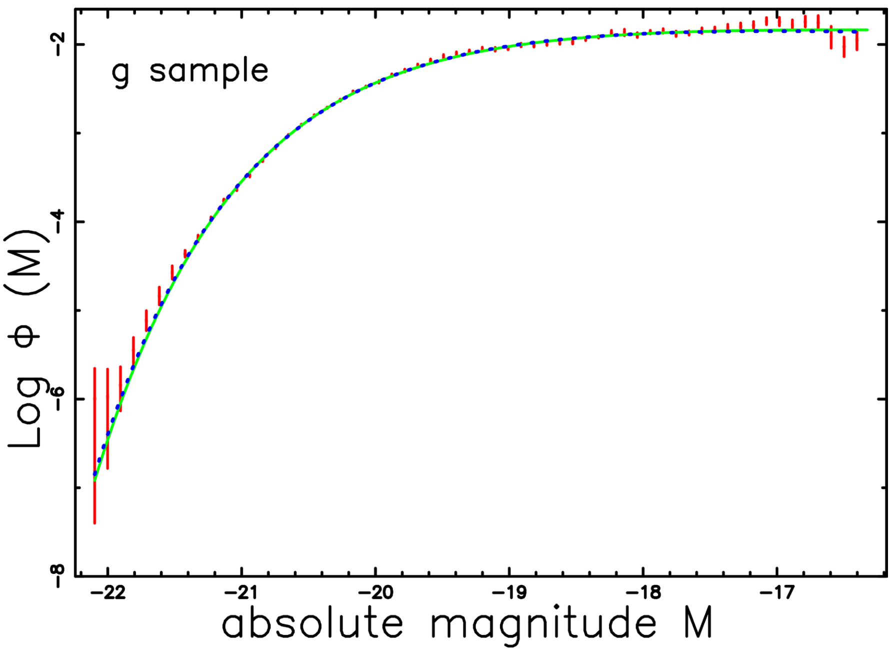

Figure 6. The luminosity function data of SDSS (r^*) are represented with error bars (red). The continuous line fit (green) represents the four-parameter gamma exponentiated LF (54) and the dotted line represents the Schechter function (blue).

Table 7. Synoptic table for the number equation of the gamma family distributions here used.

Name	PDF	DF	Mean
Gamma	(11)	(19)	(13)
Gamma Truncated	(25)	(31)	(28)
Gamma Exponentiated	(37)	(36)	
Gamma Exponentiated Truncated	(47)	(46)	

The increase from three to five parameters in going from the exponentiated gamma distribution to the truncated exponentiated gamma distribution produces an increase of P_{KS} in four cases out of five of the clusters of stars here analyzed, see Table 3 and Table 4.

Comparison with other distributions for stars

The results for the mass distribution of the γ Velorum cluster compared with other distributions are shown in Table 8. The exponentiated gamma with truncation occupies the 12th position.

Table 8. Numerical values of D, the maximum distance between the theoretical and observed DFs, and PKS, the significance level in the K-S test, for different distributions in the case of γ Velorum cluster.

Distribution	Reference	D	PKS
Exponentiated Gamma with Truncation	here	0.068	0.205
Exponentiated Gamma	here	0.103	1.210^{-2}
MLP	[25]	0.037	0.89
MLP Truncated	[25]	0.052	0.53
Benini	[26]	0.0372	0.89
Benini Right Truncated	[26]	0.042	0.779
Truncated Gompertz	[27]	0.173	9.2710^{-7}
Truncated Topp-Leone	[28]	6.0910^{-2}	0.25
Frèchet	[29]	0.125	3.1310^{-4}
Truncated Frèchet	[29]	0.077	0.07
Truncated Weibull	[30]	0.046	0.576
Truncated Sujatha	[31]	0.0485	0.534
Truncated Lindley	[32]	0.11	0.48
Generalized Gamma	[33]	0.11	1.2410^{-3}
Truncated Generalized Gamma	[33]	0.062	0.24
Lognormal	[34]	0.0729	0.11
Truncated Lognormal	[34]	0.047	0.55
Gamma	[13]	0.059	0.28
Truncated Gamma	[13]	0.0754	0.08
Beta	[35]	0.059	0.28

Luminosity function for the exponentiated gamma

The LF for galaxies for the exponentiated gamma distribution is derived both in the luminosity form, see Equation (56), and in the magnitude form, see Equation (58). The test was done on the five bands of SDSS galaxies, see Table 5. According to the above table, in three cases out of five, the χ^2 is smaller than that of the Schechter LF.

Luminosity function for the exponentiated gamma with truncation

The LF for galaxies for the exponentiated gamma distribution with truncation was derived both in the luminosity form, see Equation (59), and in the magnitude form, see Equation (60). It was tested on the five bands of the SDSS galaxies, see Table 6. According to the above Table, in five cases out of five the χ^2 is smaller than that of the Schechter LF.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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