International Journal of Astronomy and Astrophysics, 2015, 5, 275-290
Published Online December 2015 in SciRes. http://www.scirp.org/journal/ijaa
http://dx.doi.org/10.4236/ijaa.2015.54031

Effect of Perturbations in Coriolis and Centrifugal Forces on the Non-Linear Stability of L_4 in the Photogravitational Restricted Three Body Problem

Kavita Chauhan[1], S. N. Rai[2], Rajiv Aggarwal[3]

[1]Department of P. G. Department of Mathematics, V.K.S. University, Ara, India
[2]Department of Department of Mathematics, S.B. College, Ara, India
[3]Department of Mathematics, Sri Aurobindo College, University of Delhi, Delhi, India
Email: kavitachauhan908@gmail.com, rajiv_agg1973@yahoo.com

Received 16 July 2015; accepted 21 December 2015; published 24 December 2015

Abstract

Effect of perturbations in Coriolis and centrifugal forces on the non-linear stability of the libration point L_4 in the restricted three body problem is studied when both the primaries are axis symmetric bodies (triaxial rigid bodies) and the bigger primary is a source of radiation. Moser's conditions are utilized in this study by employing the iterative scheme of Henrard for transforming the Hamiltonian to the Birkhoff's normal form with the help of double D'Alembert's series. It is found that L_4 is stable for all mass ratios in the range of linear stability except for the three mass ratios μ_{c1}, μ_{c2} and μ_{c3}, which depend upon the perturbations ε_1 and ε_2 in the Coriolis and centrifugal forces respectively and the parameters A_1, A_2, A_3 and A_4 which depend upon the semi-axes $a_1, b_1, c_1; a_2, b_2, c_2$ of the triaxial rigid bodies and *p*, the radiation parameter.

Keywords

Restricted Three Body Problem, Axis Symmetric Bodies; Non-Linear Stability, Libration Point L_4, Double D'Alembert's Series Method

1. Introduction

We propose to study the effect of perturbations in Coriolis (ε_1) and centrifugal forces (ε_2) on the non-linear

How to cite this paper: Chauhan, K., Rai, S.N. and Aggarwal, R. (2015) Effect of Perturbations in Coriolis and Centrifugal Forces on the Non-Linear Stability of L_4 in the Photogravitational Restricted Three Body Problem. *International Journal of Astronomy and Astrophysics*, **5**, 275-290. http://dx.doi.org/10.4236/ijaa.2015.54031

stability of libration point (L_4) when both the primaries are axis symmetric bodies and the bigger primary is a source of radiation. We use Moser's conditions by employing the iterative scheme of Henrard (Deprit and Deprit-Bartholome [1]), for transforming the involved Hamiltonian to the Birkhoff's normal form with the help of double D'Alembert's series. In the year 1983, Bhatnagar and Hallan [2] investigated the perturbation effects in Coriolis and centrifugal forces in the non-linear aspect of stability of L_4. Rajiv Aggarwal *et al.* [3] studied the non-linear stability of L_4 in the restricted three body problem for radiated axes symmetric primaries with resonances. Mamta Jain and Rajiv Aggarwal [4] investigated the existence of non-collinear libration points and their stability (in linear sense) in the circular restricted three body problem, in which they had considered the smaller primary as an oblate spheroid and the bigger one as a point mass including the effect of dissipative force especially Stokes drag. Bhavneet Kaur and Rajiv Aggarwal [5] studied the Robe's restricted problem of 2 + 2 bodies when the bigger primary was a Roche ellipsoid. Jagadish Singh [6] investigated the combined effects of perturbations, radiation and oblateness on the non-linear stability of triangular points. We have extended this study by taking the primaries as axis symmetric bodies. In the present paper, our aim is to examine the effect of perturbations in Coriolis and centrifugul forces in the non-linear stability of the libration point L_4 of the restricted three body problem when both the primaries are axis symmetric bodies and the bigger primary is a source of radiation with its equatorial plane coincident with the plane of motion.

2. Equations of Motions and Linear Stability

We shall use dimensionless variables and adopt the notation and terminology of Szebehely [7]. The equations of motion of the infinitesimal mass m_3 in a synodic co-ordinate system (x, y) are

$$\ddot{x} - 2n\dot{y} = \Omega_x, \quad \ddot{y} + 2n\dot{x} = \Omega_y,$$

where

$$\Omega = \frac{n^2}{2}\left(x^2+y^2\right) + (1-p)\left(\frac{1-\mu}{r_1} + \frac{1-\mu}{2r_1^3}A_1 + \frac{3(1-\mu)y^2}{2r_1^5}A_2\right) + \frac{\mu}{r_2} + \frac{\mu}{2r_2^3}A_3 + \frac{3\mu y^2}{2r_2^5}A_4.$$

$$r_1^2 = (x-\mu)^2 + y^2,\ r_2^2 = (x+1-\mu)^2 + y^2,\ \mu = \frac{m_2}{m_1+m_2} \le \frac{1}{2},\ n = 1 + \frac{3}{4}A_1 + \frac{3}{4}A_3$$

$$A_1 = \frac{2a_1^2 - c_1^2 - b_1^2}{5R^2},\ A_2 = \frac{b_1^2 - a_1^2}{5R^2},\ A_3 = \frac{2a_2^2 - c_2^2 - b_2^2}{5R^2},\ A_4 = \frac{b_2^2 - a_2^2}{5R^2}.$$

R is the distance between the primaries, m_1 and m_2 ($m_1 \ge m_2$) being the masses of the primaries. a_1, b_1 and c_1 are the semi-axes of the axis symmetric body of mass m_1 and a_2, b_2 and c_2 are the semi-axes of the axis symmetric body of mass m_2. The configuration is given in **Figure 1**. Since $0 < (p, A_1, A_2, A_3, A_4) \ll 1$, we will reject second and higher order terms in p, A_1, A_2, A_3 and A_4.

We adopt the method used by Bhatnagar and Hallan [2] and give perturbation in Coriolis and centrifugal forces with the help of the parameters α and β respectively. The unperturbed value of each is unity. Consequently we take the equations of motion as

$$\ddot{x} - 2n\alpha\dot{y} - n^2\beta x = \frac{\partial F}{\partial x}, \quad \ddot{y} + 2n\alpha\dot{x} - n^2\beta y = \frac{\partial F}{\partial y},$$

where

$$\alpha = 1 + \varepsilon_1 \quad |\varepsilon_1| \ll 1, \quad \beta = 1 + \varepsilon_2 \quad |\varepsilon_2| \ll 1.$$

$$F = (1-p)\left(\frac{1-\mu}{r_1} + \frac{1-\mu}{2r_1^3}A_1 + \frac{3(1-\mu)y^2}{2r_1^5}A_2\right) + \frac{\mu}{r_2} + \frac{\mu}{2r_2^3}A_3 + \frac{3\mu y^2}{2r_2^5}A_4.$$

Equations of motion of mass m_3 can be put in the form

$$\ddot{x} - 2n\alpha\dot{y} = \Omega_x, \quad \ddot{y} + 2n\alpha\dot{x} = \Omega_y, \tag{1}$$

where

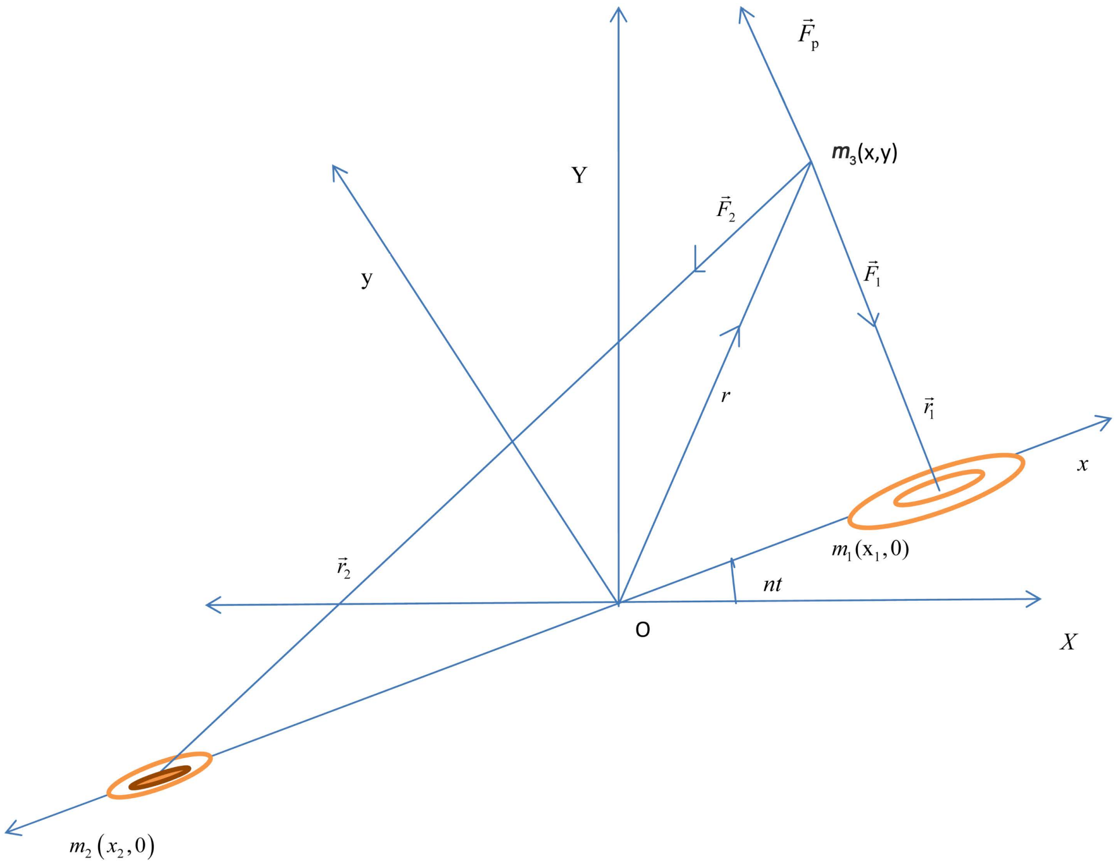

Figure 1. Configuration of the photogravitational restricted problem with both the primaries axis symmetric bodies and the bigger primary is source of radiation.

$$\Omega=\frac{\beta n^2}{2}\left(x^2+y^2\right)+(1-p)\left(\frac{1-\mu}{r_1}+\frac{(1-\mu)A_1}{2r_1^3}+\frac{3(1-\mu)A_2y^2}{2r_1^5}\right)+\left(\frac{\mu}{r_2}+\frac{\mu A_3}{2r_2^3}+\frac{3\mu A_4y^2}{2r_2^5}\right).$$

3. Location of Libration Point of L_4

At L_4, we have $\Omega_x=0, \Omega_y=0$ and $y\neq 0$.

On solving above equations, we get

$$x=-\frac{\gamma}{2}+\frac{p}{3}-\frac{1}{2}A_1-\left(\frac{7}{8}+\frac{1}{2\mu}\right)A_2+\frac{1}{2}A_3+\left(\frac{11-7\mu}{8(-1+\mu)}\right)A_4$$
$$y=\frac{\sqrt{3}}{2}-\frac{1}{3\sqrt{3}}p-\frac{2}{3\sqrt{3}}\varepsilon_2-\frac{1}{2\sqrt{3}}A_1+\left(\frac{-4+15\mu}{8\sqrt{3}\mu}\right)A_2-\frac{1}{2\sqrt{3}}A_3+\left(\frac{-11+15\mu}{8\sqrt{3}(-1+\mu)}\right)A_4 \quad (2)$$

The Lagrangian (L) of the system of equations (1) is

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right)+n\left(1+\varepsilon_1\right)\left(x\dot{y}-y\dot{x}\right)+\frac{n^2}{2}\left(1+\varepsilon_2\right)\left(x^2+y^2\right)$$
$$+(1-p)\left(\frac{1-\mu}{r_1}+\frac{1-\mu}{2r_1^3}A_1+\frac{3(1-\mu)}{2r_1^5}A_2y^2\right)+\frac{\mu}{r_2}+\frac{\mu A_3}{2r_2^3}+\frac{3\mu A_4}{2r_2^5}y^2.$$

Shift the origin to L_4 and expanding in power series of x and y, we get

$$L = L_0 + L_1 + L_2 + L_3 + L_4 + \cdots \tag{3}$$

where

$$\begin{aligned}
L_0 &= \frac{11+\gamma^2}{8} + \frac{1}{16}\left(13+4\gamma+3\gamma^2\right)A_1 + \frac{9}{16}(1+\gamma)A_2 + \frac{1}{16}\left(13-4\gamma+3\gamma^2\right)A_3 \\
&\quad + \frac{9}{16}(1-\gamma)A_4 + \frac{\varepsilon_2}{2} - \frac{1+\gamma}{2}p \\
L_1 &= -\frac{\sqrt{3}}{24}\left(12+12\varepsilon_1 - \frac{16}{3}\varepsilon_2 + \frac{7-15\gamma}{1-\gamma}A_2 + 5A_1 + 5A_3 + \frac{7+15\gamma}{1+\gamma}A_4 - \frac{8}{3}p\right)\dot{x} \\
&\quad - \frac{1}{8}\left(4\gamma + 4\gamma\varepsilon_1 - \frac{8}{3}p + (4+3\gamma)A_1 + \frac{15-7\gamma}{1-\gamma}A_2 + (-4+3\gamma)A_3 - \frac{15+7\gamma}{1+\gamma}A_4\right)\dot{y} \\
L_2 &= \frac{1}{2}\left(\dot{x}^2+\dot{y}^2\right) + \frac{1}{4}\left(4+4\varepsilon_1+3A_1+3A_3\right)\left(x\dot{y}-\dot{x}y\right) \\
&\quad + \frac{3}{16}x^2\left(2+10\varepsilon_2+(5+4\gamma)A_1 + \left(\frac{15}{4}+\frac{47\gamma}{4}+\frac{1}{2\mu}(1+7\gamma)\right)A_2\right. \\
&\quad \left. +(5-4\gamma)A_3 + \left(\frac{15}{4}-\frac{47\gamma}{4}+\frac{1}{2(1-\mu)}(1-7\gamma)\right)A_4 + \frac{2(1-3\gamma)}{3}p\right) \\
&\quad - \frac{\sqrt{3}xy}{8}\left(6\gamma - \frac{22}{3}\varepsilon_2\gamma + (6+13\gamma)A_1 + \left(\frac{87}{4}+\frac{15\gamma}{4}+\frac{1}{2\mu}(11\gamma-3)\right)A_2\right. \\
&\quad \left. -(6-13\gamma)A_3 - \left(\frac{87}{4}-\frac{15\gamma}{4}-\frac{11\gamma+3}{2(1-\mu)}\right)A_4 - \frac{2}{3}(3-\gamma)p\right) \\
&\quad + \frac{3}{16}y^2\left(6+\frac{14\varepsilon_2}{3}+11A_1 + \left(\frac{1}{4}-\frac{15\gamma}{4}+\frac{1}{2\mu}(3-11\gamma)\right)A_2\right. \\
&\quad \left. +11A_3 + \left(\frac{1}{4}+\frac{15\gamma}{4}+\frac{3+11\gamma}{2(1-\mu)}\right)A_4 - \frac{2}{3}(1-3\gamma)p\right)
\end{aligned}$$

$$\begin{aligned}
L_3 &= -\frac{1}{32}\left(14\gamma + \frac{50\varepsilon_2}{3} + (-6+25\gamma)A_1 + \left(7-\frac{15\gamma}{2}+\frac{25-37\gamma}{2\mu}\right)A_2\right. \\
&\quad \left. +(6+25\gamma)A_3 + \left(-7-\frac{15\gamma}{2}+\frac{37+25\gamma}{2(1-\mu)}\right)A_4 + \frac{4}{3}(4+\gamma)p\right)x^3 \\
&\quad - \frac{\sqrt{3}}{32}\left(6+\frac{82\varepsilon_2}{3}+(43+60\gamma)A_1 + \left(75+\frac{435\gamma}{2}+\frac{1}{2\mu}(41+75\gamma)\right)A_2\right. \\
&\quad \left. -\frac{4}{3}(-8+21\gamma)p + (43-60\gamma)A_3 + \left(75-\frac{435\gamma}{2}+\frac{1}{2(1-\mu)}(41-75\gamma)\right)A_4\right)x^2y \\
&\quad + \frac{3}{32}\left(22\gamma+30\varepsilon_2+(22+65\gamma)A_1 + \left(76+\frac{55\gamma}{2}+\frac{45\gamma-41}{2\mu}\right)A_2\right. \\
&\quad \left. +\frac{4}{3}(2+3\gamma)p - (22-65\gamma)A_3 + \left(-76+\frac{55\gamma}{2}+\frac{1}{2(1-\mu)}(41+45\gamma)\right)A_4\right)xy^2 \\
&\quad - \frac{\sqrt{3}}{32}\left(6+\frac{2}{3}\varepsilon_2+23A_1+\left(-\frac{75\gamma}{2}+\frac{1-45\gamma}{2\mu}\right)A_2 - \frac{4}{3}(2-9\gamma)p + 23A_3 + \left(\frac{75\gamma}{2}+\frac{1+45\gamma}{2(1-\mu)}\right)A_4\right)y^3
\end{aligned}$$

$$
\begin{aligned}
L_4 = & -\frac{1}{256}\left(74+190\varepsilon_2+\left(285+200\gamma\right)A_1+p_{27}A_2+q_{27}p+\left(285-200\gamma\right)A_3+p'_{27}A_4\right)x^4 \\
& +\frac{5\sqrt{3}}{192}\left(30\gamma+\frac{86\gamma}{3}\varepsilon_2+\left(-54+53\gamma\right)A_1+p_{28}A_2+\left(54+53\gamma\right)A_3+p'_{28}A_4+q_{28}p\right)x^3y \\
& +\frac{3}{128}\left(82+230\varepsilon_2+\left(405+340\gamma\right)A_1+p_{29}A_2+\left(405-340\gamma\right)A_3+p'_{29}A_4+q_{29}p\right)x^2y^2 \\
& -\frac{5\sqrt{3}}{64}\left(18\gamma+\frac{74\gamma}{3}+\left(18+71\gamma\right)A_1+p_{30}A_2+\left(-18+71\gamma\right)A_3+p'_{30}A_4+q_{30}p\right)xy^3 \\
& +\frac{3}{256}\left(2-\frac{110\varepsilon_2}{3}+65A_1+p_{31}A_2+65A_3+p'_{31}A_4+q_{31}p\right)y^4
\end{aligned}
$$

Hamiltonian function H corresponding to above Lagrangian is given by:

$$H = H_0 + H_1 + H_2 + H_3 + H_4 + \cdots, \tag{4}$$

where

$$H_0 = -L_0,\quad H_1 = 0,\quad H_3 = -L_3,\quad H_4 = -L_4$$

and

$$H_2 = \frac{1}{2}\left(p_x^2+p_y^2\right)+n\alpha\left(yp_x-xp_y\right)+Ex^2+Fy^2+Gxy,$$

$$
\begin{aligned}
E = & \frac{1}{8}+\varepsilon_1-\frac{5}{8}\varepsilon_2-\frac{3(1+4\gamma)}{16}A_1-\frac{3}{64}\left(15+47\gamma+\frac{2(1+7\gamma)}{\mu}\right)A_2-\frac{3(1-4\gamma)}{16}A_3 \\
& -\frac{3}{64}\left(15-47\gamma+\frac{2(1-7\gamma)}{1-\mu}\right)A_4-\frac{1}{8}(1-3\gamma)p,
\end{aligned}
$$

$$
\begin{aligned}
F = & -\frac{5}{8}+\varepsilon_1-\frac{7}{8}\varepsilon_2-\frac{21}{16}A_1+\frac{3}{64}\left(-1+15\gamma+\frac{2(-3+11\gamma)}{\mu}\right)A_2 \\
& -\frac{21}{16}A_3+\frac{3}{64}\left(-1-15\gamma-\frac{2(3+11\gamma)}{1-\mu}\right)A_4+\frac{1}{8}(1-3\gamma)p,
\end{aligned}
$$

$$
\begin{aligned}
G = & \frac{\gamma(9+11\varepsilon_2)}{4\sqrt{3}}+\frac{\sqrt{3}}{8}\left\{(6+13\gamma)A_1+\frac{1}{4}\left(87+15\gamma+\frac{2(-3+11\gamma)}{\mu}\right)A_2\right. \\
& \left.+(-6+13\gamma)A_3-\frac{1}{4}\left(87-15\gamma-\frac{2(3+11\gamma)}{1-\mu}\right)A_4-\frac{2(3-\gamma)}{3}p\right\}
\end{aligned}
$$

To investigate the linear stability of the motion, as in Whittaker [8], we consider the following set of linear equations in the variables x and y

$$AX = 0 \tag{5}$$

where

$$X = \begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix},\quad A = \begin{bmatrix} 2E & G & \lambda & -n \\ G & 2F & n & \lambda \\ -\lambda & n & 1 & 0 \\ -n & -\lambda & 0 & 1 \end{bmatrix}$$

The Equation (5) has a nonzero solution if and only if $\det A = 0$, which implies that

$$\lambda^4+\lambda^2\left(1+8\varepsilon_1-3\varepsilon_2-\frac{3\gamma}{2}A_1+\frac{3\gamma}{2}A_3-3(1+\gamma)A_2-3(1-\gamma)A_4\right)$$
$$+\frac{9(1-\gamma^2)}{16}\left(3+\frac{22}{3}\varepsilon_2+13A_1+13A_3+\frac{2}{3}p\right)+\frac{45}{64}(7+4\gamma-3\gamma^2)A_2 \tag{6}$$
$$+\frac{45}{64}(7-4\gamma-3\gamma^2)A_4=0.$$

Let the discriminant of the characteristic Equation (6) be denoted by *D*.

If $D<0$ then $\lambda^2<0$, it is bounded, hence stable when $0<\mu<\mu_C$ where

$$\mu_C=0.0385208965\cdots+(0.642057883\cdots)\varepsilon_1-(0.338863882\cdots)\varepsilon_2-(0.00891747\cdots)p$$
$$-(0.285001784\cdots)A_1-(1.381268434\cdots)A_2-(0.06278\cdots)A_3-(0.10349\cdots)A_4. \tag{7}$$

Let the roots of characteristic Equation (6) be $\pm\omega_1'$ and $\pm\omega_2'$. These are long term and short term perturbed frequencies, which are given by

$$\omega_1'^2+\omega_2'^2=1+8\varepsilon_1-3\varepsilon_2-\frac{3\gamma}{2}A_1+\frac{3\gamma}{2}A_3-3(1+\gamma)A_2-3(1-\gamma)A_4$$
$$\omega_1'^2\omega_2'^2=\frac{9}{16}(1-\gamma^2)\left(3+\frac{22}{3}\varepsilon_2+13A_1+13A_3+\frac{2}{3}p\right) \tag{8}$$
$$+\frac{45}{64}(7+4\gamma-3\gamma^2)A_2+\frac{45}{64}(7-4\gamma-3\gamma^2)A_4.$$

Here ω_1',ω_2' represent the perturbed basic frequencies. The unperturbed basic frequencies ω_1,ω_2, are given by

$$\omega_1^2+\omega_2^2=1,\quad \omega_1^2\omega_2^2=\frac{27}{16}(1-\gamma^2),\quad (0<\omega_2<0.5<\omega_1<1).$$

We may write

$$\omega_1'=\omega_1(1+p_1\varepsilon_1+p_2\varepsilon_2),\quad \omega_2'=\omega_2(1+q_1\varepsilon_1+q_2\varepsilon_2), \tag{9}$$

by taking perturbations in the Coriolis and centrifugal forces. Here p_1,p_2,q_1,q_2 are to be determined so that Equations (8) are satisfied. Simple calculations give

$$p_1=-q_1=\frac{4}{k_1^2}=\frac{4}{k_2^2},\quad p_2=\frac{22\omega_1^2-49}{18k_1^2}=\frac{22\omega_1^2-49}{18k_2^2},$$
$$q_2=-\frac{22\omega_2^2-49}{18k_1^2}=-\frac{22\omega_2^2-49}{18k_2^2}.$$

where

$$k_1^2=2\omega_1^2-1;\quad k_2^2=1-2\omega_2^2.$$

4. Determination of the Normal Co-Ordinates

To express H_2 in normal form, we consider the set of linear Equation (5), the solution of which can be obtained as

$$\frac{x}{2n\lambda-G}=\frac{y}{\lambda^2-n^2+2E}=\frac{p_x}{n\lambda^2-\lambda G-2nE+n^3}=\frac{p_y}{\lambda^3+n^2\lambda+2\lambda E-nG}.$$

We use the canonical transformations from the phase space (x,y,p_x,p_y) into the phase space of the angles (ϕ_1,ϕ_2) and the action moment as (I_1,I_2) *i.e.*

$$X=JT \tag{10}$$

where

$$X=\begin{bmatrix} x \\ y \\ p_x \\ p_y \end{bmatrix},\quad T=\begin{bmatrix} Q_1 \\ Q_2 \\ P_1 \\ P_2 \end{bmatrix},$$

$$Q_i=\left(\frac{2I_i}{\omega_i'}\right)^{\frac{1}{2}}\operatorname{Sin}\phi_i,$$

$$P_i=\left(2I_i\omega_i'\right)^{1/2}\operatorname{Cos}\phi_i,\quad (i=1,2),$$

$$J=\left[J_{ij}\right]=\begin{bmatrix} x_1-\dfrac{i\omega_1'x_3}{2} & x_2+\dfrac{i\omega_2'x_4}{2} & -\dfrac{ix_1}{\omega_1'}+\dfrac{x_3}{2} & \dfrac{ix_2}{\omega_2'}+\dfrac{x_4}{2} \\ y_1-\dfrac{i\omega_1'y_3}{2} & y_2+\dfrac{i\omega_2'y_4}{2} & -\dfrac{iy_1}{\omega_1'}+\dfrac{y_3}{2} & \dfrac{iy_2}{\omega_2'}+\dfrac{y_4}{2} \\ \left(p_x\right)_1-\dfrac{i\omega_1'\left(p_x\right)_3}{2} & \left(p_x\right)_2+\dfrac{i\omega_2'\left(p_x\right)_4}{2} & -\dfrac{i\left(p_x\right)_1}{\omega_1'}+\dfrac{\left(p_x\right)_3}{2} & \dfrac{i\left(p_x\right)_2}{\omega_2'}+\dfrac{\left(p_x\right)_4}{2} \\ \left(p_y\right)_1-\dfrac{i\omega_1'\left(p_y\right)_3}{2} & \left(p_y\right)_2+\dfrac{i\omega_2'\left(p_y\right)_4}{2} & -\dfrac{i\left(p_y\right)_1}{\omega_1'}+\dfrac{\left(p_y\right)_3}{2} & \dfrac{i\left(p_y\right)_2}{\omega_2'}+\dfrac{\left(p_y\right)_4}{2} \end{bmatrix}$$

Following the procedure of Bhatnagar and Hallan [2], we get the normal form of the Hamiltonian

$$H_2=\omega_1'I_1-\omega_2'I_2\,.$$

Taking

$$H=H_0+H_2\,,$$

Equations of motion

$$\frac{\mathrm{d}I_i}{\mathrm{d}t}=-\frac{\partial H}{\partial\phi_i},\quad \frac{\mathrm{d}\phi_i}{\mathrm{d}t}=\frac{\partial H}{\partial I_i},\quad (i=1,2)$$

become

$$\frac{\mathrm{d}I_i}{\mathrm{d}t}=0;\qquad (i=1,2),$$

$$\frac{\mathrm{d}\phi_1}{\mathrm{d}t}=\omega_1',\ \frac{\mathrm{d}\phi_2}{\mathrm{d}t}=-\omega_2',$$

The general solution of the equations of the motion is

$$I_i=\text{constant}\ \left(i=1,2\right),$$

$$\phi_1=\omega_1't+\text{constant}\,,$$

$$\phi_2=-\omega_2't+\text{constant}\,.$$

5. Second Order Normalization

Now, to perform Birkhoff's normalization, the coordinates (x, y) are to be expanded in double D'Alembert series:

$$x=\sum_{n\ge1}B_n^{1,0}\left(\phi_1,\phi_2,I_1,I_2\right),\ y=\sum_{n\ge1}B_n^{0,1}\left(\phi_1,\phi_2,I_1,I_2\right),\tag{11}$$

where $B_n^{1,0}$ and $B_n^{0,1}$ are homogenious functions of degree n in $I_1^{1/2}, I_2^{1/2}$ and are in the form

$$\sum_{0\le m\le n}I_1^{\frac{1}{2}(n-m)}I_2^{\frac{1}{2}m}\sum_{i,j}\left[C_{n-m,m,i,j}\operatorname{Cos}\left(i\phi_1+j\phi_2\right)+S_{n-m,m,i,j}\operatorname{Sin}\left(i\phi_1+j\phi_2\right)\right].$$

The double summation over the indices i and j is such that:

1) i runs over those integers in the interval $0 \le i \le n-m$ that have the same parity as $n-m$
2) j runs over those integers in the interval $-m \le j \le m$ that have the same parity as m.

I_1 and I_2 are to be regarded as constants of integration and ϕ_1, ϕ_2 are to be determined as linear functions of time (t) such that

$$\dot{\phi}_1 = \omega_1' + \sum_{n\ge1} f_{2n}(I_1, I_2), \quad \dot{\phi}_2 = -\omega_2' + \sum_{n\ge1} g_{2n}(I_1, I_2).$$

where f_{2n} and g_{2n} are of the form

$$f_{2n} = \sum_{n\ge m\ge 0} f'_{2(n-m),2m} I_1^{n-m} I_2^m, \quad g_{2n} = \sum_{n\ge m\ge 0} g'_{2(n-m),2m} I_1^{n-m} I_2^m.$$

According to Deprit and Deprit Bartholome [1], the canonical character of the transformation will be ensured formally by requesting that the double D'Alembert series satisfy the identities

$$(x, y) = 0, \ (x, \dot{x}) = 1, \ (y, \dot{x}) = 0, \ (x, \dot{y}) = 0, \ (y, \dot{y}) = 1, \ (\dot{x}, \dot{y}) = 0.$$

Where the left hand members stand for the Poisson's brackets with respect to the phase variables $(\phi_1, \phi_2, I_1, I_2)$. The first order components $B_1^{1,0}$ and $B_1^{0,1}$ in $I_1^{1/2}$ and $I_2^{1/2}$ are the values of x and y given by Equation (10)

$$B_1^{1,0} = J_{13}(2\omega_1')^{1/2} I_1^{1/2} \mathrm{Cos}\phi_1 + J_{14}(2\omega_2')^{1/2} I_2^{1/2} \mathrm{Cos}\phi_2,$$

$$B_1^{0,1} = J_{21}\left(\frac{2}{\omega_1'}\right)^{1/2} I_1^{1/2}\mathrm{Sin}\phi_1 + J_{22}\left(\frac{2}{\omega_2'}\right)^{1/2} I_2^{1/2}\mathrm{Sin}\phi_2 + J_{23}(2\omega_1')^{1/2} I_1^{1/2}\mathrm{Cos}\phi_1 + J_{24}(2\omega_2')^{1/2} I_2^{1/2}\mathrm{Cos}\phi_2.$$

The values of $J_{13}, J_{14}, J_{21}, J_{22}, J_{23}, J_{24}$ can be obtained from **Appendix**.

Proceeding as in Deprit and Deprit-Bartholome [1], it is observed that the second order components $B_2^{0,1}$ and $B_2^{1,0}$ are solutions of the partial differential equations

$$\Delta_1\Delta_2 B_2^{1,0} = \Phi_2, \ \Delta_1\Delta_2 B_2^{0,1} = -\Psi_2,$$

where

$$\Delta_i = D^2 + \omega_i'^2 \qquad (i = 1, 2).$$

$$\Phi_2 = \left(D^2 - \frac{3}{8}\left(6 + \frac{14}{3}\varepsilon_2 + 11A_1 + p_{10}A_2 + 11A_3 + p'_{10}A_4 + q_{10}p\right)\right)X_2 + \left[\frac{1}{2}(4 + 4\varepsilon_1 + 3A_1 + 3A_3)D - \frac{\sqrt{3}}{8}\left(6\gamma + \frac{22\gamma}{3}\varepsilon_2 + (6+13\gamma)A_1 + p_9A_2 + (-6+13\gamma)A_3 + p'_9A_4 + q_9p\right)\right]Y_2.$$

$$\Psi_2 = -\left(D^2 - \frac{3}{8}\left(2 + \frac{10}{3}\varepsilon_2 + (5+4\gamma)A_1 + p_8A_2 + (5-4\gamma)A_3 + p'_8A_4 + q_8p\right)\right)Y_2 + \left[\frac{1}{2}(4 + 4\varepsilon_1 + 3A_1 + 3A_3)D + \frac{\sqrt{3}}{8}\left(6\gamma + \frac{22\gamma}{3}\varepsilon_2 + (6+13\gamma)A_1 + p_9A_2 + (-6+13\gamma)A_3 + p'_9A_4 + q_9p\right)\right]X_2$$

$$D = \omega_1'\frac{\partial}{\partial\phi_1} - \omega_2'\frac{\partial}{\partial\phi_2}$$

X_2 and Y_2 are obtained by

$$X_2 = \left(\frac{\partial L_3}{\partial x}\right)_{\substack{x=\sum B_1^{1,0}\\ y=\sum B_1^{0,1}}}, \quad Y_2 = \left(\frac{\partial L_3}{\partial y}\right)_{\substack{x=\sum B_1^{1,0}\\ y=\sum B_1^{0,1}}}.$$

Now

$$B_2^{1,0}=r_1I_1+r_2I_2+r_3I_1\mathrm{Cos}2\phi_1+r_4I_2\mathrm{Cos}2\phi_2+r_5I_1^{1/2}I_2^{1/2}\mathrm{Cos}(\phi_1-\phi_2)+r_6I_1^{1/2}I_2^{1/2}\mathrm{Cos}(\phi_1+\phi_2)$$
$$+r_7I_1\mathrm{Sin}2\phi_1+r_8I_2\mathrm{Sin}2\phi_2+r_9I_1^{1/2}I_2^{1/2}\mathrm{Sin}(\phi_1-\phi_2)+r_{10}I_1^{1/2}I_2^{1/2}\mathrm{Sin}(\phi_1+\phi_2),$$

$$B_2^{0,1}=s_1I_1+s_2I_2+s_3\mathrm{Cos}2\phi_1I_1+s_4\mathrm{Cos}2\phi_2I_2+s_5\mathrm{Cos}(\phi_1-\phi_2)\sqrt{I_1I_2}+s_6\mathrm{Cos}(\phi_1+\phi_2)\sqrt{I_1I_2}$$
$$+s_7\mathrm{Sin}2\phi_1I_1+s_8\mathrm{Sin}2\phi_2I_2+s_9\mathrm{Sin}(\phi_1-\phi_2)\sqrt{I_1I_2}+s_{10}\mathrm{Sin}(\phi_1+\phi_2)\sqrt{I_1I_2}$$

where

$$r_i=r_{i,1}\gamma(1+\alpha_i\varepsilon_1+\alpha_i'\varepsilon_2)+(r_{i,2}+r_{i,3}\gamma)A_1+r_{i,4}A_2$$
$$+(r_{i,2}'+r_{i,3}'\gamma)A_3+r_{i,4}'A_4+r_{i,5}p \qquad i=1,2,\cdots,6$$
$$r_j=r_{j,1}+r_{j,1}\gamma(\alpha_j\varepsilon_1+\alpha_j'\varepsilon_2)+(r_{j,2}+r_{j,3}\gamma)A_1+r_{j,4}A_2$$
$$+(r_{j,2}'+r_{j,3}'\gamma)A_3+r_{j,4}{}'A_4+r_{j,5}p \qquad j=7,8,\cdots,10$$
$$s_i=s_{i,1}(1+\beta_i\varepsilon_1+\beta_i'\varepsilon_2)+(s_{i,2}+s_{i,3}\gamma)A_1+s_{i,4}A_2$$
$$+(s_{i,2}'+s_{i,3}'\gamma)A_3+s_{i,4}'A_4+s_{i,5}p \qquad i=1,2,\cdots,6$$
$$s_j=s_{j,1}\gamma(1+\beta_j\varepsilon_1+\beta_j'\varepsilon_2)+(s_{j,2}+s_{j,3}\gamma)A_1+s_{j,4}A_2$$
$$+(s_{j,2}'+s_{j,3}'\gamma)A_3+s_{j,4}'A_4+s_{j,5}p \qquad j=7,8,\cdots,10$$

The values of all r_i,r_j,s_i,s_j for $i=1,2,\cdots,6;\ j=7,8,\cdots,10$ can be obtained from the authors on request as the expressions are very long and contained in large number of pages.

6. Third-Order Terms in *H*

Following the procedure of Bhatnagar and Hallan [2], Hamiltonian H given by Equation (4) transforms to the Hamiltonian in which the 3rd order term in $I_1^{1/2}$ and $I_2^{1/2}$ is zero. That is $H_3=0$.

7. Second Order Coefficient in the Frequencies

Following the iterative procedure of Henrard, the third order homogeneous components $B_3^{1,0}$ and $B_3^{0,1}$ in Equation (11) can be obtained by partial differential equations

$$\Delta_1\Delta_2B_3^{1,0}=\Phi_3-2f_2P-2g_2Q,$$
$$\Delta_1\Delta_2B_3^{0,1}=\Psi_3-2f_2U-2g_2V,$$

where

$$\Phi_3=(D^2-\lambda_1)X_3+(\lambda_2D+\lambda_3)Y_3,$$
$$\Psi_3=(D^2-\lambda_4)Y_3-(\lambda_2D-\lambda_3)X_3,$$

$$P=\frac{\partial}{\partial\phi_1}\left[\left(\omega_1'^2\frac{\partial^2}{\partial\phi_1^2}-\lambda_1\right)\left(\omega_1'\frac{\partial B_1^{1,0}}{\partial\phi_1}-\frac{1}{2}\lambda_2B_1^{0,1}\right)+\left(\lambda_2\omega_1'\frac{\partial}{\partial\phi_1}+\lambda_3\right)\left(\omega_1'\frac{\partial B_1^{0,1}}{\partial\phi_1}+\frac{1}{2}\lambda_2B_1^{1,0}\right)\right],$$

$$Q=-\frac{\partial}{\partial\phi_2}\left[\left(\omega_2'^2\frac{\partial^2}{\partial\phi_2^2}-\lambda_1\right)\left(\omega_2'\frac{\partial B_1^{1,0}}{\partial\phi_2}+\frac{1}{2}\lambda_2B_1^{0,1}\right)+\left(\lambda_2\omega_2'\frac{\partial}{\partial\phi_2}-\lambda_3\right)\left(-\omega_2'\frac{\partial B_1^{0,1}}{\partial\phi_2}+\frac{1}{2}\lambda_2B_1^{1,0}\right)\right],$$

$$U=\frac{\partial}{\partial\phi_1}\left[\left(\omega_1'^2\frac{\partial^2}{\partial\phi_1^2}-\lambda_4\right)\left(\omega_1'\frac{\partial B_1^{0,1}}{\partial\phi_1}+\frac{1}{2}\lambda_2B_1^{1,0}\right)-\left(\lambda_2\omega_1'\frac{\partial}{\partial\phi_1}-\lambda_3\right)\left(\omega_1'\frac{\partial B_1^{1,0}}{\partial\phi_1}-\frac{1}{2}\lambda_2B_1^{0,1}\right)\right],$$

$$V=\frac{\partial}{\partial\phi_2}\left[\left(\omega_2'^2\frac{\partial^2}{\partial\phi_2^2}-\lambda_4\right)\left(-\omega_2'\frac{\partial B_1^{0,1}}{\partial\phi_2}+\frac{1}{2}\lambda_2B_1^{1,0}\right)-\left(\lambda_2\omega_2'\frac{\partial}{\partial\phi_2}+\lambda_3\right)\left(\omega_2'\frac{\partial B_1^{1,0}}{\partial\phi_2}+\frac{1}{2}\lambda_2B_1^{0,1}\right)\right],$$

$$X_3 = \frac{\partial L_3}{\partial x} + \frac{\partial L_4}{\partial x} = l_1 x^2 + l_2 xy + l_3 y^2 + l_4 x^3 + l_5 x^2 y + l_6 xy^2 + l_7 y^3,$$

$$Y_3 = \frac{\partial L_3}{\partial y} + \frac{\partial L_4}{\partial y} = m_1 x^2 + m_2 xy + m_3 y^2 + m_4 x^3 + m_5 x^2 y + m_6 xy^2 + m_7 y^3.$$

$$\lambda_1 = \frac{1}{8}\left[18 + 14\varepsilon_2 + 33A_1 - 16p_6 A_2 + 33A_3 - 16p_6' A_4 - 16q_6 p\right]$$

$$\lambda_2 = \frac{1}{2}\left[4 + 4\varepsilon_1 + 3A_1 + 3A_3\right]$$

$$\lambda_3 = -\frac{\sqrt{3}}{8}\left[6\gamma + \frac{22\gamma}{3}\varepsilon_2 + (6 + 13\gamma)A_1 + p_7 A_2 + (-6 + 13\gamma)A_3 + p_7' A_4 + q_7 p\right]$$

$$\lambda_4 = \frac{1}{8}\left[6 + 10\varepsilon_2 + (15 + 12\gamma)A_1 - 16p_5 A_2 + (15 - 12\gamma)A_3 - 16p_5' A_4 - 16q_5 p\right]$$

$$l_1 = -\frac{3}{32}\left[14\gamma + \frac{50}{3}\gamma\varepsilon_2 + (-6 + 25\gamma)A_1 + p_{11} A_2 + (6 + 25\gamma)A_3 + p_{11}' A_4 + q_{11} p\right],$$

$$l_2 = -\frac{\sqrt{3}}{16}\left[6 + \frac{82}{3}\varepsilon_2 + (43 + 60\gamma)A_1 + p_{12} A_2 + (43 - 60\gamma)A_3 + p_{12}' A_4 + q_{12} p\right],$$

$$l_3 = \frac{3}{32}\left[22\gamma + 30\gamma\varepsilon_2 + (22 + 65\gamma)A_1 + p_{13} A_2 + (-22 + 65\gamma)A_3 + p_{13}' A_4 + q_{13} p\right],$$

$$l_4 = -\frac{1}{64}\left[74 + 190\varepsilon_2 + (285 + 200\gamma)A_1 + p_{27} A_2 + (285 - 200\gamma)A_3 + p_{27}' A_4 + q_{27} p\right],$$

$$l_5 = \frac{5\sqrt{3}}{64}\left[30\gamma + \frac{86}{3}\gamma\varepsilon_2 + (-54 + 53\gamma)A_1 + p_{28} A_2 + (54 + 53\gamma)A_3 + p_{28}' A_4 + q_{28} p\right],$$

$$l_6 = \frac{3}{64}\left[82 + 230\varepsilon_2 + (405 + 340\gamma)A_1 + p_{29} A_2 + (405 - 340\gamma)A_3 + p_{29}' A_4 + q_{29} p\right],$$

$$l_7 = -\frac{5\sqrt{3}}{64}\left[18\gamma + \frac{74}{3}\gamma\varepsilon_2 + (18 + 71\gamma)A_1 + p_{30} A_2 + (-18 + 71\gamma)A_3 + p_{30}' A_4 + q_{30} p\right],$$

$$m_1 = \frac{l_2}{2}, \quad m_2 = 2l_3, \quad m_4 = \frac{l_5}{3}, \quad m_5 = l_6, \quad m_6 = 3l_7,$$

$$m_3 = -\frac{3\sqrt{3}}{32}\left[6 + \frac{2}{3}\varepsilon_2 + 23A_1 + p_{14} A_2 + 23A_3 + p_{14}' A_4 + q_{14} p\right],$$

$$m_7 = \frac{3}{64}\left[2 - \frac{110}{3}\varepsilon_2 + 65A_1 + p_{31} A_2 + 65A_3 + p_{31}' A_4 + q_{31} p\right].$$

The values of p_{ij}, p_{ij}', q_{ij} are given in **Appendix**.

The partial derivatives in the last two equations have been obtained by substituting $x = B_1^{1,0} + B_2^{1,0}$ and $y = B_1^{0,1} + B_2^{0,1}$ in L_3 and L_4. Now choosing

$$f_2 = f_{2,0}' I_1 + f_{0,2}' I_2, \qquad g_2 = g_{2,0}' I_1 + g_{0,2}' I_2.$$

We find that

$$A = f_{2,0}' = \frac{\text{coefficient of } \cos\phi_1 \text{ in } \Phi_3}{2(\text{coefficient of } \cos\phi_1 \text{ in } P)},$$

$$B = f_{0,2}' = g_{2,0}' = \frac{\text{coefficient of } \cos\phi_2 \text{ in } \Phi_3}{2(\text{coefficient of } \cos\phi_2 \text{ in } Q)},$$

$$C = g'_{0,2} = \frac{\text{coefficient of } \cos\phi_1 \text{ in } \Psi_3}{2\left(\text{coefficient of } \cos\phi_1 \text{ in } Q\right)}.$$

After simplification the values of A, B and C are given by:

$$\begin{aligned}
A = {} & \frac{\left(\omega_1^2-1\right)\left(124\omega_1^4-696\omega_1^2+81\right)}{72k_1^4\left(5\omega_1^2-1\right)}\Big[1+\left(\chi-\kappa\right)\varepsilon_1+\left(\chi'-\kappa'\right)\varepsilon_2 \\
& +\frac{1696\omega_1^6-20320\omega_1^4+14547\omega_1^2-1107}{6\left(\omega_1^2-1\right)\left(124\omega_1^4-696\omega_1^2+81\right)}A_1 \\
& +\frac{1696\omega_1^6-20320\omega_1^4+14547\omega_1^2-1107}{6\left(\omega_1^2-1\right)\left(124\omega_1^4-696\omega_1^2+81\right)}A_3 \\
& -\frac{3\left(1208\omega_1^8+2914\omega_1^6+725\omega_1^4-624\omega_1^2+45\right)}{2k_1^2\left(5\omega_1^2-1\right)\left(\omega_1^2-1\right)\left(124\omega_1^4-696\omega_1^2+81\right)}A_1\gamma \\
& +\left(\xi-\eta\right)A_2+\frac{3\left(1208\omega_1^8+2914\omega_1^6+725\omega_1^4-624\omega_1^2+45\right)}{2k_1^2\left(5\omega_1^2-1\right)\left(\omega_1^2-1\right)\left(124\omega_1^4-696\omega_1^2+81\right)}A_3\gamma \\
& +\left(\xi_1-\eta_1\right)A_4+\left(\sigma-\rho\right)p\Big] \\
B = {} & \frac{u\left(64u^2+43\right)}{6k_1^2k_2^2\left(1-5\omega_1^2\right)\left(5\omega_1^2-1\right)}\Big[1+\left(\chi_1-\kappa\right)\varepsilon_1+\left(\chi_1'-\kappa'\right)\varepsilon_2-\frac{\left(6719u^2-2319\right)}{6\left(64u^2+43\right)}A_1-\frac{\left(6719u^2-2319\right)}{6\left(64u^2+43\right)}A_3 \\
& +\frac{3\left(1116800u^8+15048088u^6-10165353u^4+1972620u^2-93312\right)}{16u^2l_1^2l_2^2\left(1-5\omega_1^2\right)\left(1-5\omega_2^2\right)\left(64u^2+43\right)}A_1\gamma \\
& -\frac{3\left(1116800u^8+15048088u^6-10165353u^4+1972620u^2-93312\right)}{16u^2l_1^2l_2^2\left(1-5\omega_1^2\right)\left(1-5\omega_2^2\right)\left(64u^2+43\right)}A_3\gamma \\
& +\left\{\left(\varsigma-\eta\right)+\frac{15\left(1+\gamma\right)}{\left(1-5\omega_1^2\right)\left(1-5\omega_2^2\right)}\right\}A_2+\left\{\left(\varsigma_1-\eta_1\right)+\frac{15\left(1-\gamma\right)}{\left(1-5\omega_1^2\right)\left(1-5\omega_2^2\right)}\right\}A_4 \\
& +\left(\tau-\rho\right)p\Big], \\
C = {} & \frac{\left(\omega_2^2-1\right)\left(124\omega_2^4-696\omega_2^2+81\right)}{72k_2^4\left(5\omega_2^2-1\right)}\Big[1+\left(\chi_2-\kappa_2\right)\varepsilon_1+\left(\chi_2'-\kappa_2'\right)\varepsilon_2 \\
& +\frac{1696\omega_2^6-20320\omega_2^4+14547\omega_2^2-1107}{6\left(\omega_2^2-1\right)\left(124\omega_2^4-696\omega_2^2+81\right)}A_1 \\
& +\frac{1696\omega_2^6-20320\omega_2^4+14547\omega_2^2-1107}{6\left(\omega_2^2-1\right)\left(124\omega_2^4-696\omega_2^2+81\right)}A_3 \\
& -\frac{3\left(1208\omega_2^8+2914\omega_2^6+725\omega_2^4-624\omega_2^2+45\right)}{2k_2^2\left(5\omega_2^2-1\right)\left(\omega_2^2-1\right)\left(124\omega_2^4-696\omega_2^2+81\right)}A_1\gamma \\
& +\left(\xi'-\eta'\right)A_2+\frac{3\left(1208\omega_2^8+2914\omega_2^6+725\omega_2^4-624\omega_2^2+45\right)}{2k_2^2\left(5\omega_2^2-1\right)\left(\omega_2^2-1\right)\left(124\omega_2^4-696\omega_2^2+81\right)}A_3\gamma \\
& +\left(\xi_1'-\eta_1'\right)A_4+\left(\sigma'-\rho'\right)p\Big].
\end{aligned} \tag{12}$$

The values of ξ,ς,η, ξ',η', $\xi_1,\varsigma_1,\eta_1,\xi_1',\eta_1',\sigma,\sigma',\rho,\rho'$ and τ can be obtained from the author on request as the expressions are again very long and contained in large number of pages. Coefficients of ε_1 and ε_2 can be obtained by Bhatnagar and Hallan [2].

8. Stability

While evaluating $B_2^{1,0}, B_2^{0,1}, B_3^{1,0}$ and $B_3^{0,1}$ the condition (i) of Moser's theorem as in Moser [9] is assumed. Now we verify that this condition is satisfied. The condition is $k_1\omega_1' + k_2\omega_2' \neq 0$ for all pairs (k_1, k_2) of rational integers such that

$$|k_1| + |k_2| \leq 4 \tag{13}$$

We note that the inequalities (13) are violated when

$$\omega_1' = 2\omega_2' \text{ and } \omega_1' = 3\omega_2' \tag{14}$$

Case (i) $\omega_1' = 2\omega_2'$.

We get

$$\omega_1' = \frac{2}{\sqrt{5}}\left(1 + 4\varepsilon_1 - \frac{3}{2}\varepsilon_2 - \frac{3\gamma}{4}A_1 + \frac{3\gamma}{4}A_3 - \frac{3}{2}(1+\gamma)A_2 - \frac{3}{2}(1-\gamma)A_4\right),$$

$$\omega_2' = \frac{1}{\sqrt{5}}\left(1 + 4\varepsilon_1 - \frac{3}{2}\varepsilon_2 - \frac{3\gamma}{4}A_1 + \frac{3\gamma}{4}A_3 - \frac{3}{2}(1+\gamma)A_2 - \frac{3}{2}(1-\gamma)A_4\right).$$

Putting these values in second of Equations (8), we get

$$\gamma^2 = \frac{611}{675} + \frac{4864}{6075}\varepsilon_2 - \frac{1024}{675}\varepsilon_1 + \frac{128p}{6075} + \left(\frac{23835 + 503\sqrt{1833}}{10125}\right)A_4 + \frac{64\left(65 + \sqrt{1833}\right)}{10125}A_3$$
$$+ \left(\frac{23835 - 503\sqrt{1833}}{10125}\right)A_2 + \frac{64\left(65 - \sqrt{1833}\right)}{10125}A_1$$

Putting $\gamma = 1 - 2\mu$ and solving for μ, denoting this value by μ_{c1}, we get

$$\begin{aligned}\mu_{c1} &= 0.02429\cdots + \frac{64}{135\sqrt{1833}}(36\varepsilon_1 - 19\varepsilon_2) - 0.17907\cdots A_1 - 1.17746\cdots A_2 \\ &\quad - 0.03685\cdots A_3 - 0.05968\cdots A_4 - 0.005536495\cdots p\end{aligned} \tag{15}$$

Case (ii) $\omega_1' = 3\omega_2'$

Proceeding as in case (i), we get $\mu = \mu_{C2}$, where

$$\begin{aligned}\mu_{c2} &= 0.01352\cdots + \frac{4}{45\sqrt{213}}(36\varepsilon_1 - 19\varepsilon_2) - 0.09938\cdots A_1 - 2.15996\cdots A_2 \\ &\quad - 0.01938\cdots A_3 - 0.03093\cdots A_4 - 0.003045283\cdots p.\end{aligned} \tag{16}$$

Hence for the value μ_{C1} and μ_{C2} of mass ratios, condition (i) of Moser's theorem is not satisfied. The normalized Hamiltonian up to fourth order is

$$H = \omega_1' I_1 - \omega_2' I_2 + \frac{1}{2}\left(AI_1^2 + 2BI_1I_2 + CI_2^2\right) + \cdots$$

where A, B, C are given by Equation (12).

Now after simplification, the determinant D occurring in condition (ii) of Moser's theorem is given by:

$$D = \det\left(b_{ij}\right), (i, j = 1, 2, 3),\; b_{ij} = \left(\frac{\partial^2 H}{\partial I_i \partial I_j}\right)_{I_i = I_j = 0} \quad (i, j = 1, 2),$$

$$b_{i3} = b_{3j} = \left(\frac{\partial H}{\partial I_i}\right)_{I_i = I_j = 0} \quad (i, j = 1, 2), b_{33} = 0.$$

That is

$$D=\begin{vmatrix} A & B & \omega_1' \\ B & C & -\omega_2' \\ \omega_1' & -\omega_2' & 0 \end{vmatrix}=-\left(A\omega_2'^2+2B\omega_1'\omega_2'+C\omega_1'^2\right).$$

Substituting the values of A,B,C from Equation (12) and ω_1',ω_2' using the Equation (8) and Equation (9), we obtain

$$D=\frac{9\left(36-541u^2+644u^4\right)+RA_1+R'A_2+R_1A_3+R_1'A_4+R''p+m_1\varepsilon_1+m_2\varepsilon_2}{72\left(4u^2-1\right)\left(25u^2-4\right)}$$

R,R',R'', R_1 and R_1' are given in the **Appendix**. Values of m_1 and m_2 can be obtained from Bhatnagar and Hallan [2]. It is seen that the condition (ii) of Moser's theorem is satisfied *i.e.* $D\neq 0$ if in the interval $0<\mu<\mu_c$, mass ratio does not take the value

$$\mu_{c3}=\mu'\left(1+\alpha''\varepsilon_1+\beta''\varepsilon_2\right)+\alpha A_1+\beta A_2+\alpha' A_3+\beta' A_4+\ell p \tag{17}$$

where

$$\mu'=0.010936677\cdots,\ \alpha=-0.02942\cdots,\ \beta=772.85704\cdots,\ \alpha'=-0.10408\cdots,$$
$$\beta'=-16.46591\cdots,\ \beta''=-166.304\cdots,\ \ell=17.63703\cdots,\ \alpha''=250.922\cdots.$$

9. Conclusions

The abscissa of L_4 is independent of the perturbation in Coriolis (ε_1) and centrifugal forces (ε_2) and ordinate of L_4 is affected by perturbation in centrifugal force (Equation (2)).

With the increase of perturbation in Coriolis force, the range of linear stability increases whereas if we increase perturbation in centrifugal force, the range of stability decreases (Equation (7)).

Values of second order coefficients (A,B,C) in the polynomials (f_2 and g_2) occurring in the frequencies $\dot{\phi}_1$ and $\dot{\phi}_2$ are affected by the perturbations in Coriolis and centrifugal forces. It is observed that if perturbation in Coriolis and centrifugal forces increase then values of second order coefficients (A,B,C) increase (Equation (12)).

μ_{c1},μ_{c2} corresponds to the resonance cases $\omega_1'=2\omega_2'$ and $\omega_1'=3\omega_2'$. Their values are given in Equation (13).

Values of $\mu_{c1},\mu_{c2},\mu_{c3}$ (values of μ at which Moser's theorem is not applicable) increase if perturbation in Coriolis force increases and decrease if perturbation in centrifugal force increases (Equations (15)-(17)).

It may be observed that values of μ_{c1},μ_{c2} decrease if parameters of axis symmetric bodies (A_1,A_2,A_3,A_4) and radiation pressure (p) increase (Equations (15) and (16)).

Moser's second condition is violated for unperturbed problem (*i.e.* for $\varepsilon_1=\varepsilon_2=A_1=A_2=A_3=A_4=p=0$) when $\mu=\mu'=0.010936677\cdots$ (Equation (17)).

It may also be observed that value of μ_{c3} increases if A_2 of the bigger primary and p increase. If A_1,A_3,A_4 increase, value of μ_{c3} decreases (Equation (17)).

By taking both the primaries as axis symmetric bodies and the bigger mass as a source of radiation, the triangular point L_4 is stable in the range of linear stability except for the three mass ratios given in Equations (15)-(17) at which Moser's theorem does not apply.

The results of Jagadish Singh [6] can be deduced by taking $a_1=b_1$ and $a_2=b_2$.

All the results of Bhatnagar and Hallan [2] can be deduced by taking $A_1=A_2=A_3=A_4=p=0$.

References

[1] Deprit, A. and Deprit-Bartholome, A. (1967) Stability of the Triangular Lagrangian Points. *Astronomical Journal*, **72**, 173-179. http://dx.doi.org/10.1086/110213

[2] Bhatnagar, K.B. and Hallan, P.P. (1983) The Effect of Perturbation in Coriolis and Centrifugal Forces on the Nonlinear Stability of Equilibrium Points in the Restricted Problem of Three Bodies. *Celestial Mechanics*, **30**, 97-114. http://dx.doi.org/10.1007/BF01231105

[3] Aggarwal, R., Taqvi, Z.A. and Ahmad, I. (2006) Non-Linear Stability of L_4 in the Restricted Three Body Problem for

radiated Axes Symmetric Primaries with Resonances. *Bulletin of Astronomical Society of India*, **34**, 327-356.

[4] Jain, M. and Aggarwal, R. (2015) A Study of Non-Collinear Libration Points in Restricted Three Body Problem with Stokes Drag Effect when Smaller Primary Is an Oblate Spheroid. *Astrophysics and Space Science*, **358**, 51. http://dx.doi.org/10.1007/s10509-015-2457-6

[5] Kaur, B. and Aggarwal, R. (2013) Robe's restricted Problem of 2+2 Bodies when the Bigger Primary Is a Roche Ellipsoid. *Acta Astronautica*, **89**, 31-37. http://dx.doi.org/10.1016/j.actaastro.2013.03.022

[6] Singh, J. (2011) Combined Effects of Perturbations, Radiation and Oblateness on the Non-Linear Stability of Triangular Points in the R3BP. *Astrophysics and Space Science*, **332**, 331-339. http://dx.doi.org/10.1007/s10509-010-0546-0

[7] Szebehely, V. (1967) Theory of Orbits. Academic Press, New York, 242-264.

[8] Whittaker, E.T. (1965) A Treatise on the Analytical Dynamics of Particles and Rigid Bodies. Cambridge University Press, London, 427-430.

[9] Moser, J. (1953) Periodische Losungen des restringierten Dreikorperproblems, die sich erst nach vielen umlaufen schliessen. *Mathematische Annalen*, **126**, 325-335. http://dx.doi.org/10.1007/BF01343166

Appendix

$$p_{27}=\frac{1}{4}\left[1485+4025\gamma+\frac{2(285+115\gamma)}{\mu}\right],\ q_{27}=-\frac{2}{3}(-87+113\gamma),\quad q_{28}=\frac{2}{3}(63-\gamma),$$

$$p_{28}=-\frac{1}{4}\left[567+291\gamma+\frac{2(171-43\gamma)}{\mu}\right],\quad q_{29}=\frac{2}{3}(111-169\gamma),\quad q_{30}=\frac{2}{3}(21+5\gamma),$$

$$p_{29}=\frac{1}{4}\left[2325+5665\gamma+\frac{2(345+215\gamma)}{\mu}\right],\quad q_{31}=\frac{2}{3}(-29+91\gamma),$$

$$p_{30}=\frac{1}{4}\left[147+111\gamma+\frac{2(-69+37\gamma)}{\mu}\right],\ p'_{27}=\frac{1}{4}\left[1485-4025\gamma+\frac{2}{1-\mu}(285-115\gamma)\right],$$

$$p_{31}=-\frac{1}{4}\left[175+1235\gamma+\frac{2(55+185\gamma)}{\mu}\right],\ p'_{28}=\frac{1}{4}\left[567-291\gamma+\frac{2}{1-\mu}(171+43\gamma)\right],$$

$$p'_{29}=-\frac{1}{4}\left[147-111\gamma+\frac{2}{1-\mu}(-69-37\gamma)\right],\ p'_{30}=-\frac{1}{4}\left[147-111\gamma+\frac{2}{1-\mu}(-69-37\gamma)\right],\ p_8=-\frac{16}{3}p_5,$$

$$p'_{31}=-\frac{1}{4}\left[175-1235\gamma+\frac{2}{1-\mu}(55-185\gamma)\right],\ p_5=-\frac{3}{64}\left[15+47\gamma+\frac{2}{\mu}(1+7\gamma)\right],\ p_9=p_7,$$

$$p_6=\frac{3}{64}\left[-1+15\gamma+\frac{2}{\mu}(-3+11\gamma)\right],\ p_7=\frac{1}{4}\left[87+15\gamma+\frac{2}{\mu}(-3+11\gamma)\right],$$

$$p_{14}=\left[-\frac{75}{2}\gamma+\frac{1}{2\mu}(1-45\gamma)\right],\ p_{10}=-\frac{16}{3}p_6,\ p'_5=-\frac{3}{64}\left[15-47\gamma+\frac{2(1-7\gamma)}{1-\mu}\right],$$

$$p'_6=\frac{3}{64}\left[-1-15\gamma-\frac{2(3+11\gamma)}{1-\mu}\right],\ p'_7=-\frac{1}{4}\left[87-15\gamma-\frac{2(3+11\gamma)}{1-\mu}\right],$$

$$p'_8=-\frac{16}{3}p'_5,\ p'_9=p'_7,\ p'_{10}=-\frac{16}{3}p'_6,\ q_5=-\frac{1}{8}(1-3\gamma),$$

$$q_6=\frac{1}{8}(1-3\gamma),\ q_7=-\frac{2}{3}(3-\gamma),\ q_8=-\frac{16}{3}q_5,\ p_{11}=\left[7-\frac{15}{2}\gamma+\frac{1}{2\mu}(-37+25\gamma)\right],$$

$$p_{14}=\left[-\frac{75}{2}\gamma+\frac{1}{2\mu}(1-45\gamma)\right],\ p'_{14}=\left[\frac{75}{2}\gamma+\frac{1}{2(1-\mu)}(1+45\gamma)\right],\ q_{14}=-\frac{4}{3}(2-9\gamma),$$

$$q_9=q_7,\ q_{10}=-\frac{16}{3}q_6,\ p_{12}=\left[75+\frac{435}{2}\gamma+\frac{1}{2\mu}(41+75\gamma)\right],\ p_{13}=\left[76+\frac{55}{2}\gamma+\frac{1}{2\mu}(-41+45\gamma)\right],$$

$$p'_{11}=\left[-7-\frac{15}{2}\gamma+\frac{1}{2(1-\mu)}(37+25\gamma)\right],\ p'_{12}=\left[75-\frac{435}{2}\gamma+\frac{1}{2(1-\mu)}(41-75\gamma)\right],$$

$$p'_{13}=\left[-76+\frac{55}{2}\gamma+\frac{1}{2(1-\mu)}(41+45\gamma)\right],\ q_{11}=\frac{4}{3}(4+\gamma),\ q_{12}=-\frac{4}{3}(-8+21\gamma),\ q_{13}=\frac{4}{3}(2+3\gamma),$$

$$R=\frac{3}{2}\left(3917u^4-14357u^2+492\right)$$
$$-27\gamma\left[\frac{1593600u^{10}+21222096u^8-13052000u^6+5408175u^4-840076u^2+23616}{2\left(4u^2-1\right)\left(25u^2-4\right)\left(16u^2+117\right)}\right]$$

$$R' = \frac{81(1+\gamma)(7-40u^2)(36-541u^2+644u^4)}{(4u^2-1)(25u^2-4)}$$
$$+\frac{(1+\gamma)(4u^2-1)(25u^2-4)(2025-26081u^2+55552u^4-2480u^6)}{k_1^4k_2^4(1-5\omega_1^2)(1-5\omega_2^2)}$$
$$+(4u^2-1)(25u^2-4)\left[\frac{\omega_2^4(124\omega_1^4-696\omega_1^2+81)(\xi-\eta)}{k_1^4(1-5\omega_1^2)}+\frac{24u^2(64u^2+43)}{k_1^2k_2^2(1-5\omega_1^2)(1-5\omega_2^2)}\right.$$
$$\left.\times\left\{\zeta-\eta+\frac{15(1+\gamma)}{(1-5\omega_1^2)(1-5\omega_2^2)}\right\}+\frac{\omega_1^4(124\omega_2^4-696\omega_2^2+81)(\xi'-\eta')}{k_2^4(1-5\omega_2^2)}\right]$$

$$R'' = (4u^2-1)(25u^2-4)\left[\frac{\omega_2^4(124\omega_1^4-696\omega_1^2+81)(\sigma-\rho)}{k_1^4(1-5\omega_1^2)}+\frac{24u^2(64u^2+43)(\tau-\rho)}{k_1^2k_2^2(1-5\omega_1^2)(1-5\omega_2^2)}\right.$$
$$\left.+\frac{\omega_1^4(124\omega_2^4-696\omega_2^2+81)(\sigma'-\rho')}{k_2^4(1-5\omega_2^2)}\right]$$

R_1 and R_1' can be obtained from R and R' respectively by replacing γ by $-\gamma$, ξ by ξ_1, η by η_1, ζ by ζ_1, ξ' by ξ_1' and η' by η_1'.

$$J_{13} = \frac{l_1}{2\omega_1k_1}\left[1+\alpha_{13}\varepsilon_1+\alpha_{13}'\varepsilon_2+\left(-\frac{3\gamma}{4k_1^2}+\frac{33}{4l_1^2}\right)A_1+p_{21}A_2+\left(\frac{3\gamma}{4k_1^2}+\frac{33}{4l_1^2}\right)A_3+p_{21}'A_4+q_{21}p\right],$$

$$J_{14} = \frac{l_2}{2\omega_2k_2}\left[1+\alpha_{14}\varepsilon_1+\alpha_{14}'\varepsilon_2+\left(\frac{3\gamma}{4k_2^2}+\frac{33}{4l_2^2}\right)A_1+p_{22}A_2+\left(-\frac{3\gamma}{4k_2^2}+\frac{33}{4l_2^2}\right)A_3+p_{22}'A_4+q_{22}p\right],$$

$$J_{21} = -\frac{4\omega_1}{l_1k_1}\left[1+\alpha_{21}\varepsilon_1+\alpha_{21}'\varepsilon_2+\left(\frac{3}{4}-\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right)A_1+p_{23}A_2+\left(\frac{3}{4}+\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right)A_3+p_{23}'A_4+q_{23}p\right],$$

$$J_{22} = \frac{4\omega_2}{l_2k_2}\left[1+\alpha_{22}\varepsilon_1+\alpha_{22}'\varepsilon_2+\left(\frac{3}{4}+\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right)A_1+p_{24}A_2+\left(\frac{3}{4}-\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right)A_3+p_{24}'A_4+q_{24}p\right],$$

$$J_{23} = \frac{3\sqrt{3}\gamma}{2\omega_1l_1k_1}\left[1+\alpha_{23}\varepsilon_1+\alpha_{23}'\varepsilon_2+\left(\frac{6+13\gamma}{6\gamma}-\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right)A_1\right.$$
$$\left.+p_{25}A_2+\left(\frac{-6+13\gamma}{6\gamma}+\frac{3\gamma}{4k_1^2}-\frac{33}{4l_1^2}\right)A_3+p_{25}'A_4+q_{25}p\right]$$

$$J_{24} = \frac{3\sqrt{3}\gamma}{2\omega_2l_2k_2}\left[1+\alpha_{24}\varepsilon_1+\alpha_{24}'\varepsilon_2+\left(\frac{6+13\gamma}{6\gamma}+\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right)A_1\right.$$
$$\left.+p_{26}A_2+\left(\frac{-6+13\gamma}{6\gamma}-\frac{3\gamma}{4k_2^2}-\frac{33}{4l_2^2}\right)A_3+p_{26}'A_4+q_{26}p\right]$$

Values of $\alpha_{13},\alpha_{13}',\alpha_{14},\alpha_{14}',\alpha_{21},\alpha_{21}',\alpha_{22},\alpha_{22}',\alpha_{23},\alpha_{23}',\alpha_{24}$ and α_{24}' can be obtained from Hallan and Bhatnagar (983).

Values of $\xi,\varsigma,\eta,\xi',\eta',\xi_1,\varsigma_1,\eta_1,\xi_1',\eta_1',\sigma,\sigma',\rho,\rho',\tau$, $p_{21},p_{22},p_{23},p_{24},p_{25},p_{26}$, $p_{21}',p_{22}',p_{23}',p_{24}',p_{25}'$ and p_{26}' can be obtained from the author on request as the expressions are very long and contained in large number of pages.