International Journal of Astronomy and Astrophysics, 2024, 14, 203-219
https://www.scirp.org/journal/ijaa
ISSN Online: 2161-4725
ISSN Print: 2161-4717

New Probability Distributions in Astrophysics: XIII. Truncation for the Benini Distribution

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How to cite this paper: Zaninetti, L. (2024) New Probability Distributions in Astrophysics: XIII. Truncation for the Benini Distribution. *International Journal of Astronomy and Astrophysics*, **14**, 203-219.
https://doi.org/10.4236/ijaa.2024.143013

Received: June 19, 2024
Accepted: August 27, 2024
Published: August 30, 2024

Abstract

In order to introduce a right truncated version of the Benini distribution, we derive its probability density function, its distribution function, its average value, its *k*th moment about the origin, its median, how to randomly generate its values, and the maximum likelihood estimator for its three unknown parameters. The astrophysical application of the Benini distribution and its right truncated version is to the initial mass function for stars.

Keywords

Stars: Normal, Stars: Luminosity Function, Mass Function Stars: Statistics

1. Introduction

The Benini distribution with three parameters was introduced in 1905 [1] in order to generalize the Pareto distribution with two parameters introduced in 1896 [2]. The Benini distribution has not been well studied, and only recently, 2013, was the sequence of its moments analysed [3] when the number of parameters is one. Another study in 2021 derived, for the Benini distribution two parameters, the following quantities: its random generation, median, and how to determine its parameters through the maximum likelihood estimator [4]. The above two references outline that the Beninini distribution was poorly analysed in the fields of economics and actuarial science and absent from the fields of physics and astrophysics. The above studies allow posing some questions:

- Is possible to derive the main statistical properties of the truncated Benini distribution?
- Can the Benini distribution model the initial mass function for stars?
- Is using the truncated Benini distribution better than using the untruncated one?

In order to answer these questions, Section 2 treats the untruncated Benini distribution and Section 3 introduces its truncation. Section 4 applies the obtained results to the initial mass function for stars.

2. The Benini Distribution

Let X be a random variable taking values x in the interval $[\sigma,\infty]$. The *Benini* probability density function (PDF), after [1] [5], is

$$f(x;\alpha,\beta,\sigma)=\frac{\left(-2\beta\ln(\sigma)+2\beta\ln(x)+\alpha\right)\mathrm{e}^{-(\ln(x)-\ln(\sigma))(-\beta\ln(\sigma)+\beta\ln(x)+\alpha)}}{x},\tag{1}$$

where α, β and σ are ≥ 0. Its distribution function (DF) is

$$F(x;\alpha,\beta,\sigma)=1-\mathrm{e}^{-\alpha(\ln(x)-\ln(\sigma))-\beta(\ln(x)-\ln(\sigma))^2}.\tag{2}$$

The genesis of this variate can be found in a generalization of the Pareto distribution, derived in 1896 [2], which has a DF

$$F_P(x;\alpha,\sigma)=1-\left(\frac{\sigma}{x}\right)^{\alpha}.\tag{3}$$

The survival function, *SF*, is defined as

$$1-F(x),\tag{4}$$

where $F(x)$ is the distribution function and the natural logarithm for the Pareto's survival function is

$$\ln\left(1-F_P(x;\alpha,\sigma)\right)=\alpha\left(-\ln(x)+\ln(\sigma)\right),\tag{5}$$

which is a polynomial of first degree in $\ln(x)$. The natural logarithm for the Benini's survival function is

$$\ln\left(1-F(x;\alpha,\beta,\sigma)\right)=\alpha\left(-\ln(x)+\ln(\sigma)\right)-\beta\left(\ln(x)-\ln(\sigma)\right)^2,\tag{6}$$

which is a polynomial of second degree in $\ln(x)$. In other words, the degree for the natural logarithm of the survival function is increased by one in the Benini distribution. The Benini PDF is presented in **Figure 1** for different parameters.

The average value or mean of the Benini distribution, μ, is

$$\mu(\alpha,\beta,\sigma)=-\frac{\sigma\left(-\sqrt{\pi}\,\mathrm{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{(\alpha-1)^2}{4\beta}}-2\sqrt{\beta}\right)}{2\sqrt{\beta}}.\tag{7}$$

The variance is derived through the formula (1) and its value is

$$\begin{aligned}Var(\alpha,\beta,\sigma)=\frac{1}{4\beta^{\frac{3}{2}}}\Bigg(-\Bigg(&\pi\sqrt{\beta}\,\mathrm{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)^2\mathrm{e}^{\frac{(\alpha-1)^2}{2\beta}}\\&-4\sqrt{\pi}\,\mathrm{e}^{\frac{(\alpha-2)^2}{4\beta}}\mathrm{erfc}\left(\frac{\alpha-2}{2\sqrt{\beta}}\right)\beta+4\sqrt{\pi}\,\mathrm{e}^{\frac{(\alpha-1)^2}{4\beta}}\mathrm{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)\beta\Bigg)\Bigg)\sigma^2,\end{aligned}\tag{8}$$

where erfc is the complementary error function

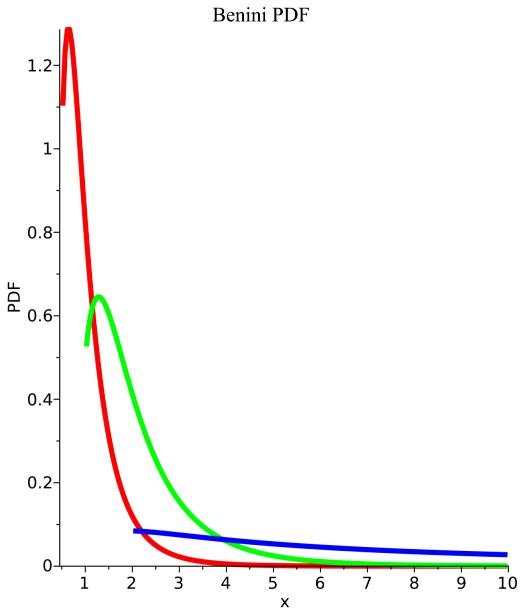

Figure 1. Benini PDF. The parameters are $\sigma = 0.5$, $\alpha = 0.5$, $\beta = 1$ for the red line, $\sigma = 1$, $\alpha = 0.5$, $\beta = 1$ for the green line and $\sigma = 1.35$, $\alpha = 0.1$, $\beta = 0.1$ for the blue line.

$$\operatorname{erfc}(z) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} \mathrm{e}^{-t^2} \mathrm{d}t = 1 - \operatorname{erf}(z), \tag{9}$$

and erf is the error function [6]. The standard deviation, *std*, is

$$std = \sqrt{Var}, \tag{10}$$

and the *k*th moment about the origin, μ'_k, is

$$\mu'_k(\alpha,\beta,\sigma) = -\frac{-\sigma^{\alpha} k \sqrt{\pi} \operatorname{erfc}\left(\frac{-k+\alpha}{2\sqrt{\beta}}\right) \mathrm{e}^{\frac{(k-\alpha)(4\beta\ln(\sigma)-\alpha+k)}{4\beta}} - 2\sigma^{k}\sqrt{\beta}}{2\sqrt{\beta}}. \tag{11}$$

The skewness can be derived through the implicit definition as in formula (2) and its explicit value is

$$\begin{aligned} \text{skewness} = {} & \frac{1}{std^3}\left(\frac{\sigma^3}{2\beta^{\frac{3}{2}}}\right)\Bigg(-3\pi\sqrt{\beta}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)\operatorname{erfc}\left(\frac{\alpha-2}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{2\alpha^2-6\alpha+5}{4\beta}} \\ & + \frac{\pi^{\frac{3}{2}}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)^3 \mathrm{e}^{\frac{3(\alpha-1)^2}{4\beta}}}{2} + 3\pi\sqrt{\beta}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)^2 \mathrm{e}^{\frac{(\alpha-1)^2}{2\beta}} \\ & - 3\Bigg(-\operatorname{erfc}\left(\frac{\alpha-3}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{(\alpha-3)^2}{4\beta}} + 2\operatorname{erfc}\left(\frac{\alpha-2}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{(\alpha-2)^2}{4\beta}} \\ & - \mathrm{e}^{\frac{(\alpha-1)^2}{4\beta}}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)\Bigg)\sqrt{\pi}\,\beta\Bigg). \end{aligned} \tag{12}$$

The kurtosis can be derived through the implicit definition as in formula (3) and its explicit value is

$$
\begin{aligned}
\mathrm{kurtosis} = {} & \frac{1}{std^4}\left(\frac{3\sigma^4}{16\beta^{\frac{5}{2}}}\right)\Bigg(8\beta\pi^{\frac{3}{2}}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)^2\operatorname{erfc}\left(\frac{\alpha-2}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{3\alpha^2-8\alpha+6}{4\beta}} \\
& +32\pi\beta^{\frac{3}{2}}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)\operatorname{erfc}\left(\frac{\alpha-2}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{2\alpha^2-6\alpha+5}{4\beta}} \\
& -16\pi\beta^{\frac{3}{2}}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)\operatorname{erfc}\left(\frac{\alpha-3}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{\alpha^2-4\alpha+5}{2\beta}} \\
& -8\beta\pi^{\frac{3}{2}}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)^3\mathrm{e}^{\frac{3(\alpha-1)^2}{4\beta}}-16\pi\beta^{\frac{3}{2}}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)^2\mathrm{e}^{\frac{(\alpha-1)^2}{2\beta}} \\
& +\frac{32\beta^2\sqrt{\pi}\operatorname{erfc}\left(\frac{\alpha-4}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{(\alpha-4)^2}{4\beta}}}{3}-32\beta^2\sqrt{\pi}\operatorname{erfc}\left(\frac{\alpha-3}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{(\alpha-3)^2}{4\beta}} \\
& +32\beta^2\sqrt{\pi}\operatorname{erfc}\left(\frac{\alpha-2}{2\sqrt{\beta}}\right)\mathrm{e}^{\frac{(\alpha-2)^2}{4\beta}} \\
& +\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)\left(-\frac{32\mathrm{e}^{\frac{(\alpha-1)^2}{4\beta}}\beta^2\sqrt{\pi}}{3}-\pi^2\mathrm{e}^{\frac{(\alpha-1)^2}{\beta}}\sqrt{\beta}\operatorname{erfc}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)^3\right)\Bigg).
\end{aligned}
\tag{13}
$$

A 3D display of the skewness is presented in Figure 2.

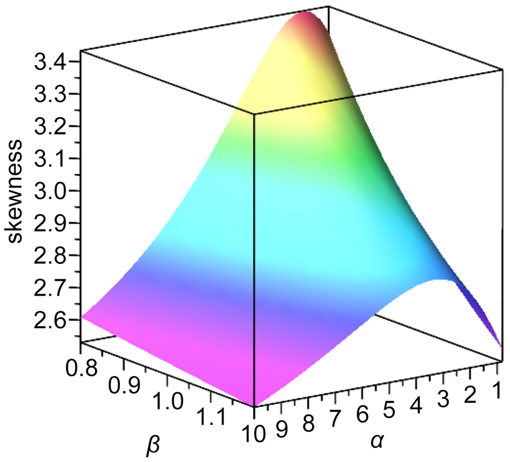

Figure 2. Benini skewness as function of α and β when $\sigma = 1$.

The random generation of the Benini variate X is given by

$$
X : \alpha, \beta, \sigma \approx \mathrm{e}^{\frac{2\beta\ln(\sigma)-\alpha+\sqrt{-4\ln(1-ran01)\beta+\alpha^2}}{2\beta}},
\tag{14}
$$

where R is the unit rectangular variate. The median, $q_{1/2}$, is at

$$q_{1/2}\left(\alpha,\beta,\sigma\right)=\mathrm{e}^{\frac{2\beta\ln(\sigma)-\alpha+\sqrt{4\ln(2)\beta+\alpha^2}}{2\beta}}, \tag{15}$$

and the mode is at

$$\mathrm{mode}\left(\alpha,\beta,\sigma\right)=\mathrm{e}^{\frac{\beta\ln(\sigma)-\frac{\alpha}{2}-\frac{1}{4}+\frac{\sqrt{8\beta+1}}{4}}{\beta}}. \tag{16}$$

The three parameters α,β and σ are obtained in the following way. Consider a sample $\mathcal{X}=x_1,x_2,\cdots,x_n$ and let $x_{(1)}\geq x_{(2)}\geq\cdots\geq x_{(n)}$ denote their order statistics, so that $x_{(1)}=\max\left(x_1,x_2,\cdots,x_n\right)$, $x_{(n)}=\min\left(x_1,x_2,\cdots,x_n\right)$. Then

$$\sigma=x_{(n)}. \tag{17}$$

The two remaining parameters α and β are found by solving the two following equations which arise from the MLE:

$$n\ln\left(\sigma\right)-\sum_{i=1}^{n}\frac{1-2\ln\left(x_i\right)^2\beta+\left(2\beta\ln\left(\sigma\right)-\alpha\right)\ln\left(x_i\right)}{2\beta\ln\left(\sigma\right)-2\beta\ln\left(x_i\right)-\alpha}=0 \tag{18}$$

$$\sum_{i=1}^{n}\frac{2\ln\left(x_i\right)^3\beta+\left(-6\beta\ln\left(\sigma\right)+\alpha\right)\ln\left(x_i\right)^2+\left(4\ln\left(\sigma\right)^2\beta-2\alpha\ln\left(\sigma\right)-2\right)\ln\left(x_i\right)+2\ln\left(\sigma\right)}{2\beta\ln\left(\sigma\right)-2\beta\ln\left(x_i\right)-\alpha}-n\ln\left(\sigma\right)^2=0. \tag{19}$$

3. The Right Truncated Benini Distribution

Let X be a random variable taking values in $\left[\sigma,x_u\right]$, where $x_u>\sigma$. The DF, $F_T\left(x\right)$, of the right truncated Bernini distribution is

$$F_T\left(x;\alpha,\beta,\sigma,x_u\right)=\frac{1-\mathrm{e}^{-(\ln(\sigma)-\ln(x))(\beta\ln(\sigma)-\beta\ln(x)-\alpha)}}{1-\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}\mathrm{e}^{-\beta\left(\ln(x_u)^2+\ln(\sigma)^2\right)}}, \tag{20}$$

and its PDF, $f_T\left(x\right)$, is

$$f_T\left(x;\alpha,\beta,\sigma,x_u\right)=\frac{\left(-2\beta\ln\left(\sigma\right)+2\beta\ln\left(x\right)+\alpha\right)\mathrm{e}^{-(\ln(x)-\ln(\sigma))(-\beta\ln(\sigma)+\beta\ln(x)+\alpha)}}{x\left(1-\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}\mathrm{e}^{-\beta\left(\ln(x_u)^2+\ln(\sigma)^2\right)}\right)}. \tag{21}$$

The survival function, SF_T is

$$SF_T\left(x;\alpha,\beta,\sigma,x_u\right)=1-F_T\left(x;\alpha,\beta,\sigma,x_u\right). \tag{22}$$

Its average value or mean, μ_T, is

$$\begin{aligned}&\mu_T\left(\alpha,\beta,\sigma,x_u\right)\\&=\frac{1}{2\sqrt{\beta}\left(-\sigma^{\alpha+2\beta\ln(x_u)}\mathrm{e}^{-\beta\left(\ln(x_u)^2+\ln(\sigma)^2\right)}+x_u^{\alpha}\right)}\\&\times\left(-x_u^{\alpha}\left(\sqrt{\pi}\sigma\mathrm{e}^{\frac{(\alpha-1)^2}{4\beta}}\operatorname{erf}\left(\frac{2\beta\ln\left(\sigma\right)-2\beta\ln\left(x_u\right)-\alpha+1}{2\sqrt{\beta}}\right)\right.\right.\\&\left.\left.+\sigma\sqrt{\pi}\mathrm{e}^{\frac{(\alpha-1)^2}{4\beta}}\operatorname{erf}\left(\frac{\alpha-1}{2\sqrt{\beta}}\right)+2x_u^{1-\alpha}\sqrt{\beta}\sigma^{\alpha+2\beta\ln(x_u)}\mathrm{e}^{-\beta\left(\ln(x_u)^2+\ln(\sigma)^2\right)}-2\sigma\sqrt{\beta}\right)\right).\end{aligned} \tag{23}$$

The increasing value of the right truncated mean as a function of the upper value in x, x_u, is shown in **Figure 3**.

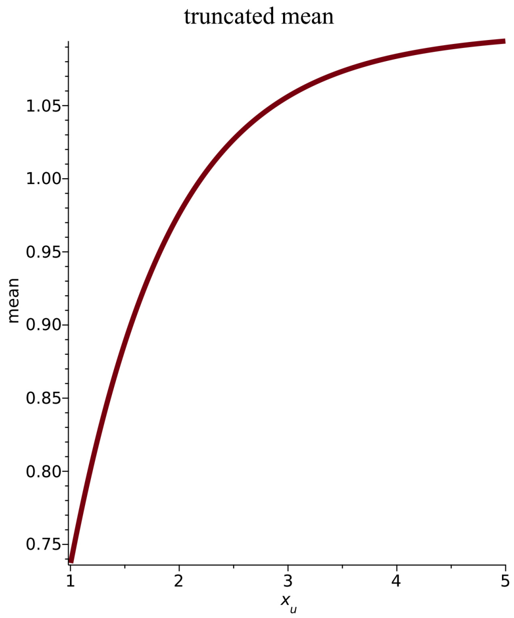

Figure 3. Mean of the right truncated Benini PDF as function of x_u when $\sigma = 0.5$, $\alpha = 0.5$, $\beta = 1$.

The *k*th moment about the origin, $\mu'_{k,T}$, is

$$\begin{aligned}&\mu'_{k,T}(\alpha,\beta,\sigma,x_u)\\&=\frac{1}{2\sqrt{\beta}\left(\sigma^{\alpha+2\beta\ln(x_u)}\mathrm{e}^{-\beta\left(\ln(\sigma)^2+\ln(x_u)^2\right)}-x_u^{\alpha}\right)}\\&\times\left(-x_u^{\alpha}\sqrt{\pi}\sigma^{\alpha}k\,\mathrm{e}^{\frac{(-\alpha+k)(4\beta\ln(\sigma)-\alpha+k)}{4\beta}}\operatorname{erf}\left(\frac{2\beta\ln(x_u)-2\beta\ln(\sigma)+\alpha-k}{2\sqrt{\beta}}\right)\right.\\&+x_u^{\alpha}\sqrt{\pi}\sigma^{\alpha}k\,\mathrm{e}^{\frac{(-\alpha+k)(4\beta\ln(\sigma)-\alpha+k)}{4\beta}}\operatorname{erf}\left(\frac{\alpha-k}{2\sqrt{\beta}}\right)\\&\left.+2\sqrt{\beta}\,x_u^{k}\sigma^{\alpha+2\beta\ln(x_u)}\mathrm{e}^{-\beta\left(\ln(\sigma)^2+\ln(x_u)^2\right)}-2\sqrt{\beta}\,x_u^{\alpha}\sigma^{k}\right).\end{aligned}\tag{24}$$

The above formula allows deriving the variance, skewness, and kurtosis, through the implicit formulas (A.1), (A.2) and (A.3) but they have complicated expressions which we do not present. The random generation of the right truncated Benini variate X is given by

$$X:\alpha,\beta,\sigma,x_u\approx\mathrm{e}^{\frac{2\beta\ln(\sigma)-\alpha+\sqrt{-4\ln\left(x_u^{-\alpha}\sigma^{\alpha+2\beta\ln(x_u)}\mathrm{e}^{-\beta\left(\ln(x_u)^2+\ln(\sigma)^2\right)}R-R+1\right)\beta+\alpha^2}}{2\beta}},\tag{25}$$

where R is the unit rectangular variate. The median, $q_{1/2}$, is at

$$q_{1/2}\left(\alpha,\beta,\sigma,x_u\right)=\mathrm{e}^{\frac{2\beta\ln(\sigma)-\alpha+\sqrt{4\ln(2)\beta-4\ln\left(1+\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}\mathrm{e}^{-\beta\left(\ln(x_u)^2+\ln(\sigma)^2\right)}\right)\beta+\alpha^2}}{2\beta}}, \quad (26)$$

and the mode is at the same position as for the standard Benini distribution, see Equation (16). We now outline how to determine the four parameters. The parameter σ is

$$\sigma = x_{(n)}, \quad (27)$$

and the parameter x_u is

$$x_u = x_{(1)}. \quad (28)$$

The two remaining parameters α and β are found by solving numerically the two following equations which arise from the MLE:

$$n\ln\left(\sigma\right)+\sum_{i=1}^{n}\left(-\ln\left(x_i\right)+\frac{\left(-\frac{1}{A}-\frac{\left(2\beta\ln\left(\sigma\right)-2\beta\ln\left(x_i\right)-\alpha\right)\left(B-C\right)}{A^2}\right)A}{2\beta\ln\left(\sigma\right)-2\beta\ln\left(x_i\right)-\alpha}\right)=0, \quad (29)$$

$$-n\ln\left(\sigma\right)^2+\sum_{i=1}^{n}\left(2\ln\left(\sigma\right)\ln\left(x_i\right)-\ln\left(x_i\right)^2+\frac{\left(\frac{2\ln\left(\sigma\right)-2\ln\left(x_i\right)}{A}-\frac{\left(2\beta\ln\left(\sigma\right)-2\beta\ln\left(x_i\right)-\alpha\right)\left(E+F\right)}{A^2}\right)A}{2\beta\ln\left(\sigma\right)-2\beta\ln\left(x_i\right)-\alpha}\right)=0, \quad (30)$$

where

$$A=-1+\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}\mathrm{e}^{-\beta\left(\ln(\sigma)^2+\ln(x_u)^2\right)}, \quad (31)$$

$$B=\sigma^{\alpha+2\beta\ln(x_u)}\ln\left(\sigma\right)x_u^{-\alpha}\mathrm{e}^{-\beta\left(\ln(\sigma)^2+\ln(x_u)^2\right)}, \quad (32)$$

$$C=\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}\ln\left(x_u\right)\mathrm{e}^{-\beta\left(\ln(\sigma)^2+\ln(x_u)^2\right)}, \quad (33)$$

$$E=2\sigma^{\alpha+2\beta\ln(x_u)}\ln\left(x_u\right)\ln\left(\sigma\right)x_u^{-\alpha}\mathrm{e}^{-\beta\left(\ln(\sigma)^2+\ln(x_u)^2\right)}, \quad (34)$$

$$F=\sigma^{\alpha+2\beta\ln(x_u)}x_u^{-\alpha}\left(-\ln\left(\sigma\right)^2-\ln\left(x_u\right)^2\right)\mathrm{e}^{-\beta\left(\ln(\sigma)^2+\ln(x_u)^2\right)}. \quad (35)$$

4. Application to the Stars

This section reviews the lognormal distribution, Salpeter's exponent, the Pareto distribution, the truncated Pareto distribution, the adopted statistics, applies the obtained results to the initial mass function for stars (IMF) and explores the survival function of massive stars.

4.1. Lognormal Distribution

Let *X* be a random variable taking values *x* in the interval $\left[0,\infty\right]$; the *first lognormal* PDF, following [7] or formula (14.2) in [8], is

$$f_{LN}\left(x;m,\sigma\right)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\frac{-\left[\ln\left(x/m\right)\right]^2}{2\sigma^2}, \tag{36}$$

its DF is

$$F_{LN}\left(x;m,\sigma\right)=\frac{1}{2}-\frac{\operatorname{erf}\left(\frac{\sqrt{2}\left(\ln\left(m\right)-\ln\left(x\right)\right)}{2\sigma}\right)}{2}, \tag{37}$$

and its SF, S_{LN} is

$$S_{LN}\left(x;m,\sigma\right)=\frac{1}{2}+\frac{\operatorname{erf}\left(\frac{\sqrt{2}\left(\ln\left(m\right)-\ln\left(x\right)\right)}{2\sigma}\right)}{2}. \tag{38}$$

The *second definition* has PDF

$$f_{LN}\left(x;\mu_{LN},\sigma\right)=\frac{1}{x\sigma\sqrt{2\pi}}\exp\frac{-\left(\ln x-\mu_{LN}\right)^2}{2\sigma^2}, \tag{39}$$

where $m=\exp\mu_{LN}$ and $\mu_{LN}=\ln m$. The DF of the second definition is

$$F_{LN}\left(x;\mu_{LN},\sigma\right)=\frac{1}{2}+\frac{\operatorname{erf}\left(\frac{\sqrt{2}\left(\ln\left(x\right)-\mu_{LN}\right)}{2\sigma}\right)}{2}, \tag{40}$$

and its SF is

$$S_{LN}\left(x;\mu_{LN},\sigma\right)=\frac{1}{2}-\frac{\operatorname{erf}\left(\frac{\sqrt{2}\left(\ln\left(x\right)-\mu_{LN}\right)}{2\sigma}\right)}{2}. \tag{41}$$

4.2. Salpeter's Exponent

The distribution in mass of the stars has been fitted with a power law starting with [9]. Salpeter suggested $\xi\left(m\right)\propto m^{-\alpha}$ where $\xi\left(m\right)$ denotes the probability of having a mass between *m* and $m+dm$. He found $\alpha=2.35$ in the range $10M_{\odot}>M\geq 1M_{\odot}$; this value has changed little with time and a recent evaluation quotes 2.35, for stars with mass greater than few $M_{\odot}$, see [10].

4.3. Pareto Distribution

Let *X* be a random variable taking values *x* in the interval $\left[a,\infty\right]$, $a>0$. The *Pareto* PDF is defined by

$$f\left(x;a,c\right)=ca^{c}x^{-(c+1)}, \tag{42}$$

with $c>0$, see formula (20.3) in [8]. The traditional Salpeter slope is therefore $-\left(c+1\right)$. Its DF is

$$F\left(x;a,c\right)=-a^{c}x^{-c}+1, \tag{43}$$

and its SF, *S*, is

$$S\left(x;a,c\right)=a^{c}x^{-c}. \tag{44}$$

4.4. Truncated Pareto distribution

An upper truncated Pareto random variable is defined in the interval $[a,b]$, and its corresponding PDF, following [11]-[14], is

$$f_T(x;a,b,c)=\frac{ca^c x^{-(c+1)}}{1-\left(\frac{a}{b}\right)^c}. \tag{45}$$

Its DF is

$$F_T(x;a,b,c)=\frac{\left(\left(\frac{x}{a}\right)^{-c}-1\right)b^c}{a^c-b^c}, \tag{46}$$

and its SF is

$$S_T(x;a,b,c)=1-\frac{\left(\frac{1}{\left(\frac{x}{a}\right)^c}-1\right)b^c}{a^c-b^c}. \tag{47}$$

4.5. Statistics

The merit function χ^2 is given by

$$\chi^2=\sum_{i=1}^{n}\frac{(T_i-O_i)^2}{T_i}, \tag{48}$$

where n is the number of bins, T_i is the theoretical value, and O_i is the experimental value as given by the frequencies. The theoretical frequency distribution is given by

$$T_i=N\Delta x_i p(x), \tag{49}$$

where N is the number of elements of the sample, Δx_i is the magnitude of the size interval, and $p(x)$ is the PDF under examination. A reduced merit function χ^2_{red} is given by

$$\chi^2_{red}=\chi^2/NF, \tag{50}$$

where $NF=n-k$ is the number of degrees of freedom, n is the number of bins, and k is the number of parameters. The goodness of the fit can be expressed by the probability Q, see equation 15.2.12 in [15], which involves the number of degrees of freedom and χ^2. According to [15] p. 658, the fit “may be acceptable” if $Q>0.001$. The Akaike information criterion (AIC), see [16], is defined by

$$\text{AIC}=2k-2\ln(L), \tag{51}$$

where L is the likelihood function and k the number of free parameters in the model. We assume a Gaussian distribution for the errors. The likelihood function can then be derived from the χ^2 statistic: $L\propto\exp\left(-\frac{\chi^2}{2}\right)$ where χ^2 is given by Equation (48), see [17] and [18]. Now the AIC becomes

$$\mathrm{AIC} = 2k + \chi^2 . \tag{52}$$

The Kolmogorov-Smirnov test (K-S), see [19]-[21], does not require the data to be binned. The K-S test, as implemented by the FORTRAN subroutine KSONE in [15], finds the maximum distance, D, between the theoretical and the astronomical DFs, as well as the significance level P_{KS}; see formulas 14.3.5 and 14.3.9 in [15]. If $P_{KS} \geq 0.1$, then the goodness of the fit is believable.

4.6. The IMF for Stars

The first test is performed on NGC 2362, where the 271 stars have a range of $1.47 M_{\odot} \geq M \geq 0.11 M_{\odot}$, see [22] and CDS catalog J/MNRAS/384/675/table1. According to [23], the distance of NGC 2362 is 1480 pc. The second test is performed on the low-mass IMF in the young cluster NGC 6611, see [24] and CDS catalog J/MNRAS/392/1034. This massive cluster has an age of 2 - 3 Myr and contains masses from $1.5 M_{\odot} \geq M \geq 0.02 M_{\odot}$. Therefore, the brown dwarfs (BD) region, $\approx 0.2 \mathcal{M}_{\odot}$, is covered. The third test is performed on the γ Velorum cluster, where the 237 stars have a range of $1.31 M_{\odot} \geq M \geq 0.15 M_{\odot}$, see [25] and CDS catalog J/A+A/589/A70/table5. The fourth test is performed on the young cluster Berkeley 59, where the 420 stars have a range of $2.24 M_{\odot} \geq M \geq 0.15 M_{\odot}$, see [26] and CDS catalog J/AJ/155/44/table3. The fifth test is performed on the Hyades, where the 602 stars have a range of $2.20 M_{\odot} \geq M \geq 0.11 M_{\odot}$, see [27] and CDS catalog J/AJ/165/108/table1.

The results are presented in **Table 1** for the Benini distribution and **Table 2** for the right truncated Benini distribution. In **Table 1** and **Table 2**, the last column shows whether the results of the K-S test are better when compared to the lognormal distribution (Y) or worse (N).

As an example, the empirical DF visualized through histograms and the theoretical Benini DF for the γ Velorum cluster are presented in **Figure 4**.

Another example is given by the PDF of the truncated Benini distribution, see **Figure 5**.

Table 1. Numerical values of χ^2_{red}, AIC, probability Q, D, the maximum distance between theoretical and observed DF, and P_{KS}, the significance level, in the K-S test of the Beninini distribution, see Equation (1), for different astrophysical environments. The last column (F) indicates a P_{KS} higher (Y) or lower (N) than that for the lognormal distribution. The number of linear bins, n, is 10.

Cluster	parameters	AIC	χ^2_{red}	Q	D	P_{KS}	F
NGC 2362	$\alpha = 6.89 \times 10^{-3}$, $\beta = 0.365$, $\sigma = 0.12$	134	18.38	1.16×10^{-24}	0.196	1.11×10^{-9}	N
NGC 6611	$\alpha = 7.45 \times 10^{-3}$, $\beta = 0.117$, $\sigma = 1.89 \times 10^{-2}$	92.61	12.37	6.11×10^{-16}	0.198	1.19×10^{-7}	N
γ Velorum	$\alpha = 0.494$, $\beta = 0.752$, $\sigma = 0.157$	20.64	2.09	4×10^{-2}	0.0372	0.89	Y
Berkeley 59	$\alpha = 0.02$, $\beta = 0.913$, $\sigma = 0.159$	27.99	3.14	2.54×10^{-3}	0.07	0.038	Y
Hyades	$\alpha = 0.094$, $\beta = 0.435$, $\sigma = 0.114$	19.89	1.98	5.3×10^{-2}	0.043	0.2	Y

Table 2. Numerical values of χ^2_{red}, AIC, probability Q, D, the maximum distance between theoretical and observed DF, and P_{KS}, the significance level, in the K-S test of the right truncated Beninini distribution, see Equation (21), for different astrophysical environments. The last column (F) indicates a P_{KS} higher (Y) or lower (N) than that for the lognormal distribution. The number of linear bins, n, is 10.

Cluster	parameters	AIC	χ^2_{red}	Q	D	P_{KS}	F
NGC 2362	$\alpha = 6.89\times10^{-3}$, $\beta = 0.365$, $\sigma = 0.12$	122.88	19.14	1.92×10^{-22}	0.252	1×10^{-15}	N
NGC 6611	$\alpha = 7.45\times10^{-3}$, $\beta = 0.117$, $\sigma = 1.89\times10^{-2}$	83.177	12.52	3.52×10^{-14}	0.261	6.09×10^{-13}	N
γ Velorum	$\alpha = 0.494$, $\beta = 0.751$, $\sigma = 0.157$	22.48	2.41	2.46×10^{-2}	0.042	0.779	Y
Berkeley 59	$\alpha = 0.02$, $\beta = 0.913$, $\sigma = 0.159$	30.07	3.67	1.17×10^{-3}	0.069	0.034	Y
Hyades	$\alpha = 0.094$, $\beta = 0.435$, $\sigma = 0.114$	22.47	2.41	2.47×10^{-2}	0.056	0.387	Y

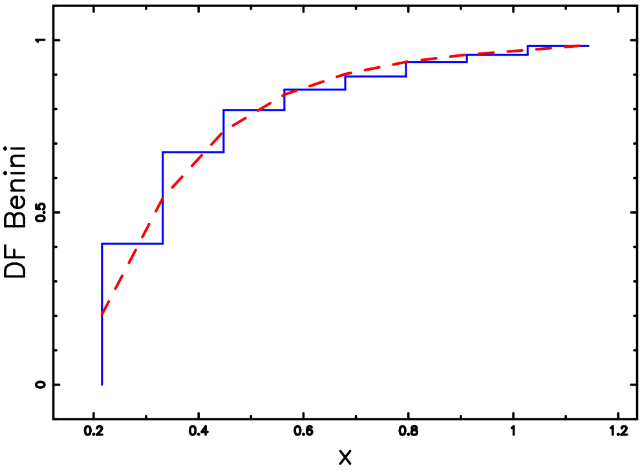

Figure 4. Empirical DF of the mass distribution for γ Velorum cluster (blue histogram) with a superposition of the Benini DF (red dashed line). Theoretical parameters as in **Table 1**.

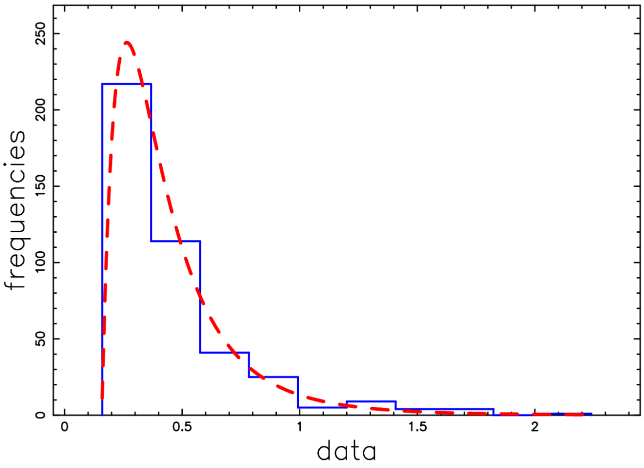

Figure 5. Empirical PDF of the mass distribution for Berkeley 59 (blue histogram) with a superposition of the truncated Benini PDF (red dashed line). Theoretical parameters as in **Table 2**.

4.7. Massive Stars

We analyse the more massive stars, $M \geq 0.5$ in the framework of the survival function (SF). We therefore analyse four different SFs:

1) the power law SF, see Equation (44),

2) the truncated power law SF, see Equation (47),

3) the lognormal SF, see Equations (38) or (41),

4) the right truncated Benini SF, see Equation (22).

The behaviour of the large masses, $M \geq 0.5 M_{\odot}$, for Hyades is presented in **Figure 6** and that for Hyades in **Figure 7**.

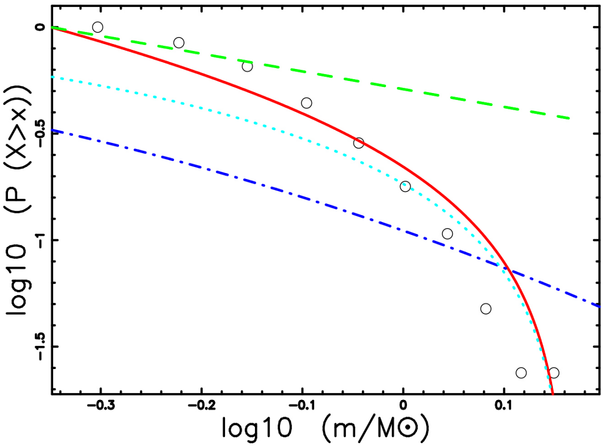

Figure 6. Survival function of NGC 6611 cluster data as $\log_{10} - \log_{10}$ plot when $M \geq 0.5 M_{\odot}$: data (empty circles), survival function of the truncated Pareto pdf (red full line) ($a = 0.43$, $b = 1.46$, $c = 1.3$) and survival function of the Pareto pdf (green dashed line) ($c = 1.3$, Salpeter slope −2.3). The lognormal (blue dot-dash-dot-dash line) and the SF of the truncated Benini distribution with parameters as in **Table 2**.

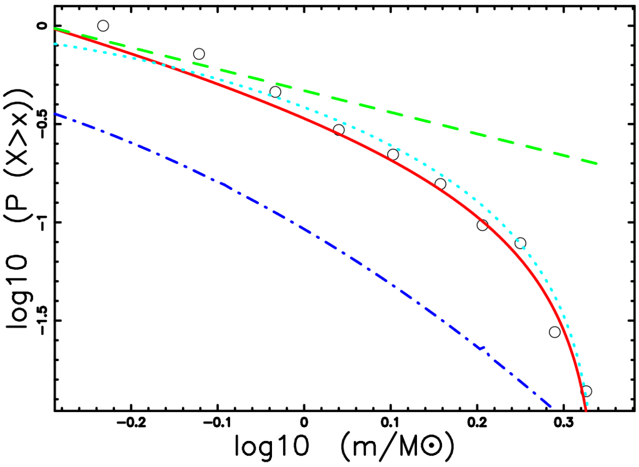

Figure 7. Survival function of Hyades data as $\log_{10} - \log_{10}$ plot when $M \geq 0.5 M_{\odot}$: data (empty circles), survival function of the truncated Pareto pdf (red full line) ($a = 0.5$, $b = 2.02$, $c = 1.09$) and survival function of the Pareto pdf (green dashed line) ($c = 1.09$, Salpeter slope −2.09). The lognormal (blue dot-dash-dot-dash line) and the SF of the truncated Benini distribution with parameters as in **Table 2**.

5. Conclusions

The truncated distribution

We derived the PDF, the DF, the average value, the kth moment about the origin, the median, a random number generator, and the MLE for the Benini distribution truncated on the right.

Application to the IMF

The application of the Benini distribution to the IMF for stars gives better results than the lognormal distribution for three out of five samples, see Table 1. The truncated Benini distribution does not improve the results over those of the regular Benini distribution for the five samples here considered, see Table 1 and Table 2.

The results for the mass distribution of the γ Velorum cluster compared with other distributions are shown in Table 3, in which the truncated Benini distribution occupies the last position.

Table 3. Numerical values of D, the maximum distance between theoretical and observed DF, and P_{KS}, the significance level, in the K–S test for different distributions in the case of γ Velorum cluster.

Distribution	Reference	D	P_{KS}
Benini	here	0.0372	0.89
Benini rigth truncated	here	0.042	0.779
truncated Gompertz	[28]	0.173	9.27×10^{-7}
truncated Topp-Leone	[29]	6.09×10^{-2}	0.25
Frècet	[30]	0.125	3.13×10^{-4}
truncated Frècet	[30]	0.077	0.07
truncated Weibull	[31]	0.046	0.576
truncated Sujatha	[32]	0.0485	0.534
truncated Lindley	[33]	0.11	0.48
generalized gamma	[34]	0.11	1.24×10^{-3}
truncated generalized gamma	[34]	0.062	0.24
lognormal	[35]	0.0729	0.11
truncated lognormal	[35]	0.047	0.55
gamma	[36]	0.059	0.28
truncated gamma	[36]	0.0754	0.08
beta	[37]	0.059	0.28

The most massive stars, see the SF reported in **Figure 6** and **Figure 7**, are better modeled by the truncated distributions, right truncated Benini and truncated Pareto, when compared to the regular distributions, lognormal and Pareto.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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Appendix

Implicit Formulas

The implicit formulae for the variance, skewness and kurtosis are

$$Var = -\left(\mu_1'\right)^2 + \mu_2', \tag{A.1}$$

$$\text{skewness} = \frac{2\left(\mu_1'\right)^3 - 3\mu_1'\mu_2' + \mu_3'}{\left(-\left(\mu_1'\right)^2 + \mu_2'\right)^{\frac{3}{2}}}, \tag{A.2}$$

$$\text{kurtosis} = \frac{-3\left(\mu_1'\right)^4 + 6\left(\mu_1'\right)^2 \mu_2' - 4\mu_1'\mu_3' + \mu_4'}{\left(-\left(\mu_1'\right)^2 + \mu_2'\right)^2}, \tag{A.3}$$

where μ_k' is the *k*th moment about the origin.