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A Mixed Finite Element Method for Vibration Problems of Non-Homogeneous Damped Beams

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Abstract

Beam is one of the common structures in engineering, with the development of technology, homogeneous beams no longer meet the needs of engineering structural design, for this reason, people have researched the non-homogeneous beams. In this paper, we study the mixed finite element method for the vibration problem of non-homogeneous damped beams. The fourth-order differential equations are transformed into a system of low-order partial differential equations by introducing intermediate variables, constructing a semi-discrete extended mixed finite element format, proving the existence and uniqueness of the solution of the format, and utilizing the elliptic projection operator for the error estimation. The time derivative term is discretized by the central difference, and the fully discrete mixed element format is given to prove the stability and convergence of the format. The feasibility and effectiveness of the mixed method are verified by numerical examples, and the effects of different damping coefficients μ on beam vibration are investigated.

Keywords

Non-Homogeneous, Damped Beams, Mixed Finite Element Methods, Error Estimation

1. Introduction

Consider the following non-homogeneous damped beam vibration problem:

$$\begin{cases} \rho_0 S u_{tt} + \mu u_t + \left(E(x) I u_{xx}\right)_{xx} = f(x,t), & (x,t) \in \Omega \times (0,T], \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(0,t) = u(L,t) = 0,\ u_{xx}(0,t) = u_{xx}(L,t) = 0, & t \in [0,T]. \end{cases} \tag{1}$$

where $\Omega = [0, L]$. $E(x), f(x,t), u_0(x)$ and $u_1(x)$ are sufficiently smooth known functions. ρ_0, S, μ and I are positive constant coefficients. There exist positive constants e_0, e_1, satisfying $e_0 \le E(x) \le e_1$.

Beam is one of the common structures in engineering, with the development of technology, homogeneous beams can not meet the needs of actual projects. Non-homogeneous beams are the organic combination of various materials, optimizing the properties of beams to meet the needs of various projects. The vibration of beams can lead to changes in their physical properties, which can affect the safety of people's lives and properties. Therefore, it is important to study the problem of beam vibration. Gupta numerically computed the vibration of conical beams and investigated the effect of taper on convergence and solution accuracy [1]. The problem of non-homogeneous damped beam vibration with different boundary conditions to obtain its analytical solution [2]-[5]. Awrejcewicz *et al.* studied the vibration of flexible beams under harmonic loading using finite difference method and finite element method [6]. Alotta *et al.* used the finite element method to analyze the fractional order Timoshenko beam vibration equations [7]. Dönmez Demir *et al.* develop general models of beams and rods with fractional order derivatives [8]. Wang *et al.* proposed a mixed finite volume element method for beam vibration equations with structural damping [9]. Zhang *et al.* used a mixed element method to numerically simulate the damped plate vibration problem and gave an optimal order error estimate [10]. Yuan *et al.* used the H^1-Galerkin mixed finite element method to solve the vibration problem of a damped beam simply supported at both ends [11]. Meng *et al.* investigated a mixed virtual element method to solve the vibration problem of a clamped Kirchhoff plate [12].

The mixed finite element method is a powerful tool for solving differential equations and provides a flexible and effective numerical way to solve complex structural problems. In the early 1970s, Babuška and Fortin *et al.* established the general theory of the mixed finite element method [13] [14]. Makridakis discussed the application of the mixed finite element method to linear elastic dynamics problems [15]. Burger *et al.* studied the mixed finite element method for nonlinear diffusion equations [16]. Lamichhane studied the mixed finite element method based on the dual harmonic equations for biorthogonal systems [17]. Liu *et al.* proposed a mixed finite element method for nonlinear time-fractional order stochastic fourth-order reaction-diffusion equations [18]. Meng *et al.* investigated the optimal order convergence of the lowest-order mixed finite element method for the biharmonic sum eigenvalue problem [19]. Huang *et al.* performed local H^1 norm error analysis [20] and alpha-robust error analysis [21] for the mixed finite element method for the time-fractional order biharmonic sum equation. Cowsat *et al.* obtained priori estimates for the second-order hyperbolic equations by the mixed finite element method [22]. Li applied the mixed finite element method to solve fourth-order elliptic and parabolic problems on quasi-uniform rectangular networks [23]. He *et al.* investigated a class of fourth-order fluctuation equations using a mixed explicit and implicit finite element method [24]. The mixed finite element method reduces fourth-order differential equations to lower-order systems through multi-variable coupling, circumventing the complex discretization of high-order derivatives in traditional approaches while effectively capturing the dynamic behavior of non-homogeneous material damping. No study using the mixed finite element method for the vibration problem of inhomogeneously damped beams has been found yet, therefore, in this paper, the mixed finite element method is used for the numerical simulation of problem (1).

The paper is organized as follows: in Section 2, intermediate variables are introduced to establish a mixed meta-weak form for problem (1). The semi-discrete mixed element format is constructed, the uniqueness and convergence of the solution are proved, and the error estimation is performed using the elliptic projection operator. In Section 3, the time term is discretized using central differences to construct the fully discrete mixed meta-format, and a proof of stability of the format and an error analysis are given. In Section 4, the non-homogeneous damped beam vibration problem is solved numerically to verify the feasibility and validity of the mixed finite element method, and to investigate the effect of different damping coefficients μ on the beam vibration.

2. Semi-Discrete Mixed Finite Element Scheme

In this section, for the problem (1), the fourth-order equations are transformed into a system of lower-order partial differential equations by introducing intermediate variables. The weak form is obtained by using Green's formula to establish a semi-discrete mixed element format. The existence of the solution is proved to be unique, and the elliptic projection operator is used for error estimation.

Let $v=-E(x)Iu_{xx}, b(x)=\dfrac{1}{E(x)I}$, then (1) has the following equivalent form:

$$\begin{cases} b(x)v+u_{xx}=0, & (x,t)\in\Omega\times(0,T], \\ \rho_0 S u_{tt}+\mu u_t-v_{xx}=f, & (x,t)\in\Omega\times(0,T], \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega, \\ u(0,t)=u(L,t)=0,\ v(0,t)=v(L,t)=0, & t\in[0,T]. \end{cases} \tag{2}$$

where $b(x)$ satisfies $b_0\le b(x)\le b_1$, b_0,b_1 are positive constants.

Using Green's formula, we obtain the weak form of (2), which is to find $\{u,v\}:[0,T]\to H_0^1(\Omega)\times H_0^1(\Omega)$, such that

$$\begin{cases} \text{(a)}\ (b(x)v,\psi)-(u_x,\psi_x)=0, & \forall\psi\in H_0^1(\Omega), \\ \text{(b)}\ \rho_0 S(u_{tt},\varphi)+\mu(u_t,\varphi)+(v_x,\varphi_x)=(f,\varphi), & \forall\varphi\in H_0^1(\Omega), \\ u(x,0)=u_0(x),\ u_t(x,0)=u_1(x), & x\in\Omega. \end{cases} \tag{3}$$

Define the finite element space: Let $0=x_0<x_1<\cdots<x_{N-1}<x_N=L$ be a partition of the interval $\Omega=(0,L)$ with step size $h=\dfrac{L}{N}$, where $x_n=nh$, for $n=0,1,2,\cdots,N$. Let Γ_h be the group of dissecting units and E be the dissecting unit. Let

$$S_h^0 = \left\{\psi_h \in H_0^1(\Omega); \psi_h\big|_E \in P_k(E), \forall E \in \Gamma_h\right\},$$

where $P_k(E)$ is the entire set of polynomials on the cell E whose number does not exceed $k \ge 1$.

For error estimation, the elliptic projection operator R_h [25]: $H_0^1(\Omega) \to S_h^0$, is introduced, which satisfies

$$\left((u - R_h u)_x, \psi_{hx}\right) = 0,\ \forall \psi_h \in S_h^0. \tag{4}$$

The approximation properties satisfied by the projection operator are as follows.

Lemma 1. $\forall u \in S_h^0 \cap H^{k+1}(\Omega)$,

$$\|u - R_h u\|_0 + h\|u - R_h u\|_1 \le Ch^{k+1}\|u\|_{k+1}. \tag{5}$$

We obtain the semi-discrete mixed element scheme of (3), which is to find $\{u_h, v_h\}:[0,T] \to S_h^0 \times S_h^0$, such that

$$\begin{cases} (\mathrm{a})\ \left(b(x)v_h, \psi_h\right) - \left(u_{hx}, \psi_{hx}\right) = 0, & \forall \psi_h \in S_h^0, \\ (\mathrm{b})\ \rho_0 S\left(u_{htt}, \varphi_h\right) + \mu\left(u_{ht}, \varphi_h\right) + \left(v_{hx}, \varphi_{hx}\right) = \left(f, \varphi_h\right), & \forall \varphi_h \in S_h^0, \\ u_h(x,0) = R_h u_0,\ u_{ht}(x,0) = R_h u_1,\ v_h(x,0) = R_h\left(-E(x) I u_{0xx}\right), & x \in \Omega. \end{cases} \tag{6}$$

Theorem 1. The solution of (6) exists and is unique.

Proof. Let $\{\phi_i\}_{i=1}^M$ be a set of bases of S_h^0, then $u_h = \sum_{j=1}^M u_j \phi_j$, $v_h = \sum_{j=1}^M v_j \phi_j$. From (6), we have

$$A\boldsymbol{V}(t) - B\boldsymbol{U}(t) = 0, \tag{7}$$

$$\rho_0 SC \frac{\mathrm{d}^2 \boldsymbol{U}(t)}{\mathrm{d}t^2} + \mu C \frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t} + B\boldsymbol{V}(t) = \boldsymbol{F}, \tag{8}$$

where

$$A = \left(b(x)\phi_j, \phi_i\right)_{M\times M}, B = \left(\phi_{jx}, \phi_{ix}\right)_{M\times M}, C = \left(\phi_j, \phi_i\right)_{M\times M}, \boldsymbol{F} = \left(f, \phi_i\right)_{M\times 1},$$

$$\boldsymbol{U}(t) = \left(u_1(t), u_2(t), \cdots, u_M(t)\right)', \boldsymbol{V}(t) = \left(v_1(t), v_2(t), \cdots, v_M(t)\right)'.$$

Since A is a symmetric positive definite matrix, it follows from (7) that

$$\boldsymbol{V}(t) = A^{-1} B \boldsymbol{U}(t), \tag{9}$$

Substituting (9) into (8) gives

$$\rho_0 SC \frac{\mathrm{d}^2 \boldsymbol{U}(t)}{\mathrm{d}t^2} + \mu C \frac{\mathrm{d}\boldsymbol{U}(t)}{\mathrm{d}t} + BA^{-1}B\boldsymbol{U}(t) = \boldsymbol{F}, \tag{10}$$

The vector $\boldsymbol{U}(0)$ can be determined by $u_h(x,0)$. Equation (10) represents an ordinary differential equation for the vector $\boldsymbol{U}(t)$, where C and $BA^{-1}B$ are symmetric positive definite matrices. According to the theory of ordinary differential equations, the solution to (10) is unique; hence, the solution to the semi-discrete mixed finite element method (6) is also unique.

Theorem 2. Let $\{u, v\}$ and $\{u_h, v_h\}$ be the solutions to weak form (3) and semi-discrete mixed finite element format (6), respectively, such that

$$\|u-u_h\|_0+\|v-v_h\|_0\le Ch^{k+1}, \tag{11}$$

$$\|u-u_h\|_1+\|v-v_h\|_1\le Ch^{k}. \tag{12}$$

Proof. Let $u-u_h=(u-R_hu)+(R_hu-u_h)=\rho+\theta$, $v-v_h=(v-R_hv)+(R_hv-v_h)=\eta+\xi$. Easy to know, $\theta(0)=\theta_t(0)=0$, $\xi(0)=0$. From the weak form (3) and semi-discrete mixed finite element format (6), we can obtain the error equation:

$$\begin{cases}\text{(a)}\ (b(x)\xi,\psi_h)-(\theta_x,\psi_{hx})=-(b(x)\eta,\psi_h),\\ \text{(b)}\ \rho_0S(\theta_{tt},\varphi_h)+\mu(\theta_t,\varphi_h)+(\xi_x,\varphi_{hx})=-\rho_0S(\rho_{tt},\varphi_h)-\mu(\rho_t,\varphi_h).\end{cases} \tag{13}$$

The derivative of (13)(a) with respect to t yields

$$(b(x)\xi_t,\psi_h)-(\theta_{xt},\psi_{hx})=-(b(x)\eta_t,\psi_h). \tag{14}$$

In (14) and (13)(b), taking the summation of $\psi_h=\xi,\varphi_h=\theta_t$ yields

$$\begin{aligned}&\rho_0S(\theta_{tt},\theta_t)+\mu(\theta_t,\theta_t)+(b(x)\xi_t,\xi)\\ &=-(b(x)\eta_t,\xi)-\rho_0S(\rho_{tt},\theta_t)-\mu(\rho_t,\theta_t).\end{aligned} \tag{15}$$

The left end of (15) satisfies

$$\begin{aligned}&\rho_0S(\theta_{tt},\theta_t)+\mu(\theta_t,\theta_t)+(b(x)\xi_t,\xi)\\ &=\frac{\rho_0S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|_0^2+\mu\|\theta_t\|_0^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(b(x)\xi,\xi).\end{aligned} \tag{16}$$

The following estimates in turn are the right end of (15).

Using the Cauchy-Schwarz inequality and Young's inequality with ε, we have

$$-(b(x)\eta_t,\xi)\le C\left(\|\eta_t\|_0^2+\|\xi\|_0^2\right), \tag{17}$$

$$-\rho_0S(\rho_{tt},\theta_t)\le C\|\rho_{tt}\|_0^2+\frac{\mu}{2}\|\theta_t\|_0^2, \tag{18}$$

$$-\mu(\rho_t,\theta_t)\le C\|\rho_t\|_0^2+\frac{\mu}{2}\|\theta_t\|_0^2. \tag{19}$$

Substituting the estimate of the right end of (15) and (16) into (15), we have

$$\frac{\rho_0S}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta_t\|_0^2+\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(b(x)\xi,\xi)\le C\left(\|\rho_{tt}\|_0^2+\|\rho_t\|_0^2+\|\eta_t\|_0^2+\|\xi\|_0^2\right) \tag{20}$$

Integrating the left and right ends of inequality (20) from 0 to t, and with $\xi(0)=\theta_t(0)=0$, it can be obtained that

$$\|\theta_t\|_0^2+\|\xi\|_0^2\le C\int_0^t\left(\|\rho_{tt}\|_0^2+\|\rho_t\|_0^2+\|\eta_t\|_0^2\right)\mathrm{d}s+C\int_0^t\|\xi\|_0^2\,\mathrm{d}s. \tag{21}$$

Noting that $\theta(0)=0$, $\|\theta\|_0^2\le T\int_0^t\|\theta_t\|_0^2\,\mathrm{d}s$, then from (21), we have

$$\|\theta_t\|_0^2+\|\theta\|_0^2+\|\xi\|_0^2\le C\int_0^t\left(\|\rho_{tt}\|_0^2+\|\rho_t\|_0^2+\|\eta_t\|_0^2\right)\mathrm{d}s+C\int_0^t\left(\|\theta_t\|_0^2+\|\xi\|_0^2\right)\mathrm{d}s. \tag{22}$$

Using Gronwall's inequality and Lemma 1, then we have

$$\|\theta_t\|_0^2 + \|\theta\|_0^2 + \|\xi\|_0^2 \le Ch^{2k+2}. \tag{23}$$

Using Lemma 1, (23) and the Triangle inequality, (11) holds true.
The following estimate $\|u-u_h\|_1$ and $\|v-v_h\|_1$.

Using (23) and the inverse Holder inequality, we have

$$\|\theta_x\|_0 \le Ch^{-1}\|\theta\|_0 \le Ch^k, \tag{24}$$

$$\|\xi_x\|_0 \le Ch^{-1}\|\xi\|_0 \le Ch^k. \tag{25}$$

Noting that $\theta \in H_0^1(\Omega)$, $\xi \in H_0^1(\Omega)$, using Lemma 1, the Triangle inequality and Poncaré inequality, we have

$$\|u-u_h\|_1 \le \|\rho\|_1 + \|\theta\|_1 \le \|\rho\|_1 + C\|\theta_x\|_0 \le Ch^k,$$
$$\|v-v_h\|_1 \le \|\eta\|_1 + \|\xi\|_1 \le \|\eta\|_1 + C\|\xi_x\|_0 \le Ch^k.$$

Theorem 3. Let $\{u,v\}$ and $\{u_h,v_h\}$ be the solutions to weak form (3) and semi-discrete mixed finite element format (6), respectively. When $\{u,v\}$ is smooth enough, there are

$$\|R_h u_{tt} - u_{htt}\|_0^2 + \|R_h u_t - u_{ht}\|_0^2 + \|R_h v_t - v_{ht}\|_0^2 \le Ch^{2k+2}. \tag{26}$$

Proof. The derivative of error Equation (12) with respect to t yields

$$\begin{cases}(\text{a})\ \left(b(x)\xi_t,\psi_h\right)-\left(\theta_{xt},\psi_{hx}\right)=-\left(b(x)\eta_t,\psi_h\right),\\ (\text{b})\ \rho_0 S\left(\theta_{ttt},\varphi_h\right)+\mu\left(\theta_{tt},\varphi_h\right)+\left(\xi_{xt},\varphi_{hx}\right)=-\rho_0 S\left(\rho_{ttt},\varphi_h\right)-\mu\left(\rho_{tt},\varphi_h\right).\end{cases} \tag{27}$$

The derivative of (27)(a) with respect to t yields

$$\left(b(x)\xi_{tt},\psi_h\right)-\left(\theta_{xtt},\psi_{hx}\right)=-\left(b(x)\eta_{tt},\psi_h\right). \tag{28}$$

In (27)(b) and (28), taking the summation of $\psi_h=\xi,\varphi_h=\theta_t$ yields

$$\begin{aligned}&\rho_0 S\left(\theta_{ttt},\theta_{tt}\right)+\mu\left(\theta_{tt},\theta_{tt}\right)+\left(b(x)\xi_{tt},\xi_t\right)\\ &=-\left(b(x)\eta_{tt},\xi_t\right)-\rho_0 S\left(\rho_{ttt},\theta_{tt}\right)-\mu\left(\rho_{tt},\theta_{tt}\right).\end{aligned} \tag{29}$$

Using the same method as in Theorem 2, it is obtained that

$$\|R_h u_{tt} - u_{htt}\|_0^2 + \|R_h u_t - u_{ht}\|_0^2 + \|R_h v_t - v_{ht}\|_0^2 \le Ch^{2k+2}.$$

3. Fully-Discrete Mixed Finite Element Scheme

In this section, the fully-discrete mixed finite element scheme is created by discretizing the time terms, and its stability is proven. The error estimation of the unknown variable u and the intermediate variable v is then performed.

Let $0=t_0<t_1<\cdots<t_{N-1}<t_N=T$ be a partition of the time interval $[0,T]$ with step size $\tau=\frac{T}{N}$, where $t_n=n\tau$, $n=0,1,2,\cdots,N$. U^n and V^n represents the approximation of u and v at time $t=t_n$. For the sake of simplicity, let $z^n=z(t_n)$, other notations are as follows:

$$z^{n+\frac{1}{2}} = \frac{z^{n+1}+z^n}{2},\ z^{n,\frac{1}{2}} = \frac{z^{n+1}+z^{n-1}}{2},\ \bar{\partial}_t z^n = \frac{z^{n+1}-z^{n-1}}{2\tau},$$

$$\bar{\partial}_t z^{n+\frac{1}{2}} = \frac{z^{n+1}-z^n}{\tau},\ \bar{\partial}_{tt} z^n = \frac{z^{n+1}-2z^n+z^{n-1}}{\tau^2}.$$

The following relations can be obtained through the use of notation.

$$\bar{\partial}_t z^n = \frac{\bar{\partial}_t z^{n+\frac{1}{2}} + \bar{\partial}_t z^{n-\frac{1}{2}}}{2},\ \bar{\partial}_{tt} z^n = \frac{\bar{\partial}_t z^{n+\frac{1}{2}} - \bar{\partial}_t z^{n-\frac{1}{2}}}{\tau}.$$

We obtain the fully-discrete mixed finite element scheme of (3), which is to find $\{U^n, V^n\} \in S_h^0 \times S_h^0$, such that

$$\begin{cases} \text{(a)}\ \left(b(x)V^{n+1}, \psi_h\right) - \left(U_x^{n+1}, \psi_{hx}\right) = 0, & \forall \psi_h \in S_h^0, \\ \text{(b)}\ \rho_0 S\left(\bar{\partial}_{tt} U^n, \varphi_h\right) + \mu\left(\bar{\partial}_t U^n, \varphi_h\right) + \left(V_x^{n,\frac{1}{2}}, \varphi_{hx}\right) = \left(f^{n,\frac{1}{2}}, \varphi_h\right), & \forall \varphi_h \in S_h^0, \\ U^0 = R_h u_0,\ V^0 = R_h\left(-E(x) I u_{0xx}\right), \\ U^1 = R_h\left(u_0 + \tau u_1 + \frac{\tau^2}{2} u_{tt}(0)\right), \\ V^1 = R_h\left(-E(x) I u_{0xx} + \tau\left(-E(x) I u_{1xx}\right)\right), \end{cases} \tag{30}$$

where $u_{tt}(0) = \frac{1}{\rho_0 S}\left(f(0) - \mu u_1 - \left(E(x) I u_{0xx}\right)_{xx}\right)$.

Theorem 4. The fully-discrete mixed finite element scheme (30) is stable and for $1 \le J \le N$, satisfy

$$\left\|U^J\right\|_0^2 + \left\|V^J\right\|_0^2 \le C\left(\max_{0\le n\le N}\left\|f^n\right\|_0^2 + \left\|\bar{\partial}_t U^{\frac{1}{2}}\right\|_0^2 + \left\|V^0\right\|_0^2 + \left\|V^1\right\|_0^2\right). \tag{31}$$

Proof. According to (30)(a), when $t = t_{n-1}$, we have

$$\left(b(x)V^{n-1}, \psi_h\right) - \left(U_x^{n-1}, \psi_{hx}\right) = 0. \tag{32}$$

From (30) and (32), we have

$$\begin{cases} \text{(a)}\ \left(b(x)\bar{\partial}_t V^n, \psi_h\right) - \left(\bar{\partial}_t U_x^n, \psi_{hx}\right) = 0, \\ \text{(b)}\ \rho_0 S\left(\bar{\partial}_{tt} U^n, \varphi_h\right) + \mu\left(\bar{\partial}_t U^n, \varphi_h\right) + \left(V_x^{n,\frac{1}{2}}, \varphi_{hx}\right) = \left(f^{n,\frac{1}{2}}, \varphi_h\right). \end{cases} \tag{33}$$

where (33)(a) is obtained by subtracting the left and right ends of (30)(a) and (32) and dividing by 2τ, respectively.

In (33), let $\varphi_h = \bar{\partial}_t U^n, \psi_h = V^{n,\frac{1}{2}}$, Equation (33)(a) is to be added to (33)(b) and we have

$$\rho_0 S\left(\bar{\partial}_{tt} U^n, \bar{\partial}_t U^n\right) + \mu\left(\bar{\partial}_t U^n, \bar{\partial}_t U^n\right) + \left(b(x)\bar{\partial}_t V^n, V^{n,\frac{1}{2}}\right) = \left(f^{n,\frac{1}{2}}, \bar{\partial}_t U^n\right). \tag{34}$$

The left end of (34) satisfies

$$\rho_0 S\left(\bar{\partial}_{tt}U^n,\bar{\partial}_t U^n\right)=\rho_0 S\left(\frac{\bar{\partial}_t U^{n+\frac{1}{2}}-\bar{\partial}_t U^{n-\frac{1}{2}}}{\tau},\frac{\bar{\partial}_t U^{n+\frac{1}{2}}+\bar{\partial}_t U^{n-\frac{1}{2}}}{2}\right)$$

$$=\frac{\rho_0 S}{2\tau}\left(\left\|\bar{\partial}_t U^{n+\frac{1}{2}}\right\|_0^2-\left\|\bar{\partial}_t U^{n-\frac{1}{2}}\right\|_0^2\right),$$

$$\mu\left(\bar{\partial}_t U^n,\bar{\partial}_t U^n\right)=\mu\left\|\bar{\partial}_t U^n\right\|_0^2, \tag{35}$$

$$\left(b(x)\bar{\partial}_t V^n,V^{n,\frac{1}{2}}\right)=\left(b(x)\frac{V^{n+1}-V^{n-1}}{2\tau},\frac{V^{n+1}+V^{n-1}}{2}\right)$$

$$=\frac{1}{4\tau}\left[\left(b(x)V^{n+1},V^{n+1}\right)-\left(b(x)V^{n-1},V^{n-1}\right)\right].$$

Using the Cauchy-Schwarz inequality and Young's inequality with ε, we have

$$\left(f^{n,\frac{1}{2}},\bar{\partial}_t U^n\right)\le C\left\|f^{n,\frac{1}{2}}\right\|_0^2+\mu\left\|\bar{\partial}_t U^n\right\|_0^2. \tag{36}$$

Substituting (35) and (36) into (34) yields

$$\frac{\rho_0 S}{2\tau}\left(\left\|\bar{\partial}_t U^{n+\frac{1}{2}}\right\|_0^2-\left\|\bar{\partial}_t U^{n-\frac{1}{2}}\right\|_0^2\right)+\frac{1}{4\tau}\left[\left(b(x)V^{n+1},V^{n+1}\right)-\left(b(x)V^{n-1},V^{n-1}\right)\right]\le C\left\|f^{n,\frac{1}{2}}\right\|_0^2. \tag{37}$$

(37) is multiplied at each end by τ, summing n from 1 to $J-1$ ($1\le J\le N$) yields

$$\frac{\rho_0 S}{2}\left\|\bar{\partial}_t U^{J-\frac{1}{2}}\right\|_0^2+\frac{b_0}{4}\left\|V^J\right\|_0^2+\frac{b_0}{4}\left\|V^{J-1}\right\|_0^2$$

$$\le C\tau\sum_{n=1}^{J-1}\left\|f^{n,\frac{1}{2}}\right\|_0^2+\frac{\rho_0 S}{2}\left\|\bar{\partial}_t U^{\frac{1}{2}}\right\|_0^2+\frac{b_1}{4}\left\|V^0\right\|_0^2+\frac{b_1}{4}\left\|V^1\right\|_0^2.$$

thus have

$$\left\|\bar{\partial}_t U^{J-\frac{1}{2}}\right\|_0^2+\left\|V^J\right\|_0^2+\left\|V^{J-1}\right\|_0^2\le C\left(\max_{0\le n\le N}\left\|f^n\right\|_0^2+\left\|\bar{\partial}_t U^{\frac{1}{2}}\right\|_0^2+\left\|V^0\right\|_0^2+\left\|V^1\right\|_0^2\right). \tag{38}$$

When $n=J-1$, in (30)(a), let $\psi_h=U^J$, and using Cauchy-Schwarz inequality and Poncaré inequality, we have

$$\left\|U^J\right\|_0\le C\left\|V^J\right\|_0. \tag{39}$$

Combining (38) and (39), we get

$$\left\|U^J\right\|_0^2+\left\|V^J\right\|_0^2\le C\left(\max_{0\le n\le N}\left\|f^n\right\|_0^2+\left\|\bar{\partial}_t U^{\frac{1}{2}}\right\|_0^2+\left\|V^0\right\|_0^2+\left\|V^1\right\|_0^2\right).$$

Theorem 5. Let $\{u^n,v^n\}$ and $\{U^n,V^n\}$ be the solutions to weak form (3) and fully-discrete mixed finite element format (30), respectively, then for $1\le J\le N$, we have

$$\left\|u^J-U^J\right\|_0+\left\|v^J-V^J\right\|_0\le C\left(h^{k+1}+\tau^2\right), \tag{40}$$

$$\left\|u^J-U^J\right\|_1+\left\|v^J-V^J\right\|_1\le C\left(h^{k}+\tau^2\right). \tag{41}$$

Proof. Let $u^n-U^n=\left(u^n-R_hu^n\right)+\left(R_hu^n-U^n\right)=\rho^n+\theta^n$, $U^n-V^n=\left(v^n-R_hv^n\right)+\left(R_hv^n-V^n\right)=\eta^n+\xi^n$. Easy to know, $\theta^0=\xi^0=0$. From the weak form (3) and fully-discrete mixed finite element format (30), we can obtain the error equation:

$$\begin{cases}(\mathrm{a})\quad \left(b(x)\xi^{n+1},\psi_h\right)-\left(\theta_x^{n+1},\psi_{hx}\right)=-\left(b(x)\eta^{n+1},\psi_h\right),\\ (\mathrm{b})\quad \rho_0S\left(\bar{\partial}_{tt}\theta^n,\varphi_h\right)+\mu\left(\bar{\partial}_t\theta^n,\varphi_h\right)+\left(\xi_x^{n,\frac{1}{2}},\varphi_{hx}\right)\\ \qquad =-\rho_0S\left(\bar{\partial}_{tt}\rho^n,\varphi_h\right)-\mu\left(\bar{\partial}_t\rho^n,\varphi_h\right)+\rho_0S\left(R_1^n,\varphi_h\right)+\mu\left(R_2^n,\varphi_h\right),\end{cases} \tag{42}$$

where $R_1^n=\bar{\partial}_{tt}u^n-u_{tt}^{n,\frac{1}{2}}=O\left(\tau^2\right)$, $R_2^n=\bar{\partial}_tu^n-u_t^{n,\frac{1}{2}}=O\left(\tau^2\right)$. From (42)(a), when $t=t_{n-1}$, we have

$$\left(b(x)\xi^{n-1},\psi_h\right)-\left(\theta_x^{n-1},\psi_{hx}\right)=-\left(b(x)\eta^{n-1},\psi_h\right), \tag{43}$$

From (42) and (43), we have

$$\begin{cases}(\mathrm{a})\quad \left(b(x)\bar{\partial}_t\xi^{n},\psi_h\right)-\left(\bar{\partial}_t\theta_x^{n},\psi_{hx}\right)=-\left(b(x)\bar{\partial}_t\eta^{n},\psi_h\right),\\ (\mathrm{b})\quad \rho_0S\left(\bar{\partial}_{tt}\theta^n,\varphi_h\right)+\mu\left(\bar{\partial}_t\theta^n,\varphi_h\right)+\left(\xi_x^{n,\frac{1}{2}},\varphi_{hx}\right)\\ \qquad =-\rho_0S\left(\bar{\partial}_{tt}\rho^n,\varphi_h\right)-\mu\left(\bar{\partial}_t\rho^n,\varphi_h\right)+\rho_0S\left(R_1^n,\varphi_h\right)+\mu\left(R_2^n,\varphi_h\right),\end{cases} \tag{44}$$

where (44)(a) is obtained by subtracting the left and right ends of (42)(a) and (43) and dividing by 2τ, respectively.

In (44), let $\varphi_h=\bar{\partial}_t\theta^n,\psi_h=\xi^{n,\frac{1}{2}}$, the equation (44)(a) is to be added to (44)(b) and we have

$$\begin{aligned}&\rho_0S\left(\bar{\partial}_{tt}\theta^n,\bar{\partial}_t\theta^n\right)+\mu\left(\bar{\partial}_t\theta^n,\bar{\partial}_t\theta^n\right)+\left(b(x)\bar{\partial}_t\xi^n,\xi^{n,\frac{1}{2}}\right)\\ &=-\left(b(x)\bar{\partial}_t\eta^n,\xi^{n,\frac{1}{2}}\right)-\rho_0S\left(\bar{\partial}_{tt}\rho^n,\bar{\partial}_t\theta^n\right)-\mu\left(\bar{\partial}_t\rho^n,\bar{\partial}_t\theta^n\right)\\ &\quad+\rho_0S\left(R_1^n,\bar{\partial}_t\theta^n\right)+\mu\left(R_2^n,\bar{\partial}_t\theta^n\right)\triangleq\sum_{i=1}^{5}H_i.\end{aligned} \tag{45}$$

The left end of (45) satisfies

$$\rho_0 S\left(\bar{\partial}_{tt}\theta^n,\bar{\partial}_t\theta^n\right)=\rho_0 S\left(\frac{\bar{\partial}_t\theta^{n+\frac{1}{2}}-\bar{\partial}_t\theta^{n-\frac{1}{2}}}{\tau},\frac{\bar{\partial}_t\theta^{n+\frac{1}{2}}+\bar{\partial}_t\theta^{n-\frac{1}{2}}}{2}\right)$$

$$=\frac{\rho_0 S}{2\tau}\left(\left\|\bar{\partial}_t\theta^{n+\frac{1}{2}}\right\|_0^2-\left\|\bar{\partial}_t\theta^{n-\frac{1}{2}}\right\|_0^2\right),$$

$$\mu\left(\bar{\partial}_t\theta^n,\bar{\partial}_t\theta^n\right)=\mu\left\|\bar{\partial}_t\theta^n\right\|_0^2, \tag{46}$$

$$\left(b(x)\bar{\partial}_t\xi^n,\xi^{n,\frac{1}{2}}\right)=\left(b(x)\frac{\xi^{n+1}-\xi^{n-1}}{2\tau},\frac{\xi^{n+1}+\xi^{n-1}}{2}\right)$$

$$=\frac{1}{4\tau}\left[\left(b(x)\xi^{n+1},\xi^{n+1}\right)-\left(b(x)\xi^{n-1},\xi^{n-1}\right)\right].$$

The following estimates $H_i\,(i=1,2,3,4,5)$ in turn:

Using Cauchy-Schwarz inequality and Young's inequality with ε, we have that

$$H_1=-\left(b(x)\bar{\partial}_t\eta^n,\xi^{n,\frac{1}{2}}\right)\le C\left(\left\|\bar{\partial}_t\eta^n\right\|_0^2+\left\|\xi^{n,\frac{1}{2}}\right\|_0^2\right)$$

$$\le C\left(\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|\eta_t\|_0^2\,\mathrm{d}s+\left\|\xi^{n+1}\right\|_0^2+\left\|\xi^{n-1}\right\|_0^2\right).$$

$$H_2=-\rho_0 S\left(\bar{\partial}_{tt}\rho^n,\bar{\partial}_t\theta^n\right)\le C\left\|\bar{\partial}_{tt}\rho^n\right\|_0^2+\frac{\mu}{4}\left\|\bar{\partial}_t\theta^n\right\|_0^2$$

$$\le\frac{C}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|\rho_{tt}\|_0^2\,\mathrm{d}s+\frac{\mu}{4}\left\|\bar{\partial}_t\theta^n\right\|_0^2.$$

$$H_3=-\mu\left(\bar{\partial}_t\rho^n,\bar{\partial}_t\theta^n\right)\le C\left\|\bar{\partial}_t\rho^n\right\|_0^2+\frac{\mu}{4}\left\|\bar{\partial}_t\theta^n\right\|_0^2$$

$$\le\frac{C}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|\rho_t\|_0^2\,\mathrm{d}s+\frac{\mu}{4}\left\|\bar{\partial}_t\theta^n\right\|_0^2.$$

By the definition of $R_i^n\,(i=1,2)$, we have that

$$H_4+H_5=\rho_0 S\left(R_1^n,\bar{\partial}_t\theta^n\right)+\mu\left(R_2^n,\bar{\partial}_t\theta^n\right)$$

$$\le C\tau^4\left(\left\|u_{tttt}^n\right\|_0^2+\left\|u_{ttt}^n\right\|_0^2\right)+\frac{\mu}{2}\left\|\bar{\partial}_t\theta^n\right\|_0^2.$$

Substituting the above estimate of $H_i\,(i=1,2,3,4,5)$ and (46) into (45) yields

$$\begin{aligned}&\frac{\rho_0 S}{2\tau}\left(\left\|\bar{\partial}_t\theta^{n+\frac{1}{2}}\right\|_0^2-\left\|\bar{\partial}_t\theta^{n-\frac{1}{2}}\right\|_0^2\right)+\frac{1}{4\tau}\left[\left(b(x)\xi^{n+1},\xi^{n+1}\right)-\left(b(x)\xi^{n-1},\xi^{n-1}\right)\right]\\&\le C\left[\frac{1}{\tau}\int_{t_{n-1}}^{t_{n+1}}\left(\|\rho_{tt}\|_0^2+\|\rho_t\|_0^2+\|\eta_t\|_0^2\right)\mathrm{d}s+\tau^4\left(\left\|u_{tttt}^n\right\|_0^2+\left\|u_{ttt}^n\right\|_0^2\right)+\left\|\xi^{n+1}\right\|_0^2+\left\|\xi^{n-1}\right\|_0^2\right].\end{aligned} \tag{47}$$

Multiplying both ends of (47) by τ, and summing over n from 1 to $J-1$ ($1\le J\le N$), we obtain

$$\begin{aligned}&\left\|\bar{\partial}_t\theta^{J-\frac{1}{2}}\right\|_0^2+\left\|\xi^J\right\|_0^2+\left\|\xi^{J-1}\right\|_0^2\\&\le C\left[\int_0^{t_J}\left(\left\|\rho_{tt}\right\|_0^2+\left\|\rho_t\right\|_0^2+\left\|\eta_t\right\|_0^2\right)\mathrm{d}s+\tau^5\sum_{n=1}^{J-1}\left(\left\|u_{tttt}^n\right\|_0^2+\left\|u_{ttt}^n\right\|_0^2\right)\right]\\&\quad+C\tau\sum_{n=1}^{J}\left\|\xi^n\right\|_0^2+C\left(\left\|\bar{\partial}_t\theta^{\frac{1}{2}}\right\|_0^2+\left\|\xi^0\right\|_0^2+\left\|\xi^1\right\|_0^2\right).\end{aligned}\tag{48}$$

Noting that $\xi^0=\theta^0=0$, $\xi^1=R_hv^1-V^1=O(\tau^2)$, $\theta^1=R_hu^1-U^1=O(\tau^3)$, $\left\|\bar{\partial}_t\theta^{\frac{1}{2}}\right\|_0=\left\|\frac{\theta^1-\theta^0}{\tau}\right\|_0=O(\tau^2)$. Using1 for (48) gives

$$\left\|\xi^J\right\|_0^2\le C\left(h^{2k+2}+\tau^4\right)+C\tau\sum_{n=1}^{J}\left\|\xi^n\right\|_0^2,$$

that is

$$(1-C\tau)\left\|\xi^J\right\|_0^2\le C\left(h^{2k+2}+\tau^4\right)+C\tau\sum_{n=1}^{J-1}\left\|\xi^n\right\|_0^2.\tag{49}$$

Using Gronwall inequality for (49), we obtain

$$\left\|\xi^J\right\|_0\le C\left(h^{k+2}+\tau^4\right).\tag{50}$$

When $n=J-1$, in (42)(a) let $\psi_h=\theta^J$, using Poncaré and Cauchy-Schwarz inequality, there is

$$\left\|\theta_x^J\right\|_0\le C\left(\left\|\xi^J\right\|_0+\left\|\eta^J\right\|_0\right)\le C\left(h^{k+1}+\tau^2\right).\tag{51}$$

Using Theorem 1, Poncaré inequality and the Triangle inequality, we have that

$$\left\|u^J-U^J\right\|_1\le\left\|\rho^J\right\|_1+\left\|\theta^J\right\|_1\le\left\|\rho^J\right\|_1+C\left\|\theta_x^J\right\|_0\le C\left(h^k+\tau^2\right).\tag{52}$$

In (42)(b), let $\varphi_h=\bar{\partial}_t\xi^n$, there is

$$\begin{aligned}\left(\xi^{n,\frac{1}{2}},\bar{\partial}_t\xi_x^n\right)&=-\rho_0S\left(\bar{\partial}_{tt}\theta^n,\bar{\partial}_t\xi^n\right)-\mu\left(\bar{\partial}_t\theta^n,\bar{\partial}_t\xi^n\right)-\rho_0S\left(\bar{\partial}_{tt}\rho^n,\bar{\partial}_t\xi^n\right)\\&\quad-\mu\left(\bar{\partial}_t\rho^n,\bar{\partial}_t\xi^n\right)+\rho_0S\left(R_1^n,\bar{\partial}_t\xi^n\right)+\mu\left(R_2^n,\bar{\partial}_t\xi^n\right)\triangleq\sum_{i=1}^{6}D_i^n.\end{aligned}\tag{53}$$

The left end of (53) satisfies

$$\begin{aligned}\left(\xi^{n,\frac{1}{2}},\bar{\partial}_t\xi_x^n\right)&=\left(\frac{\xi_x^{n+1}+\xi_x^{n-1}}{2},\frac{\xi_x^{n+1}-\xi_x^{n-1}}{2\tau}\right)\\&=\frac{1}{4\tau}\left(\left\|\xi_x^{n+1}\right\|_0^2-\left\|\xi_x^{n-1}\right\|_0^2\right).\end{aligned}\tag{54}$$

Using Cauchy-Schwarz inequality and Young's inequality with ε, we have that

$$\begin{aligned}D_1+D_2&=-\rho_0S\left(\bar{\partial}_{tt}\theta^n,\bar{\partial}_t\xi^n\right)-\mu\left(\bar{\partial}_t\theta^n,\bar{\partial}_t\xi^n\right)\\&\le C\left(\left\|\bar{\partial}_{tt}\theta^n\right\|_0^2+\left\|\bar{\partial}_t\theta^n\right\|_0^2+\left\|\bar{\partial}_t\xi^n\right\|_0^2\right)\\&\le\frac{C}{\tau}\int_{t_{n-1}}^{t_{n+1}}\left(\left\|\theta_{tt}\right\|_0^2+\left\|\theta_t\right\|_0^2+\left\|\xi_t\right\|_0^2\right)\mathrm{d}s.\end{aligned}$$

$$D_3+D_4=-\rho_0S\left(\bar{\partial}_{tt}\rho^n,\bar{\partial}_t\xi^n\right)-\mu\left(\bar{\partial}_t\rho^n,\bar{\partial}_t\xi^n\right)$$
$$\leq C\left(\left\|\bar{\partial}_{tt}\rho^n\right\|_0^2+\left\|\bar{\partial}_t\rho^n\right\|_0^2+\left\|\bar{\partial}_t\xi^n\right\|_0^2\right)$$
$$\leq\frac{C}{\tau}\int_{t_{n-1}}^{t_{n+1}}\left(\|\rho_{tt}\|_0^2+\|\rho_t\|_0^2+\|\xi_t\|_0^2\right)\mathrm{d}s.$$
$$D_5+D_6=\rho_0S\left(R_1^n,\bar{\partial}_t\xi^n\right)+\mu\left(R_2^n,\bar{\partial}_t\xi^n\right)$$
$$\leq C\tau^4\left(\left\|u_{tttt}^n\right\|_0^2+\left\|u_{ttt}^n\right\|_0^2\right)+\frac{C}{\tau}\int_{t_{n-1}}^{t_{n+1}}\|\xi_t\|_0^2\,\mathrm{d}s.$$

Substituting the above estimate of $D_i^n\left(i=1,2,\cdots,6\right)$ and (54) into (53) yields

$$\frac{1}{4\tau}\left(\left\|\xi_x^{n+1}\right\|_0^2-\left\|\xi_x^{n-1}\right\|_0^2\right)$$
$$\leq C\left[\int_{n-1}^{n+1}\left(\|\theta_{tt}\|_0^2+\|\theta_t\|_0^2+\|\xi_t\|_0^2+\|\rho_{tt}\|_0^2+\|\rho_t\|_0^2\right)\mathrm{d}s+\tau^4\left(\left\|u_{tttt}^n\right\|_0^2+\left\|u_{ttt}^n\right\|_0^2\right)\right],$$

Multiplying both ends of the previous inequality by τ, and summing over n from 1 to $J-1$ ($1\leq J\leq N$), we obtain

$$\left\|\xi_x^J\right\|_0^2+\left\|\xi_x^{J-1}\right\|_0^2\leq C\left[\int_0^{t_J}\left(\|\theta_{tt}\|_0^2+\|\theta_t\|_0^2+\|\xi_t\|_0^2+\|\rho_{tt}\|_0^2+\|\rho_t\|_0^2\right)\mathrm{d}s\right.$$
$$\left.+\tau^5\sum_{n=1}^{J-1}\left(\left\|u_{tttt}^n\right\|_0^2+\left\|u_{ttt}^n\right\|_0^2\right)\right]+\left\|\xi_x^1\right\|_0^2+\left\|\xi_x^0\right\|_0^2,$$

Noting that $\xi_x^0=0$, $\xi_x^1=R_hv_x^1-V_x^1=O\left(\tau^2\right)$, using Lemma 1 and Theorem 3, we obtain

$$\left\|\xi_x^J\right\|_0^2\leq C\left(h^{2k+2}+\tau^2\right). \tag{55}$$

Using Lemma 1, Poncaré inequality and the Triangle inequality, we have that

$$\left\|v^J-V^J\right\|_1\leq\left\|\eta^J\right\|_1+\left\|\xi^J\right\|_1\leq\left\|\eta^J\right\|_1+C\left\|\xi_x^J\right\|_0\leq C\left(h^k+\tau^2\right).$$

4. Numerical Simulation

Two examples are given in this section. The first one is a numerical solution of a non-homogeneous damped beam vibration problem to verify the feasibility and effectiveness of the mixed finite element method. The second example utilizes the mixed finite element method to investigate the effect of different damping coefficients μ on the beam vibration.

Example 1. Let $\rho_oS=1$, $\mu=1$, $E\left(x\right)I=1-\frac{x}{2}$, $u_0\left(x\right)=0$, $u_1\left(x\right)=\sin\left(\pi x\right)$, in (1). The exact solution is $u\left(x,t\right)=t\cos\left(t\right)\sin\left(\pi x\right)$.

Choosing piecewise linear functions to approximate the variables, the mixed finite element method is used to solve problem (1). Let $h=\tau$, and Table 1 respectively describe errors and convergence orders of the solution of the fully discrete mixed finite element format (30) in L^2-norm. With $h=\tau^2$, Table 2 respectively describes errors and convergence orders of the solution of the fully discrete mixed finite element format (30) in H^1-norm. Figure 1 and Figure 2 show the exact and numerical solutions of u and v at $t = 0.5$, and $t = 1.0$, respectively. The spatio-temporal images of the exact and numerical solutions of u and v are shown in **Figure 3** and **Figure 4**. In the following table, the order of spatial convergence is abbreviated as $order_1$, and the order of time convergence is abbreviated as $order_2$.

According to **Table 1** and **Table 2**, it can be inferred that the spatial convergence order of u and v approximates 2 in the L^2-norm and 1 in the H^1-norm. The temporal convergence order of u and v approximates 2 in the L^2-norm and H^1-norm. These results are consistent with theoretical derivations. As can be seen in **Figures 1-4**, the numerical solution is very close to the exact solution.

Table 1. The errors and convergence orders of u and v in L^2-norm.

τ	h	$\|u-u_h\|_0$	$order_1$	$order_2$	$\|v-v_h\|_0$	$order_1$	$order_2$
$\frac{1}{2^4}$	$\frac{1}{2^4}$	2.2418e−03	-	-	9.9347e−03	-	-
$\frac{1}{2^5}$	$\frac{1}{2^5}$	5.7842e−04	1.9545	1.9545	2.5877e−03	1.9408	1.9408
$\frac{1}{2^6}$	$\frac{1}{2^6}$	1.4609e−04	1.9852	1.9852	6.5680e−04	1.9781	1.9781
$\frac{1}{2^7}$	$\frac{1}{2^7}$	3.6618e−05	1.9963	1.9963	1.6422e−04	1.9999	1.9999
$\frac{1}{2^8}$	$\frac{1}{2^8}$	9.1613e−06	1.9989	1.9989	4.1148e−05	1.9967	1.9967
$\frac{1}{2^9}$	$\frac{1}{2^9}$	2.2908e−06	1.9997	1.9997	1.0296e−05	1.9988	1.9988

Table 2. The errors and convergence orders of u and v in H^1-norm.

τ	h	$\|u-u_h\|_1$	$order_1$	$order_2$	$\|v-v_h\|_1$	$order_1$	$order_2$
$\frac{1}{2^3}$	$\frac{1}{2^6}$	1.7154e−02	-	-	1.4383e−01	-	-
$\frac{1}{2^4}$	$\frac{1}{2^8}$	4.2854e−03	1.0005	2.0010	3.5904e−02	1.0011	2.0021
$\frac{1}{2^5}$	$\frac{1}{2^{10}}$	1.0684e−03	1.0020	2.0040	8.9536e−03	1.0018	2.0036
$\frac{1}{2^6}$	$\frac{1}{2^{12}}$	2.6691e−04	1.0005	2.0011	2.2370e−03	1.0004	2.0009
$\frac{1}{2^7}$	$\frac{1}{2^{14}}$	6.6712e−05	1.0002	2.0003	5.5916e−04	1.0001	2.0002
$\frac{1}{2^8}$	$\frac{1}{2^{16}}$	1.6677e−05	1.0000	2.0001	1.3978e−04	1.0000	2.0001

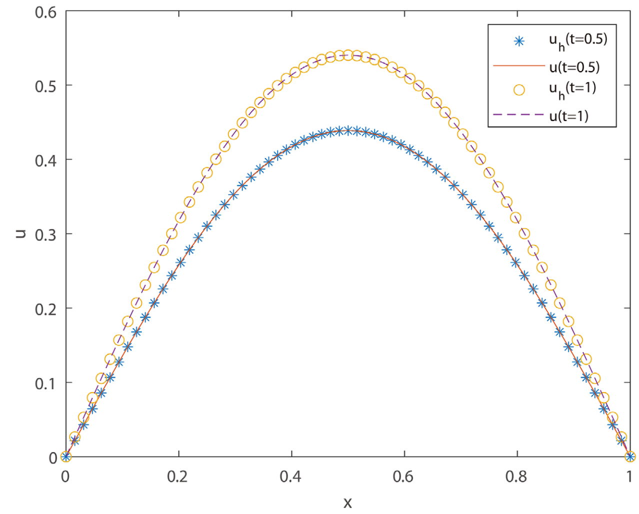


Figure 1. The numerical and exact solution plots of u at $t = 0.5, 1.0$.

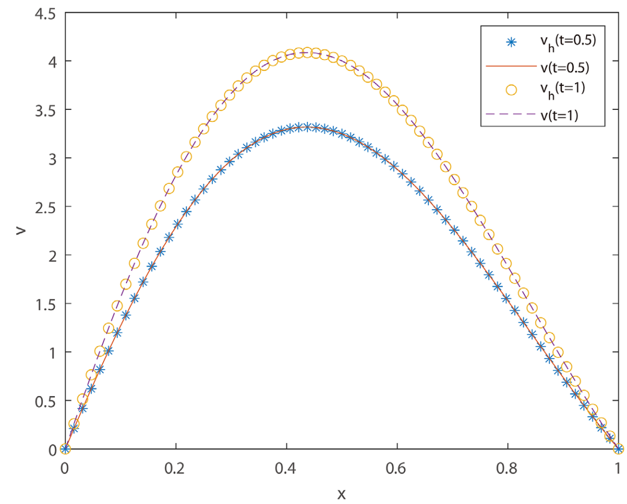
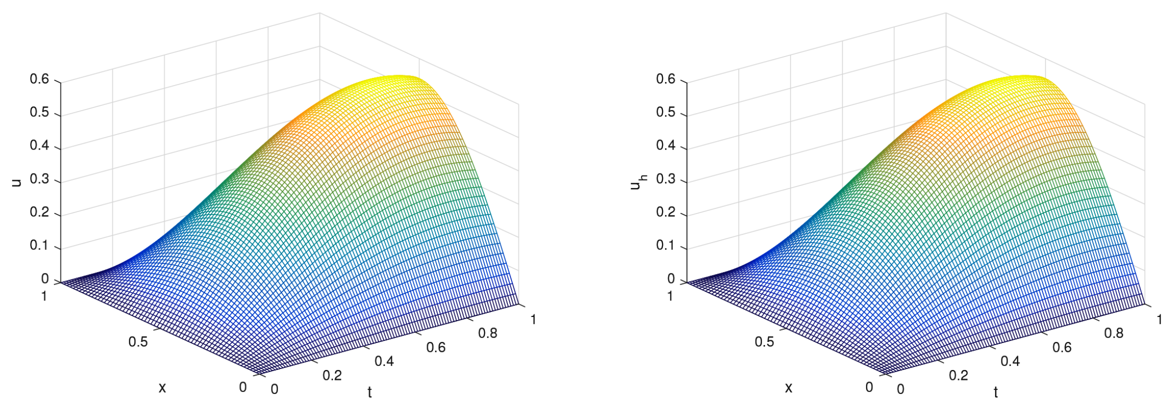


Figure 2. The numerical and exact solution plots of v at $t = 0.5, 1.0$.



(a) The space-time image of u

(b) The space-time image of u_h

Figure 3. Image of u and u_h for $h = \frac{1}{2^6}, \tau = 10^{-2}$.

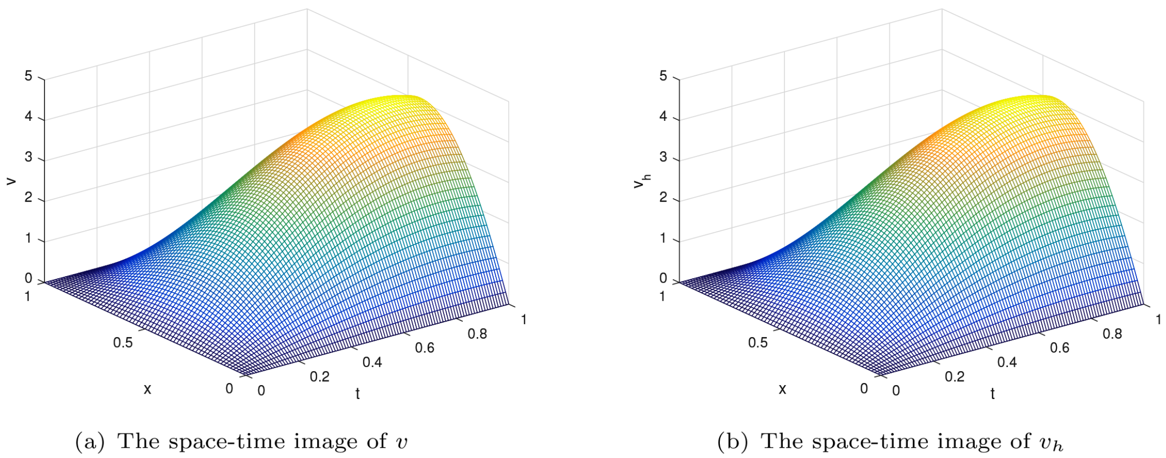


Figure 4. Image of v and v_h for $h=\frac{1}{2^6}, \tau=10^{-2}$.

Example 2. Let $\Omega=[0,1]$, $T=1$, $\rho_o S=1$, $E(x)I=\pi\left(1-\frac{x}{2}\right)$, in (1). The initial displacement is $u(x,0)=\sin(\pi x)$. The effect of different damping coefficients μ on the beam vibration is investigated when μ is 5, 10, 15, 20 respectively. With $f=0$, **Figure 5** shows the image of the vibration at the midpoint of the beam with different damping coefficients. It can be seen that, for the same initial displacement, the larger the damping coefficient μ is, the faster the beam vibration decays.

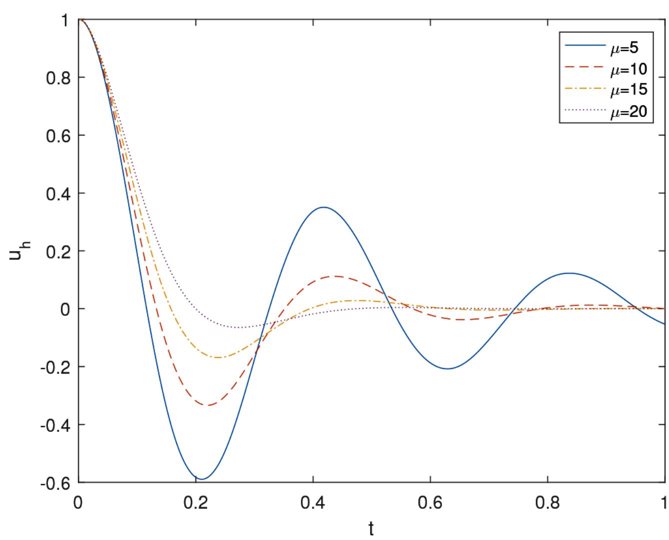


Figure 5. Displacement of beam midpoint with time for different values of damping factor.

Example 3. For practical problems, in the case of constant coefficients, let $\Omega=[0,1]$, $T=0.3$, $\rho_o S=48.4$, $EI=23000$, in (1). The initial displacement is $u(x,0)=\sin(\pi x)$. The effect of different damping coefficients on the beam vibration is investigated when μ is 2000, 3000, 5000, 7000 respectively. When $f=0$,

Figure 6 shows the vibration of the midpoint of the beam under different damping coefficients. In practical situations, the larger the damping coefficient, the faster the attenuation of beam vibration.

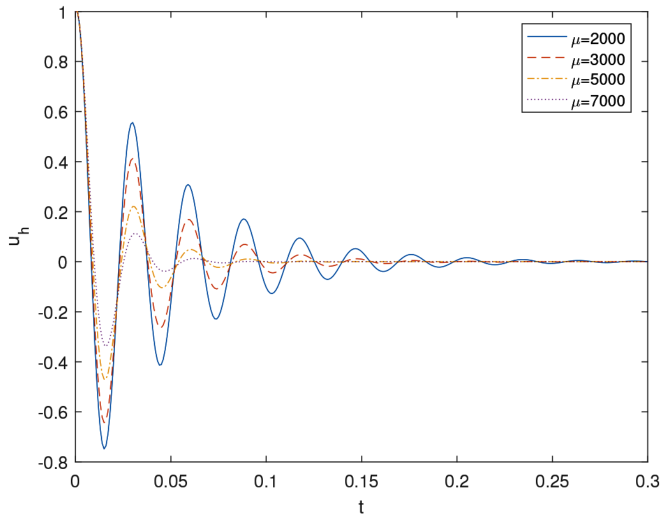


Figure 6. In practical situations, displacement of beam midpoint with time for different values of damping factor.

5. Conclusion

This paper discusses the mixed finite element method for vibration problems of non-homogeneous damped beams. The weak form of the mixed finite element is obtained by introducing intermediate variables, and the semi-discrete and fully discrete mixed finite element formats are established. The existence of a unique solution for the semi-discrete format and the stability of the fully discrete format are proved, and error estimates are given. The feasibility and validity of the mixed element method are substantiated by numerically solving a non-homogeneous damped beam vibration problem, and the impact of different damping coefficients, denoted by μ, on the beam vibration is examined.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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