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The Finite Volume Element Method for Time-Fractional Nonlinear Fourth-Order Diffusion Equation with Time Delay

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Abstract

In this article, a finite volume element algorithm is presented and discussed for the numerical solutions of a time-fractional nonlinear fourth-order diffusion equation with time delay. By choosing the second-order spatial derivative of the original unknown as an additional variable, the fourth-order problem is transformed into a second-order system. Then the fully discrete finite volume element scheme is formulated by using $L1$ approximation for temporal Caputo derivative and finite volume element method in spatial direction. The unique solvability and stable result of the proposed scheme are proved. A priori estimate of L^2-norm with optimal order of convergence $O\left(h^2+\tau^{2-\alpha}\right)$ where τ and h are time step length and space mesh parameter, respectively, is obtained. The efficiency of the scheme is supported by some numerical experiments.

Keywords

Time-Fractional Nonlinear Fourth-Order Diffusion Equation with Time Delay, Finite Volume Element Method, Caputo-Fractional Derivative, Optimal Priori Error Analysis

1. Introduction

Nowadays, researchers have placed more attention on the development of fractional differential equations as these equations are widely used in fractal media, mathematical biology, chemistry, statistical mechanics, engineering and so on [1]-[7]. Time delay occurs in many real-life applications such as population ecology, cell biology, control theory [8]-[13]. Therefore, development of numerical methods for fractional equations with time delay seems to be vital and essential.

In recent years, various numerical methods and theory of fractional differential equations have been studied extensively by researchers and their study comprises numerical methods such as finite difference, finite volume, finite element and so on. In [14], Danumjaya P *et al.* applied the mixed finite element methods to a fourth order reaction diffusion equation with different types of boundary conditions and established some priori bounds with the help of Lyapunov functional. In [15], Yang Liu *et al.* presented a finite difference/finite element algorithm, which is based on a finite difference approximation in time direction and finite element method in spatial direction, and discussed the numerical solutions of a time-fractional fourth-order reaction-diffusion problem with a nonlinear reaction term. Tie Zhang *et al.* in [16] studied the finite volume method for solving the time-fractional diffusion equations and analyzed a fully discrete numerical scheme which is based on the linear finite volume method and the L1 difference. Xinfei Liu *et al.* in [17] considered the nonlinear time-fractional stochastic fourth-order reaction-diffusion equation perturbed by noises based on the mixed finite element in spatial direction and the generalized BDF2-θ in temporal discretization, and obtained the semi- and fully-discrete schemes.

There have been many studies on nonlinear time delay differential equations with spatial second derivative [18]-[21]. However, limited work has been done for nonlinear fourth-order differential equations with time delay. Sarita Nandal *et al.* in [22] constructed a compact difference scheme for one-dimensional time-fractional fourth-order nonlinear sub-diffusion wave equation with time delay and conducted the numerical analysis of the scheme using discrete energy method. In [23] Hongxia Xie *et al.* constructed a compact difference scheme for two-dimensional time-fractional nonlinear fourth-order diffusion equation with time delay and proved the convergence rate in time and space.

In this article, we take into account the following time-fractional nonlinear fourth-order diffusion equation with time delay,

$$\begin{aligned} &{}_0^C D_t^{\alpha} u(x,y,t) + \Delta^2 u(x,y,t) \\ &= f\left(x,y,t,u(x,y,t),u(x,y,t-s)\right),(x,y,t)\in\Omega\times(0,T], \end{aligned} \tag{1.1a}$$

$$u(x,y,t) = \Delta u(x,y,t) = 0,(x,y,t)\in\partial\Omega\times[0,T], \tag{1.1b}$$

$$u(x,y,t) = g(x,y,t),(x,y,t)\in\bar{\Omega}\times[-s,0], \tag{1.1c}$$

where $\alpha\in(0,1)$, $\Omega=(0,L_1)\times(0,L_2)$, $s>0$ is delay and $f\left(x,y,t,u(x,y,t),u(x,y,t-s)\right)$ stands for nonlinear time delay source term, $g(x,y,t)$ is given and sufficiently smooth function. The fractional derivative ${}_0^c D_t^{\alpha}(x,y,t)$ is considered in Caputo sense as follows

$${}_0^c D_t^{\alpha} u(x,y,t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,y,\xi)}{\partial \xi}\frac{1}{(t-\xi)^{\alpha}}\mathrm{d}\xi.$$

Throughout the article, we assume that the source function $f(x,y,t,\mu,\nu)$ is sufficiently smooth likewise considered in the following sense:

The partial derivatives $f_{\mu}(x,y,t,\mu,\nu)$ and $f_{\nu}(x,y,t,\mu,\nu)$ are continuous in the ε_0 neighborhood of the solution and let

$$\begin{aligned}
\bar{c}_1 &= \sup_{(x,y,t)\in\Omega\times(0,T],|\varepsilon_1|\le\varepsilon_0,|\varepsilon_2|\le\varepsilon_0} \left| f_\mu\left(x,y,t,u(x,y,t)+\varepsilon_1,u(x,y,t-s)+\varepsilon_2\right)\right|,\\
\bar{c}_2 &= \sup_{(x,y,t)\in\Omega\times(0,T],|\varepsilon_1|\le\varepsilon_0,|\varepsilon_2|\le\varepsilon_0} \left| f_\nu\left(x,y,t,u(x,y,t)+\varepsilon_1,u(x,y,t-s)+\varepsilon_2\right)\right|,\\
\bar{c}_3 &= \max\left\{\bar{c}_1,\bar{c}_2\right\}.
\end{aligned} \tag{1.2}$$

2. Preliminary

To construct a finite volume element method for problem (1.1), we firstly divide the region $\bar{\Omega}\times[-s,T]$. Taking a positive integer N, we define the temporal step size $\tau=\frac{T}{N}$ and $m=\frac{s}{\tau}$.

Suppose Ω is a polygonal region with boundary $\partial\Omega$. Divide $\bar{\Omega}$ into a sum of finite number of small triangles called elements that they have no overlapping internal region. All the elements constitute a triangulation of $\bar{\Omega}$, denoted by T_h, where h is the maximum length of all the sides. Then we construct a dual decomposition T_h^* related to T_h. Let P_0 be a node of a triangle, $P_i\left(i=1,2,\cdots,6\right)$ the adjacent nodes of P_0. Choose the barycenter Q_i of the triangle $\Delta P_0P_iP_{i+1}\left(P_7=P_1\right)$ and the midpoint M_i of $\overline{P_0P_i}$ and connect successively to form a dual element $K_{P_0}^*$ as shown in **Figure 1**. All the dual elements constitute the dual grid T_h^*.

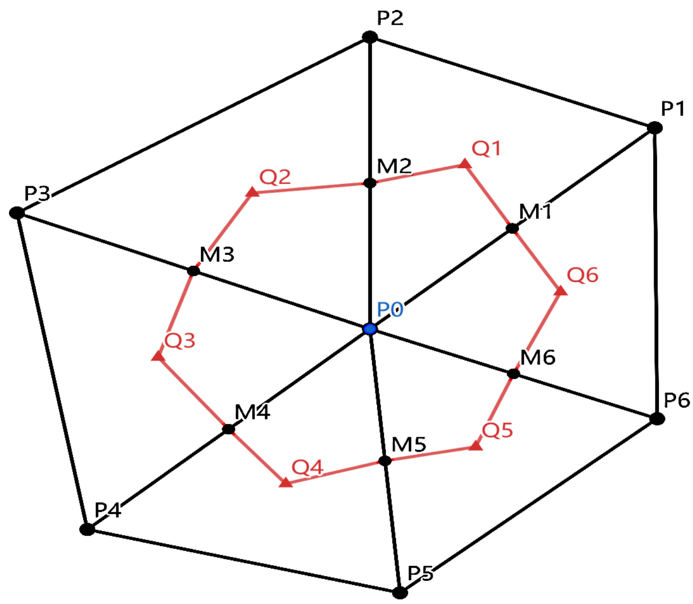


Figure 1. Barycenter dual decomposition.

In this paper, we denote by $\bar{\Omega}_h$ the set of the nodes of the decomposition T_h, $\dot{\Omega}_h=\bar{\Omega}_h\setminus\partial\Omega$ the set of the interior nodes, and Ω_h^* the set of the nodes of the dual decomposition T_h^*. We assume that T_h and T_h^* are quasi-uniform, *i.e.* let S_{K_Q} and $S_{P_0}^*$ be the areas of the triangular element K_Q and dual element $S_{P_0}^*$ respectively, there exist constants $c_1,c_2,c_3>0$ independent of h that

$$c_1 h^2 \le S_{K_Q} \le h^2,\ Q \in \Omega_h^*,$$

$$c_2 h^2 \le S_{P_0}^* \le c_3 h^2,\ P_0 \in \bar{\Omega}_h.$$

Lemma 2.1 [20]. The $L1$ approximation formula for Caputo fractional derivative of $\alpha(0<\alpha<1)$ order is given by

$$\begin{aligned} &{}_0^C D_t^\alpha f(t_n) \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\left[a_0^{(\alpha)} f(t_n) - \sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}\right) f(t_k) - a_{n-1}^{(\alpha)} f(t_0)\right] + E_0^n, \\ &:= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n} a_{n-k}^{(\alpha)}\left(f(t_k) - f(t_{k-1})\right) + E_0^n, \\ &= \partial_t^{(\alpha)} f(t_n) + E_0^n, \end{aligned} \tag{2.1}$$

where $a_l^\alpha = (l+1)^{1-\alpha} - l^{1-\alpha},\ l = 0,1,\cdots$, and

$$\left|E_0^n\right| \le \frac{1}{2\Gamma(2-\alpha)}\left[\frac{1}{4} + \frac{\alpha}{(1-\alpha)(2-\alpha)}\right]\max_{t_0 \le t \le t_n}\left|f_{tt}\right|\tau^{2-\alpha}. \tag{2.2}$$

The following statements hold for $a_l^{(\alpha)}$:

$$\begin{aligned} &1 = a_0^\alpha > a_1^\alpha > a_2^\alpha > \cdots > a_l^\alpha > 0;\ a_0^\alpha \to 0, \text{ if } l \to \infty, \\ &(1-\alpha) l^{-\alpha} < a_{l-1}^{(\alpha)} < (1-\alpha)(l-1)^{-\alpha},\ l \ge 1. \end{aligned} \tag{2.3}$$

Lemma 2.2 [24]. Let $\{y_k\}$ and $\{z_k\}$ be nonnegative sequences. If

$$z_k \le C + \sum_{0 \le k < n} y_k z_k,\ k \ge 0,\ n \ge 1,$$

then it holds that

$$z_k \le C \cdot \exp\left(\sum_{0 \le k < n} y_k\right),\ k \ge 0,\ n \ge 1,$$

where C is a nonnegative constant.

3. Formula of the Finite Volume Element Scheme

In this section, we will present the derivation of the finite volume element scheme approximating problem (1.1). Let $v = -\Delta u$ be an auxiliary variable and the problem (1.1) can be rewritten as the following system:

$$\begin{aligned} &{}_0^C D_t^\alpha u(x,y,t) - \Delta v(x,y,t) \\ &= f\left(x,y,t,u(x,y,t),u(x,y,t-s)\right),\quad (x,y,t) \in \Omega \times (0,T], \end{aligned} \tag{3.1a}$$

$$v(x,y,t) = -\Delta u(x,y,t),\quad (x,y,t) \in \Omega \times (0,T], \tag{3.1b}$$

$$u(x,y,t) = v(x,y,t) = 0,\quad (x,y,t) \in \partial\Omega \times [0,T], \tag{3.1c}$$

$$u(x,y,t) = g(x,y,t),\quad (x,y,t) \in \bar{\Omega} \times [-s,0]. \tag{3.1d}$$

Making use of Green's formula, the corresponding weak formulation of (3.1) is to seek $(u,v):[0,T] \mapsto H_0^1(\Omega) \times H_0^1(\Omega)$ satisfying, for any $(\varphi,\psi) \in H_0^1(\Omega) \times H_0^1(\Omega)$

$$\left({}_0^cD_t^\alpha u,\varphi\right)+a(v,\varphi)=(f,\varphi),\ \forall\varphi\in H_0^1(\Omega), \tag{3.2a}$$

$$a(u,\psi)=(v,\psi),\ \forall\psi\in H_0^1(\Omega), \tag{3.2b}$$

with $u(x,y,t)=g(x,y,t),(x,y,t)\in\bar{\Omega}\times[-s,0]$, where $a(u,v)=\int_\Omega\nabla u\cdot\nabla v\,\mathrm{d}x\mathrm{d}y$. Define the space U_h as the set of piecewise-linear polynomials with respect to T_h, which can be expressed by

$$U_h=\left\{u_h\in C(\bar{\Omega}):u_h|_K\in\mathcal{P}_1,\forall K\in T_h\right\},$$

where $\mathcal{P}_1$ is the set of all linear polynomials on K. It is obvious that U_h is the subspace of $H_0^1(\Omega)$. Then we choose the test function space V_h as the piecewise constant function space corresponding to T_h^*:

$$V_h=\left\{v_h:v_h\text{ is a constant on the interior of each }K^*\in T_h^*\right\}.$$

Set

$$U_{0h}=\left\{u_h\in U_h:u_h(P_0)=0,\ \forall P_0\in\bar{\Omega}_h\cap\partial\Omega\right\},$$

$$V_{0h}=\left\{v_h\in V_h:v_h(P_0)=0,\ \forall P_0\in\bar{\Omega}_h\cap\partial\Omega\right\}.$$

Let Π_h^* be the interpolation project from U_h to V_h:

$$\Pi_h^*\omega_h=\sum_{P_0\in\bar{\Omega}_h}\omega_h(P_0)\chi_{P_0},\ \omega_h\in U_h, \tag{3.3}$$

where χ_{P_0} is the characteristic function of the set $K_{P_0}^*$.

We define $u^k=u(\cdot,t_k)$. Using Taylor's series, the following equations can be easily obtained

$$u^k=2u^{k-1}-u^{k-2}+O(\tau^2),\ k\in N_+.$$

For u^{k-m} with $m=\frac{s}{\tau}$ when $m\notin N_+$:

Case 1: $m\in(0,1)$

$$u^{k-m}=(2-m)u^{k-1}-(1-m)u^{k-2}+O(\tau^2);$$

Case 2: $m>1$

$$u^{k-m}=(\lceil m\rceil-m)u^{k-\lfloor m\rfloor}+(m-\lfloor m\rfloor)u^{k-\lceil m\rceil}+O(\tau^2).$$

Denote

$$\hat{u}^k=2u^{k-1}-u^{k-2},$$

and

$$\tilde{u}^{k-m}=\begin{cases}u^{k-m},\text{ if }m\in N_+,\\(2-m)u^{k-1}-(1-m)u^{k-2},\text{ if }m\in(0,1),\\(\lceil m\rceil-m)u^{k-\lfloor m\rfloor}+(m-\lfloor m\rfloor)u^{k-\lceil m\rceil},\ if\ m>1\text{ and }m\notin N_+.\end{cases}$$

Linearization of the non-linear source term $f(x,y,t,u(x,y,t),u(x,y,t-s))$ by Taylor's series yields

$$f\left(x_i, y_j, t_k, u_{ij}^k, u_{ij}^{k-m}\right) = f\left(x_i, y_j, t_k, \hat{u}_{ij}^k, \tilde{u}_{ij}^{k-m}\right) + O\left(\tau^2\right).$$

We denote $\overline{f}_h = f\left(\cdot, \hat{u}_h, \tilde{u}_h\right)$ and $E_1 = O\left(\tau^2\right)$.

The semidiscrete finite volume scheme is to find a pair $\left(u_h, v_h\right):[0,T] \mapsto U_{0h} \times U_{0h}$ such that

$$\left({}_0^c D_t^\alpha u_h, \Pi_h^* \varphi_h\right) + a\left(v_h, \Pi_h^* \varphi_h\right) = \left(\overline{f}_h, \Pi_h^* \varphi_h\right), \forall \varphi_h \in U_{0h}, \tag{3.4a}$$

$$a\left(u_h, \Pi_h^* \psi_h\right) = \left(v_h, \Pi_h^* \psi_h\right), \forall \psi_h \in U_{0h}, \tag{3.4b}$$

$$u_h\left(\cdot, t\right) = g\left(\cdot, t\right), t \in [-s, 0], \tag{3.4c}$$

where $a\left(u_h, \Pi_h^* \psi_h\right) = -\sum_{P_0 \in \bar{\Omega}_h} \psi_h\left(P_0\right) \int_{\partial K_{P_0}^*} \frac{\partial u_h}{\partial n} \mathrm{d}s$.

Denote $\mu = \tau^\alpha \Gamma(2-\alpha)$. Applying Lemme 2.1, consider the completely discretization finite volume scheme as follows: find $\left(u_h^n, v_h^n\right) \in U_{0h} \times U_{0h}$, such that:

$$\begin{aligned} &\left(u_h^n, \Pi_h^* \varphi_h\right) + \mu a\left(v_h^n, \Pi_h^* \varphi_h\right) \\ &= a_{n-1}^{(\alpha)}\left(u_h^0, \Pi_h^* \varphi_h\right) + \sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}\right)\left(u_h^k, \Pi_h^* \varphi_h\right) \\ &\quad + \mu\left(\overline{f}_h^n, \Pi_h^* \varphi_h\right), n = 1, 2, \cdots, \forall \varphi_h \in U_{0h}, \end{aligned} \tag{3.5a}$$

$$a\left(u_h^n, \Pi_h^* \psi_h\right) = \left(v_h^n, \Pi_h^* \psi_h\right), n = 1, 2, \cdots, \forall \psi_h \in U_{0h}, \tag{3.5b}$$

$$u_h\left(x, y, t\right) = P_h g\left(x, y, t\right), t \in [-s, 0], \tag{3.5c}$$

where u_h^n and v_h^n are approximation of $u\left(\cdot, t_n\right)$ and $v\left(\cdot, t_n\right)$, respectively. The projection P_h will be defined lately.

Lemma 3.1 [25]. The bilinear form $a\left(\cdot, \Pi_h^* \cdot\right)$ is symmetric and positive definite:

$$a\left(u_h, \Pi_h^* \omega_h\right) = a\left(\omega_h, \Pi_h^* u_h\right), \forall u_h, \omega_h \in U_{0h},$$

$$a\left(u_h, \Pi_h^* u_h\right) = \left|u_h\right|_1^2, \forall u_h \in U_{0h}.$$

Lemma 3.2 [25]. There hold the following statements:

(i) $\left(u_h, \Pi_h^* \bar{u}_h\right) = \left(\bar{u}_h, \Pi_h^* u_h\right), \forall u_h, \bar{u}_h \in U_{0h}$.

(ii) Set $|||u_h||| = \left(u_h, \Pi_h^* u_h\right)^{\frac{1}{2}}$. Then $|||\cdot|||$ is equivalent to $\|\cdot\|$ on U_{0h}, that is, there exist positive constants c_1 and c_2 such that

$$c_1 \|u_h\| \le |||u_h||| \le c_2 \|u_h\|, \forall u_h \in U_{0h}.$$

Applying Hölder inequality, it's obvious that

$$\left|\left(u_h, \Pi_h^* \bar{u}_h\right)\right| \le |||u_h||| \, |||\bar{u}_h|||, \forall u_h, \bar{u}_h \in U_{0h}.$$

Theorem 3.1. The finite volume scheme (3) is uniquely solvable.

proof. Since the finite volume scheme is linear, we can obtain the unique solvability by proving that the relevant homogeneous problem:

$$\left(u_h^n, \Pi_h^* \varphi_h\right) + \mu a\left(v_h^n, \Pi_h^* \varphi_h\right) = 0, \tag{3.6a}$$

$$a\left(u_h^n, \Pi_h^* \psi_h\right) = \left(v_h^n, \Pi_h^* \psi_h\right), \tag{3.6b}$$

admits solely trivial solution. Setting $\varphi_h = u_h^n$ and $\psi_h = v_h^n$ in (3.6a) and (3.6b), respectively. Using Lemma 2.3, we have

$$a\left(u_h^n, \Pi_h^* v_h^n\right) = a\left(v_h^n, \Pi_h^* u_h^n\right).$$

Multiplying (3.6b) by μ, then we substract the resulting equation from (3.6a) to obtain

$$\left(u_h^n, \Pi_h^* u_h^n\right) = -\mu\left(v_h^n, \Pi_h^* v_h^n\right).$$

Applying Lemma 2.4, we arrive at $-\left(v_h^n, \Pi_h^* v_h^n\right) \le 0$, then it can be easily obtained that

$$\left(u_h^n, \Pi_h^* u_h^n\right) = \left\|\left|u_h^n\right|\right\|^2 = 0.$$

From the above equality, it is obvious that $u_h^n = v_h^n = 0$. Therefore we show that the solution of (3.6a)-(3.6b) is zero which implies that the scheme (3.5) is uniquely solvable. This proves the theorem.

4. Convergence and Stability Analysis

In this subsection, to analyze and discuss fully discrete a priori error results, we need to introduce an auxiliary projection $P_h : H_0^1(\Omega) \cap H^2(\Omega) \mapsto U_{0h}$ defined by

$$a\left(P_h u, \chi_h\right) = a\left(u, \chi_h\right), \forall \chi_h \in V_{0h}, \tag{4.1}$$

Lemma 4.1 [25]. Let $P_h u$ be the auxiliary projection of u defined by (4.1) and $u \in W^{3,p}(\Omega) \cap H_0^1(\Omega)$ then

$$\left\|u - P_h u\right\| \le Ch^2 \left\|u\right\|_{3,p}, (p > 1).$$

Theorem 4.1. Let $u \in C^2\left([0,T], H_0^1(\Omega) \cap H^2(\Omega)\right)$ be the solution of (1) and u_h^n be the solution of the finite volume scheme (3) with $u_h^0 = P_h u^0$ respectively, then the optimal error result in L^2-norm hold

$$\left\|u(t_n) - u_h^n\right\| \le C\left(h^2 + \tau^{2-\alpha}\right), \tag{4.2}$$

where C is independent of h and τ.

proof. To simplify the process of writing in the proof, we now split the errors as

$$u^n - u_h^n = \left(u^n - P_h u^n\right) + \left(P_h u^n - u_h^n\right) = \rho^n + \eta^n,$$

$$v^n - v_h^n = \left(v^n - P_h v^n\right) + \left(P_h v^n - v_h^n\right) = \xi^n + \theta^n,$$

$$f^n - \bar{f}_h^n = \left(f^n - \bar{f}^n\right) + \left(\bar{f}^n - \bar{f}_h^n\right) = E_1 + \left(\bar{f}^n - \bar{f}_h^n\right),$$

$$\sigma^n = \partial_t^{(\alpha)} P_h u^n - {}_0^C D_t^\alpha u^n.$$

Using (4.1), we note that

$$a\left(\rho, \Pi_h^* \chi_h\right) = 0,\ a\left(\xi, \Pi_h^* \chi_h\right) = 0,\ \forall \chi_h \in U_h.$$

Substracting the Equation (3.5) from (3.1), we obtain the error equations as follows

$$\begin{aligned}&\left({}_0^cD_t^\alpha u^n-\partial_t^{(\alpha)}u_h^n,\Pi_h^*\varphi_h\right)+a\left(\theta^n,\Pi_h^*\varphi_h\right)\\&=\left(\bar{f}^n-\bar{f}_h^n,\Pi_h^*\varphi_h\right)+\left(E_1,\Pi_h^*\varphi_h\right),\ \forall\varphi_h\in U_{0h},\end{aligned}\tag{4.3a}$$

$$a\left(\eta^n,\Pi_h^*\psi_h\right)=\left(\xi^n+\theta^n,\Pi_h^*\psi_h\right),\ \forall\psi_h\in U_{0h}.\tag{4.3b}$$

Setting $\varphi_h=\eta^n$ and $\psi_h=\theta^n$, substracting the Equations (4.3b) from (4.3a), we arrive at

$$\begin{aligned}&\left(\partial_t^{(\alpha)}\eta^n,\Pi_h^*\eta^n\right)+\left(\theta^n,\Pi_h^*\theta^n\right)\\&=\left(\sigma^n,\Pi_h^*\eta^n\right)-\left(\xi^n,\Pi_h^*\theta^n\right)+\left(\bar{f}^n-\bar{f}_h^n,\Pi_h^*\eta^n\right)+\left(E_1,\Pi_h^*\eta^n\right).\end{aligned}\tag{4.4}$$

based on the result $a\left(\theta^n,\Pi_h^*\eta^n\right)=a\left(\eta^n,\Pi_h^*\theta^n\right)$ obtained by Lemma 3.1.

Substituting the definition of $\partial_t^{(\alpha)}\eta^n$ and multiplying the equations by μ, we obtain that

$$\begin{aligned}&\left(\eta^n,\Pi_h^*\eta^n\right)+\mu\left(\theta^n,\Pi_h^*\theta^n\right)\\&=\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)}\right)\left(\eta^k,\Pi_h^*\eta^n\right)+a_{n-1}^{(\alpha)}\left(\eta^0,\Pi_h^*\eta^n\right)\\&\quad-\mu\left(\xi^n,\Pi_h^*\theta^n\right)+\mu\left(\sigma^n,\Pi_h^*\eta^n\right)+\mu\left(\bar{f}^n-\bar{f}_h^n,\Pi_h^*\eta^n\right)+\mu\left(E_1,\Pi_h^*\eta^n\right).\end{aligned}\tag{4.5}$$

Making use of Cauchy-Schwarz inequality and Lemma 3.2 and multiplying the equations by 4, we get

$$\begin{aligned}&4|||\eta^n|||^2+4\mu|||\theta^n|||^2\\&\le2\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)}\right)\left(|||\eta^k|||^2+|||\eta^n|||^2\right)+2a_{n-1}^{(\alpha)}\left(|||\eta^0|||^2+|||\eta^n|||^2\right)\\&\quad+2\mu\left(|||\xi^n|||^2+|||\theta^n|||^2\right)+4\mu\left(\sigma^n,\Pi_h^*\eta^n\right)+4\mu\left(\bar{f}^n-\bar{f}_h^n,\Pi_h^*\eta^n\right)\\&\quad+4\mu\left(E_1,\Pi_h^*\eta^n\right).\end{aligned}\tag{4.6}$$

From Lemma 2.1, we see that

$$\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)}\right)=1-a_{n-1}^{(\alpha)}.$$

Hence we can rewrite (4.6) as follows

$$\begin{aligned}2|||\eta^n|||^2&\le2\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)}\right)|||\eta^k|||^2+2a_{n-1}^{(\alpha)}|||\eta^0|||^2+2\mu|||\xi^n|||^2\\&\quad+4\mu\left(\sigma^n,\Pi_h^*\eta^n\right)+4\mu\left(\bar{f}^n-\bar{f}_h^n,\Pi_h^*\eta^n\right)+4\mu\left(E_1,\Pi_h^*\eta^n\right).\end{aligned}\tag{4.7}$$

Choosing $\varepsilon=\frac{3}{4}$, using ε-Cauchy inequality and (4.7), we obtain

$$\begin{aligned}|||\eta^n|||^2&\le2\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)}\right)|||\eta^k|||^2+2a_{n-1}^{(\alpha)}|||\eta^0|||^2+2\mu|||\xi^n|||^2\\&\quad+3\mu\left(|||\sigma^n|||^2+|||\bar{f}^n-\bar{f}_h^n|||^2+|||E_1|||^2\right).\end{aligned}\tag{4.8}$$

Applying Lemma 3.2 yields

$$\begin{aligned}\left\|\eta^n\right\|^2 &\le 2\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)}\right)\left\|\eta^k\right\|^2+2a_{n-1}^{(\alpha)}\left\|\eta^0\right\|^2+2\mu\left\|\xi^n\right\|^2\\ &\quad+3\mu\left(\left\|\sigma^n\right\|^2+\left\|\overline{f}^n-\overline{f}_h^n\right\|^2+\left\|E_1\right\|^2\right),\\ &:=2\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)}-a_{n-k}^{(\alpha)}\right)\left\|\eta^k\right\|^2+e_1+e_2+e_3+e_4+e_5.\end{aligned} \tag{4.9}$$

For the next process, now we need to consider e_1 - e_5.

Considering e_1, based on the initial condition, it's clear that

$$\eta^0=P_hu^0-u_h^0=0. \tag{4.10}$$

For e_2, we can easily get the following inequality by Lemma 4.1,

$$\left\|\xi^n\right\|\le Ch^2\max_{0\le t\le t_n}\left\|v\right\|_{3,p}. \tag{4.11}$$

For e_3, we rewrite σ^n as the following form

$$\begin{aligned}\sigma^n&=\left(\partial_t^{(\alpha)}P_hu^n-\partial_t^{(\alpha)}u^n\right)+\left(\partial_t^{(\alpha)}u^n-{}_0^cD_t^\alpha u^n\right)\\ &:=\sigma_1^n+\sigma_2^n.\end{aligned}$$

Using Lemma 2.1, we obtain

$$\begin{aligned}\left|\sigma_1^n\right|&=\left|\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}a_{n-k}^{(\alpha)}\int_{t_{k-1}}^{t_k}\left(P_h-I\right)u_t\,\mathrm{d}t\right|\\ &\le\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)}\sum_{k=1}^{n}a_{n-k}^{(\alpha)}\,\tau\max_{0\le t\le t_n}\left|\left(P_h-I\right)u_t\right|\\ &\le C\tau^{1-\alpha}\sum_{k=1}^{n}a_{n-k}^{(\alpha)}\max_{0\le t\le t_n}\left|\left(P_h-I\right)u_t\right|\\ &=C\left(n\tau\right)^{1-\alpha}\max_{0\le t\le t_m}\left|\left(P_h-I\right)u_t\right|,\end{aligned}$$

and

$$\left|\sigma_2^n\right|\le\frac{1}{2\Gamma(1-\alpha)}\left[\frac{1}{4}+\frac{\alpha}{(1-\alpha)(2-\alpha)}\right]\max_{0\le t\le t_n}\left|u_{tt}\right|\tau^{2-\alpha}.$$

Applying Lemma 4.1, it yields

$$\left\|\sigma_1^n\right\|=\left(\int_\Omega\left|\sigma_1^n\right|^2\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\le C\left(n\tau\right)^{1-\alpha}\max_{0\le t\le t_n}\left\|\left(P_h-I\right)u_t\right\|\le Ch^2\max_{0\le t\le t_n}\left\|u_t\right\|_{3,p},$$

and

$$\left\|\sigma_2^n\right\|=\left(\int_\Omega\left|\sigma_2^n\right|^2\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}}\le C\tau^{2-\alpha}\max_{0\le t\le t_n}\left\|u_{tt}\right\|.$$

Then we use the triangle inequality to derive

$$\left\|\sigma^n\right\|^2\le2\left(\left\|\sigma_1^n\right\|^2+\left\|\sigma_2^n\right\|^2\right)\le Ch^4\max_{0\le t\le t_n}\left\|u_t\right\|_{3,p}^2+C\tau^{4-2\alpha}\max_{0\le t\le t_n}\left\|u_{tt}\right\|^2. \tag{4.12}$$

For e_4, it follows from the definition of $\hat{u}^n$:

$$\begin{aligned}\left|\hat{u}^n-\hat{u}_h^n\right| &= \left|\left(2u^{n-1}-u^{n-2}\right)-\left(2u_h^{n-1}-u_h^{n-2}\right)\right| \\ &\le 2\left|u^{n-1}-u_h^{n-1}\right|+\left|u^{n-1}-u_h^{n-2}\right|.\end{aligned}$$

Similarly,

$$\left|\tilde{u}^{n-m}-\tilde{u}_h^{n-m}\right|$$
$$\le \begin{cases}\left|u^{n-m}-u_h^{n-m}\right|, \text{ if } m\in N_+ \\ (2-m)\left|u^{n-1}-u_h^{n-1}\right|+(1-m)\left|u^{n-2}-u_h^{n-2}\right|, \text{ if } m\in(0,1) \\ \left(\lceil m\rceil-m\right)\left|u^{k-\lfloor m\rfloor}-u_h^{k-\lfloor m\rfloor}\right|+\left(m-\lfloor m\rfloor\right)\left|u^{k-\lceil m\rceil}-u_h^{k-\lceil m\rceil}\right|, \text{ if } m>1 \text{ and } m\notin N_+.\end{cases}$$

Noting that $m-\lfloor m\rfloor<2$ and $\lceil m\rceil-m<2$, by the triangle inequality and Taylor's series, we can check that the following inequalities hold :

Case 1: if $m>1$ and $m\notin N_+$

$$\begin{aligned}\left\|\bar{f}^n-\bar{f}_h^n\right\| &= \left\|\bar{c}_1\left(\hat{u}^n-\hat{u}_h^n\right)+\bar{c}_2\left(\tilde{u}^{n-m}-\tilde{u}_h^{n-m}\right)\right\| \\ &\le 2\bar{c}_3\left(\left\|u^{n-1}-u_h^{n-1}\right\|+\left\|u^{n-2}-u_h^{n-2}\right\|+\left\|u^{n-\lceil m\rceil}-u_h^{n-\lceil m\rceil}\right\|+\left\|u^{n-\lfloor m\rfloor}-u_h^{n-\lfloor m\rfloor}\right\|\right) \\ &\le 2\bar{c}_3\left(\left\|\rho^{n-1}\right\|+\left\|\rho^{n-2}\right\|+\left\|\rho^{n-\lceil m\rceil}\right\|+\left\|\rho^{n-\lfloor m\rfloor}\right\|\right. \\ &\quad \left.+\left\|\eta^{n-1}\right\|+\left\|\eta^{n-2}\right\|+\left\|\eta^{n-\lceil m\rceil}\right\|+\left\|\eta^{n-\lfloor m\rfloor}\right\|\right).\end{aligned}$$

Thus

$$\begin{aligned}\left\|\bar{f}^n-\bar{f}_h^n\right\|^2 \le 16\bar{c}_3 &\left(\left\|\rho^{n-1}\right\|^2+\left\|\rho^{n-2}\right\|^2+\left\|\rho^{n-\lceil m\rceil}\right\|^2+\left\|\rho^{n-\lfloor m\rfloor}\right\|^2\right. \\ &\left.+\left\|\eta^{n-1}\right\|^2+\left\|\eta^{n-2}\right\|^2+\left\|\eta^{n-\lceil m\rceil}\right\|^2+\left\|\eta^{n-\lfloor m\rfloor}\right\|^2\right);\end{aligned}$$

Case 2: if $m\in(0,1)$

$$\left\|\bar{f}^n-\bar{f}_h^n\right\|^2 \le 16\bar{c}_3\left(\left\|\rho^{n-1}\right\|^2+\left\|\rho^{n-2}\right\|^2+\left\|\eta^{n-1}\right\|^2+\left\|\eta^{n-2}\right\|^2\right);$$

Case 3: if $m\in N_+$

$$\left\|\bar{f}^n-\bar{f}_h^n\right\|^2 \le 16\bar{c}_3\left(\left\|\rho^{n-1}\right\|^2+\left\|\rho^{n-2}\right\|^2+\left\|\rho^{n-m}\right\|^2+\left\|\eta^{n-1}\right\|^2+\left\|\eta^{n-2}\right\|^2+\left\|\eta^{n-m}\right\|^2\right).$$

Applying Lemma 4.1, we obtain that

$$\left\|\rho^k\right\| \le Ch^2\max_{0\le t\le t_k}\|u\|_{3,p},$$

for $k=n-1, n-2, n-\lceil m\rceil, n-\lfloor m\rfloor$.

Based on the above derivation, we reach that

$$\begin{aligned}&\left\|\bar{f}^n-\bar{f}_h^n\right\|^2 \\ &\le Ch^4\max_{0\le t\le t_n}\|u\|_{3,p}^2+16\bar{c}_3\left(\left\|\eta^{n-1}\right\|^2+\left\|\eta^{n-2}\right\|^2+\left\|\eta^{n-m}\right\|^2+\left\|\eta^{n-\lceil m\rceil}\right\|^2+\left\|\eta^{n-\lfloor m\rfloor}\right\|^2\right).\end{aligned} \tag{4.13}$$

For e_5, we have

$$\left\|E_1\right\|^2 \le C\tau^4. \tag{4.14}$$

Substitute the results for e_1 - e_5 into (4.9) to obtain

$$\begin{aligned}\left\|\eta^n\right\|^2 \le\ & 2\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}\right)\left\|\eta^k\right\|^2 \\ & + 48\overline{c}_3\tau^\alpha\left(\left\|\eta^{n-1}\right\|^2 + \left\|\eta^{n-2}\right\|^2 + \left\|\eta^{n-m}\right\|^2 + \left\|\eta^{n-\lceil m\rceil}\right\|^2 + \left\|\eta^{n-\lfloor m\rfloor}\right\|^2\right) \\ & + C\tau^\alpha\left(h^4\max_{0\le t\le t_n}\left\|v\right\|_{3,p}^2 + h^4\max_{0\le t\le t_n}\left\|u\right\|_{3,p}^2 + h^4\max_{0\le t\le t_n}\left\|u_t\right\|_{3,p}^2\right. \\ & \left. + \tau^{4-2\alpha}\max_{0\le t\le t_n}\left\|u_{tt}\right\|^2 + \tau^4\right).\end{aligned} \tag{4.15}$$

Using Lemma 2.2, we have

$$\begin{aligned}\left\|\eta^n\right\|^2 &\le C\left(h^4 + \tau^{4-2\alpha}\right)\exp\left(2\sum_{k=1}^{n-1}\left(a_{n-k-1}^{(\alpha)} - a_{n-k}^{(\alpha)}\right) + 240\overline{c}_3\right) \\ &= C\left(h^4 + \tau^{4-2\alpha}\right)\exp\left(2\left(1 - a_{n-1}^{(\alpha)}\right) + 240\overline{c}_3\right) \\ &\le C\left(h^4 + \tau^{4-2\alpha}\right)\exp\left(2 + 240\overline{c}_3\right).\end{aligned} \tag{4.16}$$

Denoting $\overline{C} = \left[C\exp\left(2 + 240\overline{c}_3\right)\right]^{\frac{1}{2}}$, we obtain that

$$\left\|\eta^n\right\| \le \overline{C}\left(h^2 + \tau^{2-\alpha}\right). \tag{4.17}$$

Applying Lemma 4.1, we conclude that

$$\left\|u(t_n) - u_h^n\right\| \le \left\|\rho^n\right\| + \left\|\eta^n\right\| \le C\left(h^2 + \tau^{2-\alpha}\right). \tag{4.18}$$

Thus the proof of the theorem is completed.

Next, we analyze the numerical stability of the finite volume scheme (3.5). The numerical stability means that a small perturbation of the initial value implies a small perturbation of the numerical solution.

Theorem 4.2. we suppose that $\left(U^n, V^n\right)$ is the solution of perturbation equation and ι is a small perturbation of g, denoting $\lambda_h^k = u_h^k - U^k$, $\gamma_h^k = v_h^k - V^k$, the following stability result hold

$$\left\|\lambda_h^n\right\|^2 + \mu\left\|\gamma_h^n\right\|^2 \le C\max_{-m\le k\le 0}\left\|\iota^k\right\|^2. \tag{4.19}$$

This theorem can be proved by using the same way as the proof of Theorem 4.1.

5. Numerical Experiments

We now present some numerical experiments to verify our theoretical statements. For the purpose of manifesting the stability and convergence rate of the proposed scheme, we first consider an example in which the exact solutions are known. Then in the second example, we consider a more realistic problem for which the exact solution is not given beforehand. In addition, we choose $\Omega = (0,1)\times(0,1)$ and $T = 1$ for all examples.

Example 1. Choose the exact solution for the problem (1.1) to be

$$u = t^3\sin(\pi x)\sin(\pi y), (x, y, t) \in \overline{\Omega}\times[-s, T].$$

associated with the following source function

$$f = u(x, y, t-s) + u^2(x, y, t) + \left[\frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha} - (t-s)^3 + 4\pi^4 t^3\right]\sin(\pi x)\sin(\pi y) - t^6 \sin^2(\pi x)\sin^2(\pi y)$$

In Table 1 and Table 2, the error in L^2-norms are listed with delay parameter $s = 0.2$. To testify the convergence order in spatial direction, we keep varying h and $\tau = h^2$. Similarly, to testify the convergence order in temporal direction, we keep varying τ and $h = \tau$. From the numerical results we can see that the scheme (3) is stable and has the convergence rate of $O\left(h^2 + \tau^{2-\alpha}\right)$. In order to present a comparison with our scheme, this example is also numerically solved by a central difference scheme and the results of it are listed in Table 3 and Table 4. By comparison, we can find that our scheme is more advantageous in accuracy, the comparison between the two methods is presented in Figure 2. Furthermore, The numerical solutions and exact solutions are plotted in Figure 3.

Table 1. Errors and spatial convergence orders of finite volume scheme with $h = \sqrt{\tau}$ at $T = 1$.

s	α	h	$\lVert u - u_h \rVert$	order
0.2	0.3	$\frac{1}{10}$	3.6198E−03	
		$\frac{1}{20}$	9.0193E−04	2.00
		$\frac{1}{30}$	4.0061E−04	2.00
		$\frac{1}{40}$	2.2529E−04	2.00
	0.5	$\frac{1}{10}$	3.6134E−03	
		$\frac{1}{20}$	9.0018E−04	2.01
		$\frac{1}{30}$	3.9981E−04	2.00
		$\frac{1}{40}$	2.2485E−04	2.00
	0.7	$\frac{1}{10}$	3.6089E−03	
		$\frac{1}{20}$	8.9874E−04	2.01
		$\frac{1}{30}$	3.9910E−04	2.00
		$\frac{1}{40}$	2.2449E−04	2.00

Table 2. Errors and time convergence orders of finite volume scheme with $h = \tau$ at $T = 1$.

s	α	τ	$\|u - u_h\|$	order
0.2	0.3	$\frac{1}{10}$	3.5423E−03	
		$\frac{1}{20}$	8.8172E−04	2.01
		$\frac{1}{30}$	3.9178E−04	2.00
		$\frac{1}{40}$	2.2045E−04	2.00
	0.5	$\frac{1}{10}$	3.5672E−03	
		$\frac{1}{20}$	8.9237E−04	2.00
		$\frac{1}{30}$	3.9809E−04	1.99
		$\frac{1}{40}$	2.2476E−04	1.99
	0.9	$\frac{1}{10}$	3.7612E−03	
		$\frac{1}{20}$	9.9265E−04	1.92
		$\frac{1}{30}$	4.6512E−04	1.87
		$\frac{1}{40}$	2.7484E−04	1.83

Table 3. Errors and spatial convergence orders of central difference scheme with $h = \sqrt{\tau}$ at $T = 1$.

s	α	h	$\|u - u_h\|$	order
0.2	0.3	$\frac{1}{10}$	8.3102E−03	
		$\frac{1}{20}$	2.0635E−03	2.01
		$\frac{1}{30}$	9.1595E−04	2.00
		$\frac{1}{40}$	5.1498E−04	2.00
	0.5	$\frac{1}{10}$	8.3036E−03	

Continued

		$\frac{1}{20}$	2.0618E−03	2.01
		$\frac{1}{30}$	9.1516E−04	2.00
		$\frac{1}{40}$	5.1456E−04	2.00
	0.7	$\frac{1}{10}$	8.2989E−03	
		$\frac{1}{20}$	2.0603E−03	2.01
		$\frac{1}{30}$	9.1444E−04	2.00
		$\frac{1}{40}$	5.1419E−04	2.00

Table 4. Errors and time convergence orders of central difference scheme with $h = \tau$ at $T = 1$.

s	α	τ	$\|u - u_h\|$	order
0.2	0.3	$\frac{1}{10}$	8.2269E−03	
		$\frac{1}{20}$	2.0429E−03	2.01
		$\frac{1}{30}$	9.0707E−04	2.00
		$\frac{1}{40}$	5.1014E−04	2.00
	0.5	$\frac{1}{10}$	8.2519E−03	
		$\frac{1}{20}$	2.0536E−03	2.01
		$\frac{1}{30}$	9.1338E−04	2.00
		$\frac{1}{40}$	5.1445E−04	2.00
	0.9	$\frac{1}{10}$	8.4475E−03	
		$\frac{1}{20}$	2.1541E−03	1.97
		$\frac{1}{30}$	9.8047E−04	1.94
		$\frac{1}{40}$	5.6455E−04	1.92

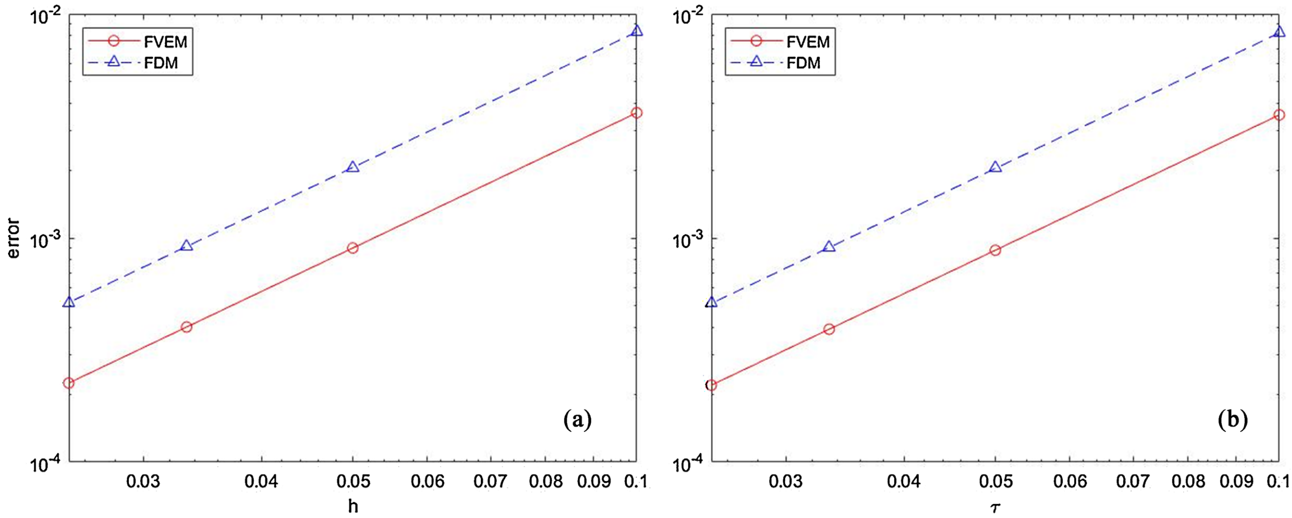


Figure 2. Error comparison between the two methods when $\alpha = 0.3$. (a) Spatial errors for u_h; (b) Temporal errors for u_h.

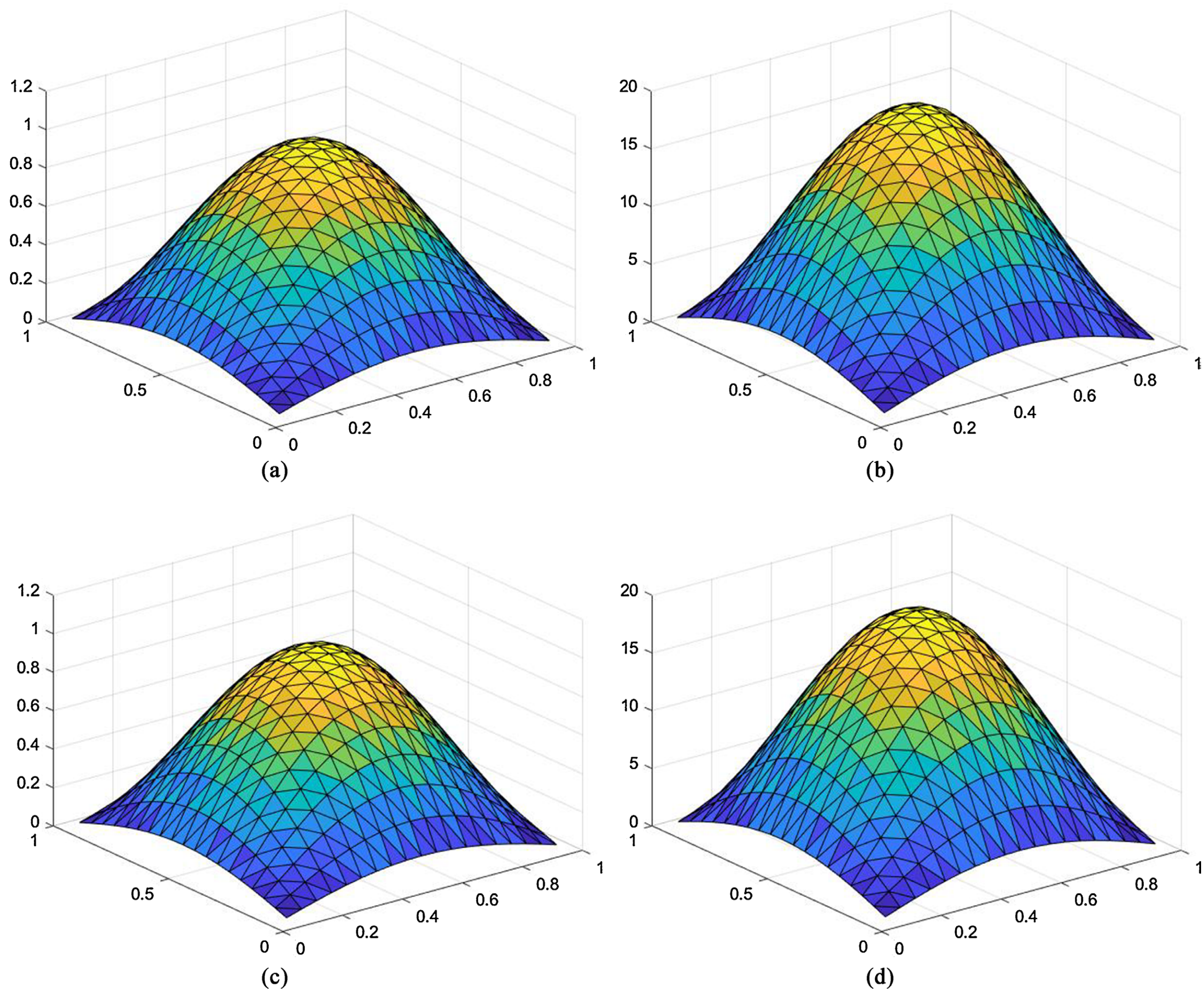


Figure 3. Numerical solution and exact solution in Example 1 at $T = 1$ with $\alpha = 0.3$, $\tau = h^2 = \frac{1}{400}$. (a) Numerical solution u_h; (b) Numerical solution v_h; (c) Exact solution u; (d) Exact solution v.

Example 2. We consider the problem (1.1) with the following initial and boundary conditions

$$u(x,y,t)=0,(x,y,t)\in\partial\Omega\times[0,T],$$
$$\Delta u(x,y,t)=0,(x,y,t)\in\partial\Omega\times[0,T],$$
$$u(x,y,t)=2(1-t^4)\sin(5\pi x)\sin(5\pi y),(x,y,t)\in\bar{\Omega}\times[-s,0],$$
$$\Delta u(x,y,t)=100\pi^2(1-t^4)\sin(5\pi x)\sin(5\pi y),(x,y,t)\in\bar{\Omega}\times[-s,0],$$

and source function

$$f(x,y,t)=u^2(x,y,t-s)+u(x,y,t).$$

We choose the numerical solution with $h=\frac{1}{120}$ and $\tau=h^2$ as the approximating exact solution.

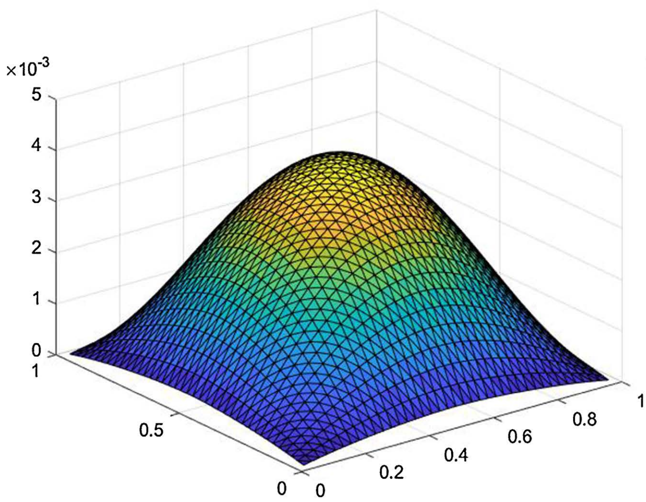
In Table 5 and Table 6, the error in L^2-norms are listed with delay parameter

Table 5. Errors and spatial convergence orders of finite volume scheme with $h=\sqrt{\tau}$ at $T=1$.

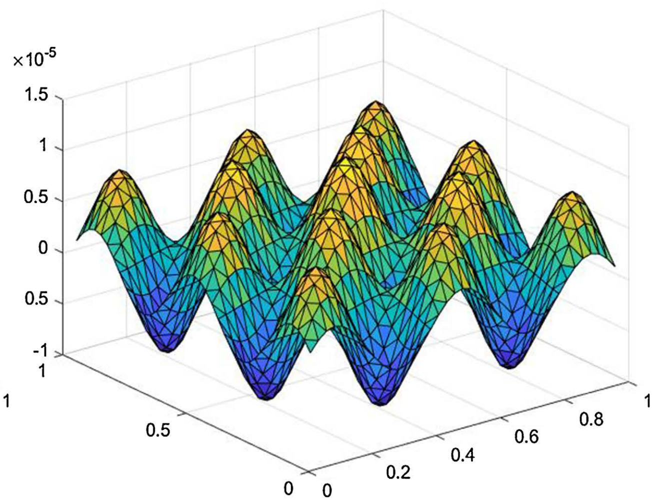
s	α	h	$\|u-u_h\|$	order
0.3	0.2	$\frac{1}{10}$	1.5841E−06	
		$\frac{1}{20}$	6.8489E−07	1.21
		$\frac{1}{30}$	3.2876E−07	1.81
		$\frac{1}{40}$	1.8257E−07	2.04
0.5	0.2	$\frac{1}{10}$	1.1614E−06	
		$\frac{1}{20}$	5.0213E−07	1.21
		$\frac{1}{30}$	2.4103E−07	1.81
		$\frac{1}{40}$	1.3385E−07	2.04
0.7	0.2	$\frac{1}{10}$	6.8891E−07	
		$\frac{1}{20}$	2.9784E−07	1.21
		$\frac{1}{30}$	1.4297E−07	1.81
		$\frac{1}{40}$	7.9394E−08	2.04

Table 6. Errors and time convergence orders of finite volume scheme with $h=\tau$ at $T=1$.

s	α	τ	$\|u-u_h\|$	order
0.3	0.2	$\frac{1}{10}$	1.5708E−06	
		$\frac{1}{20}$	6.7389E−07	1.22
		$\frac{1}{30}$	3.2117E−07	1.83
		$\frac{1}{40}$	1.7739E−07	2.06
0.5	0.2	$\frac{1}{10}$	1.1452E−06	
		$\frac{1}{20}$	4.8876E−07	1.23
		$\frac{1}{30}$	2.3183E−07	1.84
		$\frac{1}{40}$	1.2759E−07	2.08
0.7	0.2	$\frac{1}{10}$	6.7562E−07	
		$\frac{1}{20}$	2.8683E−07	1.24
		$\frac{1}{30}$	1.3542E−07	1.85
		$\frac{1}{40}$	7.4271E−08	2.09



(a)



(b)

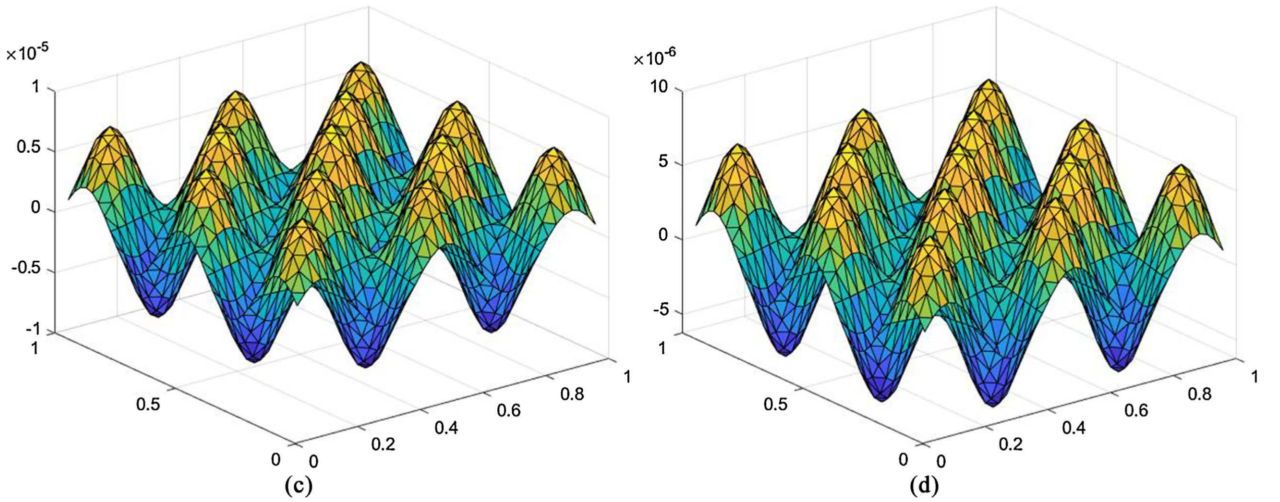


Figure 4. Numerical solution in Example 2 with $\alpha = 0.3$, $s = 0.2$. (a) $t = 0.2$; (b) $t = 0.4$; (c) $t = 0.6$; (d) $t = 0.8$.

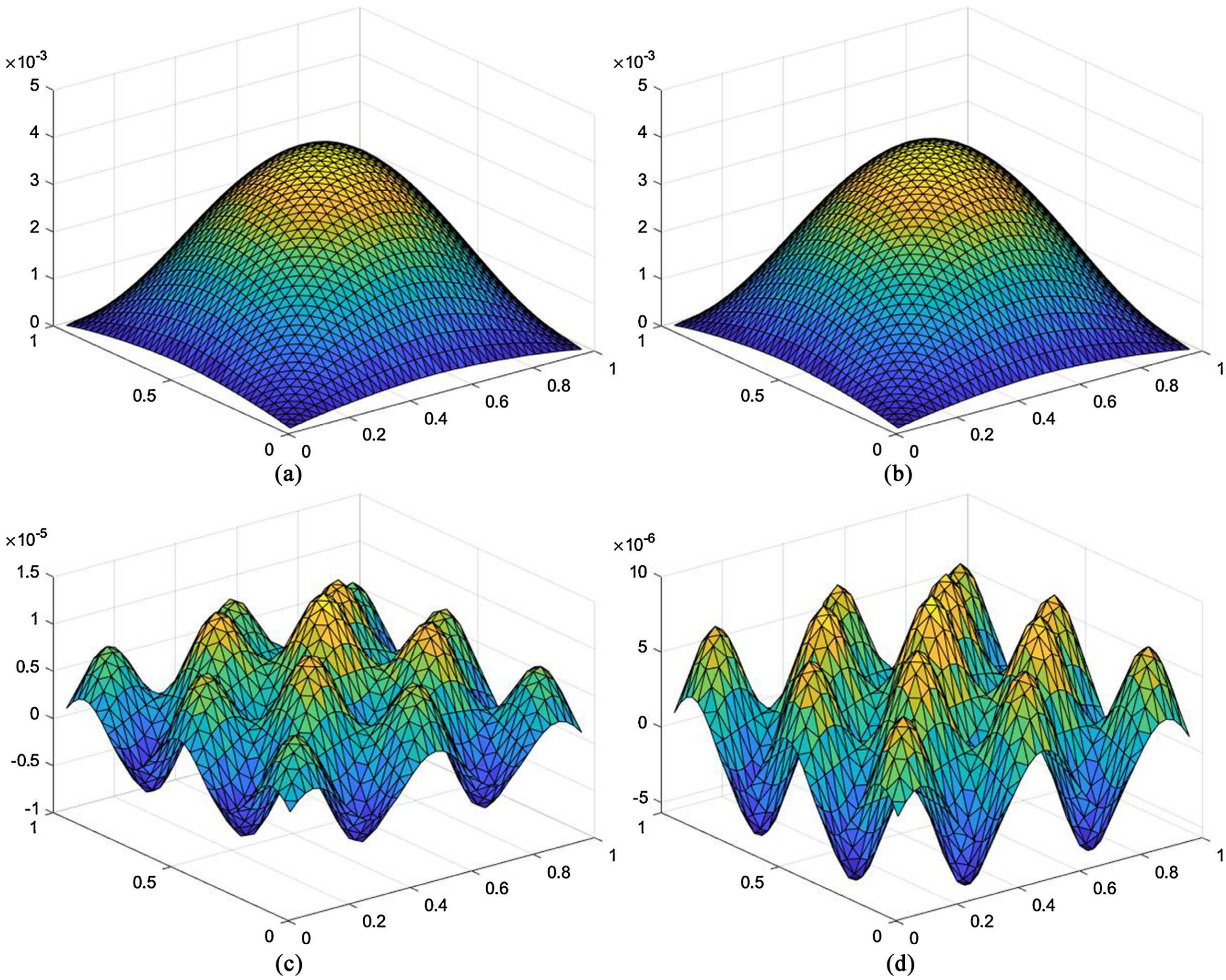


Figure 5. Numerical solution in Example 2 with $\alpha = 0.3$, $s = 0.5$. (a) $t = 0.2$; (b) $t = 0.4$; (c) $t = 0.6$; (d) $t = 0.8$.

$s = 0.2$. To testify the convergence order in spatial direction, we keep varying h

and $\tau = h^2$. Similarly, to testify the convergence order in temporal direction, we keep varying τ and $h = \tau$. From tables we can see that numerical results are consistent with the theoretical results. In **Figure 4** and **Figure 5**, the numerical solution with the time evolution are plotted when the delay s = 0.2, 0.5, respectively. It indicate that the delay effect on the behavior of the numerical solution.

6. Conclusion

In this article, a finite volume element scheme, which can achieve the convergence rate of $O\left(h^2 + \tau^{2-\alpha}\right)$, has been derived for the two-dimensional time-fractional nonlinear fourth-order diffusion equation with time delay. The stability and convergence analyses of our scheme are proved by Gronwall lemma. Then, the numerical experiments are given to verify the effectiveness of the proposed scheme.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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