

# Multi Parameter Adaptive Estimation of Reaction-Diffusion Equation

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## Abstract

This study addresses the problem of parameter estimation for a one-dimensional reaction-diffusion equation, involving both unknown domain parameters and unknown boundary parameters. The proposed approach utilizes the least-squares method to design an adaptive law for parameter estimation. The convergence analysis demonstrates that under persistent excitation conditions, the adaptive law converges exponentially to zero, indicating that the estimated parameters converge exponentially to their true values. Numerical simulations confirm the effectiveness. Furthermore, it is shown that within a certain range of the reaction coefficient, the auxiliary system acts as a state observer, providing an accurate estimate of the system state at an exponential rate.

## Keywords

Parameter Estimation, Adaptive Law, Backstepping Transformation

## 1. Introduction

Reaction-diffusion equations, as an important representative of parabolic equations, describe substance diffusion and reaction processes in partial differential equations (PDEs). In practical applications, reaction-diffusion equations are extensively used and play a vital role in multiple fields ([1]-[8]). For example, in the field of biomedical science, a reaction-diffusion model with PDE constraints was proposed for low-grade glioma growth in [9]. Gholami *et al.* used a simplified spatial discretization of the Gauss-Newton method to estimate the spatial distribution and diffusion extent of tumor concentration at different locations through parameter estimation. In the field of chemical engineering, a reaction-diffusion PDE system was developed to model chemical composition and morphological changes in electrodeposition by Sgura *et al.* in [10]. Using singular value decomposition (SVD) and gradient descent algorithm, parameter estimation was performed. The research findings can guide experimental design in

various fields, including electrodeposition and electrochemical modification of alloys under different conditions.

In the past decade, there have been developments in the research of parameter estimation problems for parabolic equations, particularly in the field of adaptive online estimation ([11]-[15]). For example, in [12], Ji and Zhang applied finite-dimensional and infinite-dimensional backstepping transformations to reaction-diffusion PDEs with unknown parameters which were fully coupled. They designed a least-squares adaptive parameter estimator and an adaptive observer, achieving exponential convergence of the parameter estimator and state observer under persistent excitation conditions. In [15], the distributed state and unknown parameters in infinite-dimensional systems were estimated by employing an adaptive observer and sampled data. Ahmed-Ali *et al.* provided a persistent excitation condition for the exponential convergence of the observer and gave guidelines for selecting the observer gain based on the space sampling interval. Different parameters in these equations reflect various physical and chemical processes. For example, the diffusion coefficient indicates substance propagation speed and extent in space. Thus, studying parameter estimation problems in reaction-diffusion equations is of great significance ([16]-[18]). Adaptive methods can adjust parameter estimates in real-time when parameters are unknown or time-varying, allowing them to better adapt to system changes. This is useful for complex PDE systems. This paper will employ an adaptive least-squares method for parameter estimation. Compared to traditional least-squares, it has higher adaptability, better robustness, and stronger anti-interference capability.

In the design and analysis process of the parameter estimator in this paper, the backstepping transformation is of great importance. It is a highly significant method that plays a crucial role in addressing a variety of challenges related to PDEs such as state observation, parameter estimation, output regulation, and more ([13] [19]-[24]). Specifically, in [13], to address the state observation and online parameter estimation problems of an ODE-PDE coupled system, Ahmed-Ali *et al.* combined the backstepping method and Kalman design method to design an observer and parameter estimator. They successfully overcame the challenge of unmeasurable connection points between the ODE and PDE components. In [24], the SOC and SOH estimation problem of batteries was modeled as a PDE model. Krstic *et al.* combined the backstepping method and Padé-based parameter identification to propose an adaptive observer. By utilizing real-time state and parameter information, they achieved an accurate estimation of the battery state and parameters.

In this paper, we consider the following reaction-diffusion equation

$$\begin{cases} \xi_t(\vartheta, t) = \xi_{\vartheta\vartheta}(\vartheta, t) + a\xi(\vartheta, t) + \phi(\vartheta, t)^T k_0, & \vartheta \in (0, 1), t > 0, \\ \xi_\vartheta(0, t) = -k_1\xi(0, t) + \psi(\vartheta, t)^T k_2, & t \geq 0, \\ \xi(1, t) = U(t), & t \geq 0, \\ y(t) = \xi(0, t), & t \geq 0. \end{cases} \quad (1)$$

where  $\xi_{gg}(\mathcal{G}, t)$  represents the diffusion term, and  $a\xi(\mathcal{G}, t)$  represents the reaction term.  $\xi(\mathcal{G}, t)$  is the unknown state of the system, where  $\mathcal{G}$  and  $t$  denote space and time, respectively. The coefficient  $a$  is a known reaction coefficient,  $\phi(\mathcal{G}, t) \in C^1([0, 1] \times [0, +\infty); \mathbf{R}^{n_0})$  and  $\psi(\mathcal{G}, t) \in C^2([0, 1] \times [0, +\infty); \mathbf{R}^{n_2})$  known vector functions,  $U(t)$  the system control, and  $y(t)$  the system measurement.  $k_0 \in \mathbf{R}^{n_0}$ ,  $k_1 \in \mathbf{R}$ , and  $k_2 \in \mathbf{R}^{n_2}$  are the unknown constant parameters that we want to estimate.

The objective of this paper is to utilize adaptive methods to find appropriate online estimation functions for accurate estimation of unknown parameters  $k_0, k_1, k_2$ .

The contributions include the following two aspects:

- Compared to [15], where the number of sensors is required to be equal to the number of unknown parameters, this paper achieves the online estimation of multiple parameters by utilizing a unique observation at the zero point, with the number of unknown parameters exceeding the number of sensors.
- Compared to the parabolic equation in [25], the reaction-diffusion equation investigated in this paper has an additional linear term, that is the reaction term. This introduces increased complexity to the parameter estimation task, and at the same time, the number of unknown parameters has also increased.

Our study proceeds in the following order. Firstly, we define the error system and filters, followed by the design of the parameter adaptive estimation law in Section 2. Then, in Section 3, the convergence analysis of the parameter adaptive estimation law is conducted. Additionally, the convergence condition for the state observer is provided when the reaction coefficient is within a specific range. In Section 4, we validate the effectiveness of parameter and state estimation through numerical simulations. Finally, in Section 5, a summary and an outlook for future research are given.

## 2. Parameter Adaptive Law Design

For convenience, we introduce the following  $(n_0 + n_2 + 1)$ -dimensional column vector

$$\kappa = \begin{bmatrix} k_0 \\ k_1 \\ k_2 \end{bmatrix}. \quad (2)$$

Define two functions

$$g(\mathcal{G}, \zeta) = -(a+c)(1-\mathcal{G}) \frac{I_1\left(\sqrt{(a+c)(\mathcal{G}-\zeta)(2-\mathcal{G}-\zeta)}\right)}{\sqrt{(a+c)(\mathcal{G}-\zeta)(2-\mathcal{G}-\zeta)}}, \quad (3)$$

$$\begin{aligned} \alpha(\mathcal{G}) &= g_\zeta(\mathcal{G}, 0) \\ &= \frac{(a+c)(1-\mathcal{G})}{\mathcal{G}(2-\mathcal{G})} I_2\left(\sqrt{(a+c)\mathcal{G}(2-\mathcal{G})}\right), \end{aligned} \quad (4)$$

where the constant  $c > 0$  and  $I_n$  represents the modified Bessel function of

order  $n$ , which can be expressed as a series

$$I_n(\vartheta) = \sum_{h=0}^{\infty} \frac{\left(\frac{\vartheta}{2}\right)^{n+2h}}{h3!(h+n)!}. \quad (5)$$

Next, we design an auxiliary system as follows

$$\begin{cases} \hat{\xi}_t(\vartheta, t) = \hat{\xi}_{\vartheta\vartheta}(\vartheta, t) + a\hat{\xi}(\vartheta, t) + \phi(\vartheta, t)^T \hat{k}_0(t) - \alpha(\vartheta)(\hat{\xi}(0, t) - y(t)) + \rho(\vartheta, t), \\ \hat{\xi}_{\vartheta}(0, t) = -\hat{k}_1(t)\xi(0, t) + \psi(\vartheta, t)^T \hat{k}_2(t), \\ \hat{\xi}(1, t) = U(t), \end{cases} \quad (6)$$

where  $\alpha(\vartheta)$  is defined by (4) and serves as the auxiliary system gain term. In addition,  $\rho(\vartheta, t)$  is an unknown additional gain term, and  $\hat{k}_0(t)$ ,  $\hat{k}_1(t)$ , and  $\hat{k}_2(t)$  are unknown estimation functions for the corresponding parameters, which will be designed using the least-squares method in this section. For simplicity, let

$$\hat{\kappa}(t) = \begin{bmatrix} \hat{k}_0(t) \\ \hat{k}_1(t) \\ \hat{k}_2(t) \end{bmatrix}. \quad (7)$$

The error functions are defined as follows

$$\begin{aligned} \tilde{\xi}(\vartheta, t) &= \hat{\xi}(\vartheta, t) - \xi(\vartheta, t), \\ \tilde{k}_0(t) &= \hat{k}_0(t) - k_0, \quad \tilde{k}_1(t) = \hat{k}_1(t) - k_1, \quad \tilde{k}_2(t) = \hat{k}_2(t) - k_2, \end{aligned} \quad (8)$$

$$\tilde{\kappa}(t) = \hat{\kappa}(t) - \kappa = \begin{bmatrix} \tilde{k}_0(t) \\ \tilde{k}_1(t) \\ \tilde{k}_2(t) \end{bmatrix}.$$

Then we obtain the following error system

$$\begin{cases} \tilde{\xi}_t(\vartheta, t) = \tilde{\xi}_{\vartheta\vartheta}(\vartheta, t) + a\tilde{\xi}(\vartheta, t) + \phi(\vartheta, t)^T \tilde{k}_0(t) - \alpha(\vartheta)\tilde{\xi}(0, t) + \rho(\vartheta, t), \\ \tilde{\xi}_{\vartheta}(0, t) = -\tilde{k}_1(t)\xi(0, t) + \psi(\vartheta, t)^T \tilde{k}_2(t), \\ \tilde{\xi}(1, t) = 0. \end{cases} \quad (9)$$

To better investigate the convergence property of the error system (9) and design parameter adaptive law, three filters are designed.

$$\begin{cases} \mu_{0,t}(\vartheta, t) = \mu_{0,\vartheta\vartheta}(\vartheta, t) + a\mu_0(\vartheta, t) - \alpha(\vartheta)\mu_0(0, t) + \phi^T(\vartheta, t), \\ \mu_{0,\vartheta}(0, t) = 0, \\ \mu_0(1, t) = 0. \end{cases} \quad (10)$$

$$\begin{cases} \mu_{1,t}(\vartheta, t) = \mu_{1,\vartheta\vartheta}(\vartheta, t) + a\mu_1(\vartheta, t) - \alpha(\vartheta)\mu_1(0, t), \\ \mu_{1,\vartheta}(0, t) = -\xi(0, t), \\ \mu_1(1, t) = 0. \end{cases} \quad (11)$$

$$\begin{cases} \mu_{2,t}(\mathcal{G},t) = \mu_{2,gg}(\mathcal{G},t) + a\mu_2(\mathcal{G},t) - \alpha(\mathcal{G})\mu_2(0,t), \\ \mu_{2,g}(0,t) = \psi(\mathcal{G},t)^\top, \\ \mu_2(1,t) = 0. \end{cases} \quad (12)$$

Then we set

$$\theta(x,t) = [\mu_0(\mathcal{G},t) \quad \mu_1(\mathcal{G},t) \quad \mu_2(\mathcal{G},t)]. \quad (13)$$

**Remark.** Inspired by lemma 2 of [26], when  $\phi(\mathcal{G},t)$  belongs to  $C^1([0,1] \times \mathbf{R}^+; \mathbf{R})$ ,  $U(t)$  and  $\psi(\mathcal{G},t)$  belong to  $C^2([0,1] \times \mathbf{R}^+; \mathbf{R})$ , for any initial value  $\xi(\cdot, 0) \in H^1(0,1)$ , system (1) has a unique strong solution. Similarly, it can be shown that the filter (10) - (12) also have unique strong solutions respectively.

The following finite-dimensional backstepping-like transformation is designed as

$$\begin{aligned} \sigma(\mathcal{G},t) &= \tilde{\xi}(\mathcal{G},t) - \mu_0(\mathcal{G},t)\tilde{k}_0(t) - \mu_1(\mathcal{G},t)\tilde{k}_1(t) - \mu_2(\mathcal{G},t)\tilde{k}_2(t) \\ &= \tilde{\xi}(\mathcal{G},t) - \theta(\mathcal{G},t)\tilde{\kappa}(t). \end{aligned} \quad (14)$$

Taking the derivative of the aforementioned transformation (14) with respect to time  $t$ , we have

$$\begin{aligned} \sigma_t(\mathcal{G},t) &= \tilde{\xi}_t(\mathcal{G},t) - \theta_t(\mathcal{G},t)\tilde{\kappa}(t) - \theta(\mathcal{G},t)\dot{\tilde{\kappa}}(t) \\ &= \tilde{\xi}_{gg}(\mathcal{G},t) + a\tilde{\xi}(\mathcal{G},t) + \phi(\mathcal{G},t)^\top \tilde{k}_0(t) - \alpha(\mathcal{G})\tilde{\xi}(0,t) + \rho(\mathcal{G},t) \\ &\quad - \theta_t(\mathcal{G},t)\tilde{\kappa}(t) - \theta(\mathcal{G},t)\dot{\tilde{\kappa}}(t) \\ &= \sigma_{gg}(\mathcal{G},t) + a\sigma(\mathcal{G},t) - \alpha(\mathcal{G})\sigma(0,t) + \rho(\mathcal{G},t) - \theta(\mathcal{G},t)\dot{\tilde{\kappa}}(t) \\ &\quad + [\mu_{0,gg}(\mathcal{G},t) + a\mu_0(\mathcal{G},t) - \alpha(\mathcal{G})\mu_0(0,t) - \mu_{0,t}(\mathcal{G},t) + \phi^\top(\mathcal{G},t)]\tilde{k}_0(t) \\ &\quad + [\mu_{1,gg}(\mathcal{G},t) + a\mu_1(\mathcal{G},t) - \alpha(\mathcal{G})\mu_1(0,t) - \mu_{1,t}(\mathcal{G},t)]\tilde{k}_1(t) \\ &\quad + [\mu_{2,gg}(\mathcal{G},t) + a\mu_2(\mathcal{G},t) - \alpha(\mathcal{G})\mu_2(0,t) - \mu_{2,t}(\mathcal{G},t)]\tilde{k}_2(t). \end{aligned} \quad (15)$$

Furthermore, the boundary conditions satisfy

$$\begin{aligned} \sigma(1,t) &= \tilde{\xi}(1,t) - \theta(1,t)\tilde{\kappa}(t) \\ &= \tilde{\xi}(1,t) - \mu_0(1,t)\tilde{k}_0(t) - \mu_1(1,t)\tilde{k}_1(t) - \mu_2(1,t)\tilde{k}_2(t). \end{aligned} \quad (16)$$

$$\begin{aligned} \sigma_g(0,t) &= \tilde{\xi}_g(0,t) - \theta_g(0,t)\tilde{\kappa}(t) \\ &= -\mu_{0,g}(0,t)\tilde{k}_0(t) + [-\xi(0,t) - \mu_{1,g}(0,t)]\tilde{k}_1(t) \\ &\quad + [\psi(\mathcal{G},t)^\top - \mu_{2,g}(0,t)]\tilde{k}_2(t), \end{aligned} \quad (17)$$

Define the additional feedback term  $\rho(\mathcal{G},t)$  as follows

$$\rho(\mathcal{G},t) = \theta(\mathcal{G},t)\dot{\tilde{\kappa}}(t). \quad (18)$$

As a result, system (9) is transformed into the following form of target system,

$$\begin{cases} \sigma_t(\mathcal{G},t) = \sigma_{gg}(\mathcal{G},t) + a\sigma(\mathcal{G},t) - \alpha(\mathcal{G})\sigma(0,t), \\ \sigma_g(0,t) = 0, \\ \sigma(1,t) = 0. \end{cases} \quad (19)$$

According to lemma 1 in Section 3,  $\sigma(0, t)$  decays exponentially. Therefore, by the transformation (14), when  $t$  is sufficiently large, we have the following approximation,

$$\tilde{\xi}(0, t) \approx \theta(0, t) \tilde{\kappa}(t) \quad (20)$$

Applying the least-squares method with the forgetting factor described in Section 4.3 of [27], we design the following parameter adaptive law,

$$\begin{aligned} \dot{\tilde{\kappa}}(t) &= -\frac{\lambda(t)\theta^T(0, t)}{1+\theta(0, t)\theta^T(0, t)} \tilde{\xi}(0, t), \\ \dot{\lambda}(t) &= \lambda(t) - \frac{\lambda(t)\theta^T(0, t)\theta(0, t)\lambda(t)}{1+\theta(0, t)\theta^T(0, t)}, \end{aligned} \quad (21)$$

where  $\lambda(0) = \lambda_0, \lambda_0 = \lambda_0^T > 0$ .

During this process, we also obtain the following

$$\dot{\lambda}^{-1}(t) = -\lambda^{-1}(t) + \frac{\theta^T(0, t)\theta(0, t)}{1+\theta(0, t)\theta^T(0, t)}. \quad (22)$$

We can summarize the parameter estimator in **Table 1**.

**Table 1.** The structure of parameter estimator.

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**Auxiliary system:**

$$\begin{aligned} \dot{\hat{\xi}}_i(\mathcal{G}, t) &= \hat{\xi}_{i, \mathcal{G}\mathcal{G}}(\mathcal{G}, t) + \alpha \hat{\xi}(\mathcal{G}, t) + \phi(\mathcal{G}, t)^T \hat{k}_0(t) - \alpha(\mathcal{G}) (\hat{\xi}(0, t) - y(t)) + \theta(\mathcal{G}, t) \hat{\kappa}(t), \\ \dot{\hat{\xi}}_{\mathcal{G}}(0, t) &= -\hat{k}_1(t) \xi(0, t) + \psi(\mathcal{G}, t)^T \hat{k}_2(t), \\ \hat{\xi}(1, t) &= U(t). \end{aligned}$$

$$\alpha(\mathcal{G}) = \frac{(a+c)(1-\mathcal{G})}{\mathcal{G}(2-\mathcal{G})} I_2 \left( \sqrt{(a+c)\mathcal{G}(2-\mathcal{G})} \right),$$

$$\text{with } y(t) = \xi(0, t), \theta(\mathcal{G}, t) = [\mu_0(\mathcal{G}, t) \quad \mu_1(\mathcal{G}, t) \quad \mu_2(\mathcal{G}, t)].$$

**Filters:**

$$\mu_{0,i}(\mathcal{G}, t) = \mu_{0,\mathcal{G}\mathcal{G}}(\mathcal{G}, t) + a\mu_0(\mathcal{G}, t) - \alpha(\mathcal{G})\mu_0(0, t) + \phi^T(\mathcal{G}, t),$$

$$\mu_{0,\mathcal{G}}(0, t) = 0, \quad \mu_0(1, t) = 0.$$

$$\mu_{1,i}(\mathcal{G}, t) = \mu_{1,\mathcal{G}\mathcal{G}}(\mathcal{G}, t) + a\mu_1(\mathcal{G}, t) - \alpha(\mathcal{G})\mu_1(0, t),$$

$$\mu_{1,\mathcal{G}}(0, t) = -\xi(0, t), \quad \mu_1(1, t) = 0.$$

$$\mu_{2,i}(\mathcal{G}, t) = \mu_{2,\mathcal{G}\mathcal{G}}(\mathcal{G}, t) + a\mu_2(\mathcal{G}, t) - \alpha(\mathcal{G})\mu_2(0, t),$$

$$\mu_{2,\mathcal{G}}(0, t) = \psi(\mathcal{G}, t)^T, \quad \mu_2(1, t) = 0.$$

**Parameter estimator:**

$$\dot{\tilde{\kappa}}(t) = -\frac{\lambda(t)\theta^T(0, t)}{1+\theta(0, t)\theta^T(0, t)} \tilde{\xi}(0, t),$$

$$\dot{\lambda}(t) = \lambda(t) - \frac{\lambda(t)\theta^T(0, t)\theta(0, t)\lambda(t)}{1+\theta(0, t)\theta^T(0, t)},$$

$$\lambda(0) = \lambda_0; \lambda_0 = \lambda_0^T > 0.$$


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**Remark.** By the existence and uniqueness theorem for ODEs, we can conclude that  $\tilde{\kappa}(t)$  and  $\lambda(t)$  in the parameter adaptive law (21) both exist and are unique.

### 3. Parameter Adaptive Law Analysis

To justify the application of the least-squares method in designing the parameter adaptive law, we first prove the exponential convergence property of the system (19).

**Lemma 1.** For any initial value  $\sigma(\cdot, 0) \in L^2(0, 1)$ , the system defined by Equation (19) has a unique solution  $\sigma(\cdot, t) \in C([0, \infty); L^2(0, 1))$ , and this solution is exponentially stable.

*Proof.* Introduce the following backstepping transformation which is bounded and invertible,

$$\sigma(\vartheta, t) = \eta(\vartheta, t) - \int_0^\vartheta g(\vartheta, \zeta) \eta(\zeta, t) d\zeta, \quad (23)$$

where the kernel function  $g(\vartheta, \zeta)$  satisfies the following PDE,

$$\begin{cases} g_{\vartheta\vartheta}(\vartheta, \zeta) - g_{\zeta\zeta}(\vartheta, \zeta) = -(a+c)g(\vartheta, \zeta), \\ g(\vartheta, \vartheta) = \frac{a+c}{2}(\vartheta-1), \\ g(1, \zeta) = 0. \end{cases} \quad (24)$$

By [28], the aforementioned function (3) is the unique solution of the system (24). Thus, we can obtain the following target system,

$$\begin{cases} \eta_t(\vartheta, t) = \eta_{\vartheta\vartheta}(\vartheta, t) - c\eta(\vartheta, t), \\ \eta_\vartheta(0, t) = 0, \\ \eta(1, t) = 0. \end{cases} \quad (25)$$

By computation, the bounded inverse backstepping transformation of (23) is as follows

$$\eta(\vartheta, t) = \sigma(\vartheta, t) - \int_0^\vartheta h(\vartheta, \zeta) \sigma(\zeta, t) d\zeta, \quad (26)$$

where  $h(\vartheta, \zeta)$ , as the kernel function of the inverse transformation, satisfies the following PDE,

$$\begin{cases} h_{\vartheta\vartheta}(\vartheta, \zeta) - h_{\zeta\zeta}(\vartheta, \zeta) = (a+c)h(\vartheta, \zeta), \\ h(\vartheta, \vartheta) = \frac{a+c}{2}(\vartheta-1), \\ h(1, \zeta) = 0. \end{cases} \quad (27)$$

The system (27) has a unique solution, which can be explicitly expressed as

$$h(\vartheta, \zeta) = -(a+c)(1-\vartheta) \frac{J_1\left(\sqrt{(a+c)(\vartheta-\zeta)(2-\vartheta-\zeta)}\right)}{\sqrt{(a+c)(\vartheta-\zeta)(2-\vartheta-\zeta)}}, \quad (28)$$

where  $J_n$  satisfies

$$J_n(\vartheta) = \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{\vartheta}{2}\right)^{n+2p}}{p!(p+n)!}. \quad (29)$$

Inspired by [29], regardless of the initial values  $\eta(\cdot, 0) \in L^2(0, 1)$ , the system (25) has a unique solution  $\eta(\cdot, t) \in C([0, \infty); L^2(0, 1))$  that exhibits exponential convergence. Moreover, the convergence speed is dependent on the magnitude of the constant parameter  $c$ , with larger values of  $c$  resulting in faster convergence.

Therefore, due to the bounded and invertible transformation (23) and (26), system (19) also has a unique solution that exhibits exponential convergence. Moreover, the convergence speed also increases as  $c$  becomes larger. Hence, it is reasonable to use the least-squares method to design the parameter adaptive law (21).  $\square$

Now, we give the definition of persistent excitation assumption (**PE assumption**) motivated by [27] as follows.

**PE assumption.**  $\theta(0, t)$  is referred to persistently exciting, if and only if

$$\exists \delta, \varepsilon > 0, \forall t > 0: \int_0^{t+\delta} \frac{\theta^T(0, \zeta)\theta(0, \zeta)}{1+\theta(0, \zeta)\theta^T(0, \zeta)} d\zeta > \varepsilon I. \quad (30)$$

where  $I$  is the identity matrix.

The PE assumption ensures that  $\lambda^{-1}(t)$  is a positive definite matrix, and there exist two positive constants  $m_0, m_1$  that satisfy the following equation,

$$m_0 \leq \lambda^{-1}(t) \leq m_1, \quad t \geq 0. \quad (31)$$

**Remark.** Satisfying the PE condition is very important for the stability and parameter convergence of adaptive control systems. It ensures that adaptive algorithms can correctly identify and track changes in system parameters.

Choose the following Lyapunov function

$$\Upsilon(t) = \tilde{\kappa}^T(t) \lambda^{-1}(t) \tilde{\kappa}(t). \quad (32)$$

Differentiating  $\Upsilon(t)$ , we obtain the following,

$$\begin{aligned} \dot{\Upsilon}(t) &= \tilde{\kappa}^T(t) \dot{\lambda}^{-1}(t) \tilde{\kappa}(t) + 2\tilde{\kappa}^T(t) \lambda^{-1}(t) \dot{\tilde{\kappa}}(t) \\ &= \tilde{\kappa}^T(t) \left[ -\lambda^{-1}(t) + \frac{\theta^T(0, t)\theta(0, t)}{1+\theta(0, t)\theta^T(0, t)} \right] \tilde{\kappa}(t) - 2\tilde{\kappa}^T(t) \frac{\theta^T(0, t)\tilde{\xi}(0, t)}{1+\theta(0, t)\theta^T(0, t)} \\ &= -\tilde{\kappa}^T(t) \lambda^{-1}(t) \tilde{\kappa}(t) + \tilde{\kappa}^T(t) \frac{\theta^T(0, t)\theta(0, t)}{1+\theta(0, t)\theta^T(0, t)} \tilde{\kappa}(t) \\ &\quad - 2\tilde{\kappa}^T(t) \frac{\theta^T(0, t)}{1+\theta(0, t)\theta^T(0, t)} [\sigma(0, t) + \theta(0, t) \tilde{\kappa}(t)] \\ &= -\tilde{\kappa}^T(t) \lambda^{-1}(t) \tilde{\kappa}(t) - \tilde{\kappa}^T(t) \frac{\theta^T(0, t)\theta(0, t)}{1+\theta(0, t)\theta^T(0, t)} \tilde{\kappa}(t) - 2 \frac{\tilde{\kappa}^T(t)\theta^T(0, t)\sigma(0, t)}{1+\theta(0, t)\theta^T(0, t)} \\ &\leq -\tilde{\kappa}^T(t) \lambda^{-1}(t) \tilde{\kappa}(t) - \frac{\tilde{\kappa}^T(t)\theta^T(0, t)\theta(0, t)\tilde{\kappa}(t)}{1+\theta(0, t)\theta^T(0, t)} + \sigma^2(0, t) \\ &\quad + \left\| \frac{\tilde{\kappa}^T(t)\theta^T(0, t)}{1+\theta(0, t)\theta^T(0, t)} \right\|^2 \\ &\leq -\Upsilon(t) + \sigma^2(0, t). \end{aligned} \quad (33)$$

The first inequality is derived from Young's inequality here. By lemma 3.1,

$\sigma(0, t)$  converges to zero at an exponential rate, so as  $\sigma^2(0, t)$ . So based on the above inequality,  $\Upsilon(t)$  exhibits exponential stability by comparison lemma ([30]). Therefore, under the assumption of persistent excitation, due to the positive definite and bounded nature of  $\lambda^{-1}(t)$ , it can be concluded that  $\tilde{\kappa}(t)$  also converges to zero at an exponential rate. This is equivalent to the estimation function  $\hat{\kappa}(t)$  converging exponentially to the parameter  $\kappa$  itself. By employing the Lyapunov function method, we have demonstrated the effectiveness of our parameter adaptive estimation law (21).

Let  $\alpha_M = \max_{0 \leq \vartheta \leq 1} \alpha(\vartheta)$ . The following lemma is provided and its proof will be presented in the appendix.

**Lemma 2.** When the reaction coefficient  $a < \frac{\pi^2}{4}$ , it is possible to choose a constant parameter  $c$  to ensure that the filter has the property of bounded  $\int_0^1 \|\theta(\vartheta, t)\|^2 dx$ . Specifically,

1) When  $0 \leq a < \frac{\pi^2}{4}$ , by choosing a positive constant  $c$  such that

$0 < \alpha_M < \pi^2 - 4a$ , we can obtain the boundedness of  $\int_0^1 \|\theta(\vartheta, t)\|^2 dx$ .

2) When  $a < 0$ , by choosing any  $0 < c < -a$ , or selecting a positive constant  $c$  such that  $0 < \alpha_M < \pi^2 - 4a$  and  $c > -a$ , we can achieve the boundedness of  $\int_0^1 \|\theta(\vartheta, t)\|^2 dx$ .

Furthermore, from (14), the following inequality can be obtained,

$$\sqrt{\int_0^1 \tilde{\xi}^2(\vartheta, t) d\vartheta} \leq \sqrt{\int_0^1 \sigma^2(\vartheta, t) d\vartheta} + \|\tilde{\kappa}(t)\| \sqrt{\int_0^1 \|\theta(\vartheta, t)\|^2 d\vartheta}. \tag{34}$$

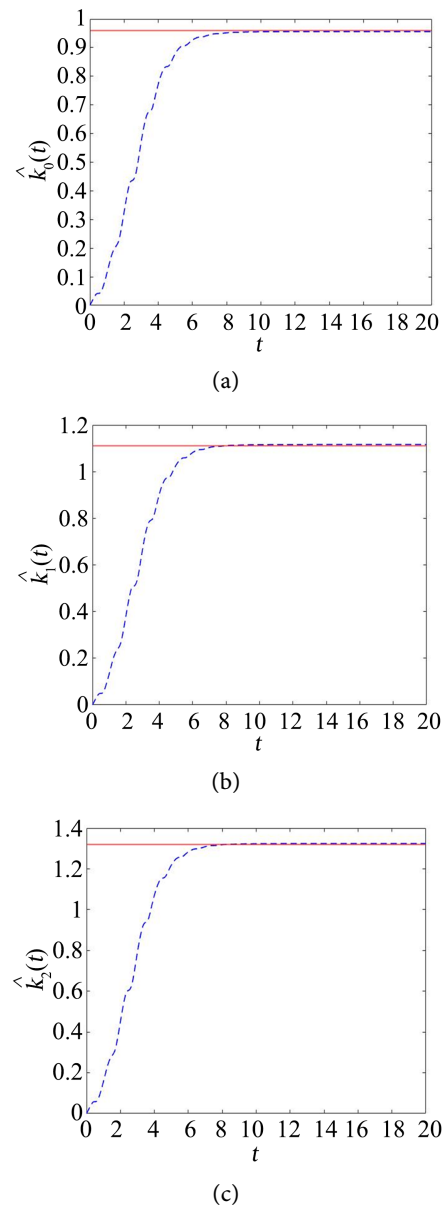
From the lemma 2, when  $a < \frac{\pi^2}{4}$ , by selecting an appropriate constant  $c$ , we can obtain that  $\int_0^1 \tilde{\xi}^2(\vartheta, t) d\vartheta$  exponentially approaches zero, and the convergence speed increases with the increase of  $c$ . In this case, the designed auxiliary system (6) becomes an observer for the original system (1), enabling accurate estimation of the unknown states of system (1). The state observer can provide real-time information, providing accurate state feedback for subsequent control design. The combination of adaptive parameter estimation and state observation can enable further research, such as in the area of fault detection. This is beneficial for fault diagnosis, as it can better identify abnormal changes in system parameters, and thus quickly locate the fault.

### 4. Numerical Simulation

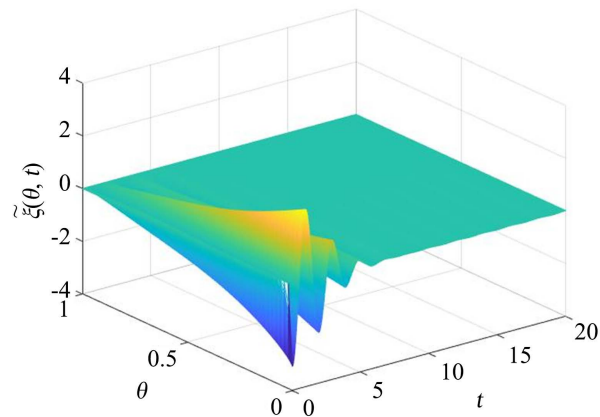
To demonstrate the effectiveness of parameter estimation and state observation, numerical simulations are conducted. Set  $a = \frac{\pi^2}{6}$ ,  $\phi^T(\vartheta, t) = 2e^{0.2\vartheta} \xi(0, t)$  and  $\psi^T(\vartheta, t) = -\frac{3}{2} \sin(\pi t)$  in (1). Thus we consider the system as follows,

$$\begin{aligned} \xi_t(\vartheta, t) &= \xi_{\vartheta\vartheta}(\vartheta, t) + \frac{\pi^2}{6} \xi(\vartheta, t) + 2k_0 e^{0.2\vartheta} \xi(0, t), \quad \vartheta \in (0, 1), t > 0, \\ \xi_\vartheta(0, t) &= -k_1 \xi(0, t) - \frac{3}{2} k_2 \sin(\pi t), \quad t \geq 0. \end{aligned} \tag{35}$$

where  $k_0$ ,  $k_1$  and  $k_2$  are set to 0.96, 1.11 and 1.32 respectively, assuming them as unknown parameters. With  $c = 6$ ,  $\lambda_0 = I_2$  and utilizing the parameter estimation law provided in **Table 1**, we obtained the parameter estimation functions  $\hat{k}_0(t)$ ,  $\hat{k}_1(t)$  and  $\hat{k}_2(t)$ . The three graphs in **Figure 1** depict the evolution of the three-parameter estimation functions over time. The images demonstrate that after 8 seconds, the parameter estimation functions converge well to their respective parameter values. Additionally, by using the state observer (6), we plot the evolution of the error system over time in **Figure 2**. From the graph, it can be observed that after 7 seconds, the error system tends to zero, indicating that the state observer accurately infers the true state of the system.



**Figure 1.** The evolution of the parameter estimation functions over time. In the graph, the three parts labeled as (a), (b), and (c) correspond to the evolution of the estimation functions for  $k_0$ ,  $k_1$  and  $k_2$  respectively.



**Figure 2.** The evolution of the error system state over time.

## 5. Conclusions

The paper has addressed the problem of estimating multiple parameters in reaction-diffusion equations using only a single measurement. We employed a combination of backstepping transformation, forgetting factor least-squares, Lyapunov function, and other methods. The exponential convergence of the parameter adaptive law was demonstrated. Additionally, after establishing the boundedness of the filters, we also confirmed the exponential convergence of the state observer when the reaction coefficient was within a specific range. Finally, the convergence properties of both the parameter adaptive law and the state observer were validated through MATLAB simulations using the differential method. This confirmed the accuracy of our estimation.

Of course, this paper also has certain limitations, such as the fact that the state measurement does not cover all situations, and the reaction coefficient needs to be confined within a specified range. Based on this, we can explore the future research possibilities of utilizing the backstepping method and least-squares method for the estimation of reaction coefficients or diffusion coefficients in reaction-diffusion equations. Furthermore, we can attempt to overcome the limitations on the range of reaction coefficients in the state observation problem of Equation (1).

## Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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## Appendix: Proof of Lemma 2

First, let us consider the boundedness of  $\int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta$ . The following Lyapunov function is defined as

$$\Upsilon_0(\mu_0) = \frac{1}{2} \int_0^1 \mu_0(\vartheta, t) \mu_0^T(\vartheta, t) d\vartheta. \quad (\text{A1})$$

Taking the derivative of the function (A1), we have

$$\begin{aligned} \dot{\Upsilon}_0(\mu_0) &= \int_0^1 \mu_0(\vartheta, t) \mu_{0,t}^T(\vartheta, t) d\vartheta \\ &= \int_0^1 \mu_0(\vartheta, t) \mu_{0,\vartheta\vartheta}^T(\vartheta, t) d\vartheta + a \int_0^1 \mu_0(\vartheta, t) \mu_0^T(\vartheta, t) d\vartheta \\ &\quad - \int_0^1 \alpha(\vartheta) \mu_0(\vartheta, t) \mu_0^T(0, t) d\vartheta + \int_0^1 \mu_0(\vartheta, t) \phi(\vartheta, t) d\vartheta \\ &= - \int_0^1 \mu_{0,\vartheta\vartheta}(\vartheta, t) \mu_{0,\vartheta}^T(\vartheta, t) d\vartheta + a \int_0^1 \mu_0(\vartheta, t) \mu_0^T(\vartheta, t) d\vartheta \\ &\quad - \int_0^1 \alpha(\vartheta) \mu_0(\vartheta, t) [\mu_0^T(0, t) - \mu_0^T(\vartheta, t)] d\vartheta \\ &\quad - \int_0^1 \alpha(\vartheta) \mu_0(\vartheta, t) \mu_0^T(\vartheta, t) d\vartheta + \int_0^1 \mu_0(\vartheta, t) \phi(\vartheta, t) d\vartheta \end{aligned} \quad (\text{A2})$$

The last equation above is derived using the method of integration by parts. By the Young's inequality, for any  $m > 0$  and  $n > 0$ , we have

$$\begin{aligned} \dot{\Upsilon}_0(\mu_0) &\leq - \int_0^1 \|\mu_{0,\vartheta\vartheta}(\vartheta, t)\|^2 d\vartheta + a \int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta - \int_0^1 \alpha(\vartheta) \|\mu_0(\vartheta, t)\|^2 d\vartheta \\ &\quad + \frac{1}{2m} \int_0^1 \alpha(\vartheta) \|\mu_0(0, t) - \mu_0(\vartheta, t)\|^2 d\vartheta + \frac{m}{2} \int_0^1 \alpha(\vartheta) \|\mu_0(\vartheta, t)\|^2 d\vartheta \\ &\quad + \frac{1}{2n} \int_0^1 \|\phi(\vartheta, t)\|^2 d\vartheta + \frac{n}{2} \int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta \\ &\leq - \int_0^1 \|\mu_{0,\vartheta\vartheta}(\vartheta, t)\|^2 d\vartheta + a \int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta - \int_0^1 \alpha(\vartheta) \|\mu_0(\vartheta, t)\|^2 d\vartheta \\ &\quad + \frac{2}{m\pi^2} \cdot \alpha_M \int_0^1 \|\mu_{0,\vartheta}(\vartheta, t)\|^2 d\vartheta + \frac{m}{2} \int_0^1 \alpha(\vartheta) \|\mu_0(\vartheta, t)\|^2 d\vartheta \\ &\quad + \frac{1}{2n} \int_0^1 \|\phi(\vartheta, t)\|^2 d\vartheta + \frac{2n}{\pi^2} \int_0^1 \|\mu_{0,\vartheta}(\vartheta, t)\|^2 d\vartheta, \end{aligned} \quad (\text{A3})$$

where the last inequality is derived from the Wirtinger's inequality presented in [31]. Thus,

$$\begin{aligned} \dot{\Upsilon}_0(\mu_0) &\leq - \left(1 - \frac{2\alpha_M}{m\pi^2} - \frac{2n}{\pi^2}\right) \int_0^1 \|\mu_{0,\vartheta\vartheta}(\vartheta, t)\|^2 d\vartheta + a \int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta \\ &\quad - \left(1 - \frac{m}{2}\right) \int_0^1 \alpha(\vartheta) \|\mu_0(\vartheta, t)\|^2 d\vartheta + \frac{1}{2n} \int_0^1 \|\phi(\vartheta, t)\|^2 d\vartheta \end{aligned} \quad (\text{A4})$$

Let  $m = 2$ . From lemma 2, the choice of parameter  $c$  ensures that  $0 < 1 - \frac{\alpha_M}{\pi^2}$  and  $0 < 1 - \frac{\alpha_M}{\pi^2} - \frac{4a}{\pi^2}$ . Choose  $n$  such that  $0 < \frac{2n}{\pi^2} < \min\left\{1 - \frac{\alpha_M}{\pi^2}, 1 - \frac{\alpha_M}{\pi^2} - \frac{4a}{\pi^2}\right\}$ . Then, we obtain  $1 - \frac{\alpha_M}{\pi^2} - \frac{2n}{\pi^2} > 0$  and  $1 - \frac{\alpha_M}{\pi^2} - \frac{2n}{\pi^2} - \frac{4a}{\pi^2} > 0$ .

Therefore, by the Wirtinger's inequality,

$$\begin{aligned}
\dot{\Upsilon}_0(\mu_0) &\leq -\frac{\pi}{4} \left(1 - \frac{\alpha_M}{\pi^2} - \frac{2n}{\pi^2}\right) \int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta \\
&\quad + a \int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta + \frac{1}{2n} \int_0^1 \|\phi(\vartheta, t)\|^2 d\vartheta \\
&= -\frac{\pi^2}{4} \left(1 - \frac{\alpha_M}{\pi^2} - \frac{2n}{\pi^2} - \frac{4a}{\pi^2}\right) \int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta + \frac{1}{2n} \int_0^1 \|\phi(\vartheta, t)\|^2 d\vartheta
\end{aligned} \tag{A5}$$

This is equivalent to

$$\dot{\Upsilon}_0(\mu_0) \leq -\frac{\pi^2}{2} \left(1 - \frac{\alpha_M}{\pi^2} - \frac{2n}{\pi^2} - \frac{4a}{\pi^2}\right) \Upsilon_0(\mu_0) + \frac{1}{2n} \int_0^1 \|\phi(\vartheta, t)\|^2 d\vartheta. \tag{A6}$$

Based on the assumption that  $\phi(\vartheta, t)$  is bounded, we can conclude that  $\int_0^1 \|\mu_0(\vartheta, t)\|^2 d\vartheta$  is also bounded.

Next, we consider the boundedness of  $\int_0^1 \mu_1^2(\vartheta, t) d\vartheta$ . Let us choose the following Lyapunov function

$$\Upsilon_1(\mu_1) = \frac{1}{2} \int_0^1 \mu_1^2(\vartheta, t) d\vartheta. \tag{A7}$$

Taking the derivative and using the method of integration by parts, we obtain

$$\begin{aligned}
\dot{\Upsilon}_1(\mu_1) &= \int_0^1 \mu_1(\vartheta, t) \mu_{1,t}(\vartheta, t) d\vartheta \\
&= \int_0^1 \mu_1(\vartheta, t) [\mu_{1,\vartheta\vartheta}(\vartheta, t) + a\mu_1(\vartheta, t) - \alpha(\vartheta)\mu_1(0, t)] d\vartheta \\
&= \xi(0, t)\mu_1(0, t) - \int_0^1 \mu_{1,\vartheta\vartheta}^2(\vartheta, t) d\vartheta + a \int_0^1 \mu_1^2(\vartheta, t) d\vartheta \\
&\quad - \int_0^1 \alpha(\vartheta)\mu_1(\vartheta, t) [\mu_1(0, t) - \mu_1(\vartheta, t)] d\vartheta - \int_0^1 \alpha(\vartheta)\mu_1^2(\vartheta, t) d\vartheta.
\end{aligned} \tag{A8}$$

By the Young's inequality, for any  $p > 0$  and  $q > 0$ , we have

$$\begin{aligned}
\dot{\Upsilon}_1(\mu_1) &\leq -\int_0^1 \mu_{1,\vartheta\vartheta}^2(\vartheta, t) d\vartheta + a \int_0^1 \mu_1^2(\vartheta, t) d\vartheta - \int_0^1 \alpha(\vartheta)\mu_1^2(\vartheta, t) d\vartheta \\
&\quad + \frac{p}{2} \mu_1^2(0, t) + \frac{1}{2p} \xi^2(0, t) + \frac{q}{2} \int_0^1 \alpha(\vartheta)\mu_1^2(\vartheta, t) d\vartheta \\
&\quad + \frac{1}{2q} \int_0^1 \alpha(\vartheta) [\mu_1(0, t) - \mu_1(\vartheta, t)]^2 d\vartheta \\
&\leq -\int_0^1 \mu_{1,\vartheta\vartheta}^2(\vartheta, t) d\vartheta + a \int_0^1 \mu_1^2(\vartheta, t) d\vartheta - \left(1 - \frac{q}{2}\right) \int_0^1 \alpha(\vartheta)\mu_1^2(\vartheta, t) d\vartheta \\
&\quad + \frac{1}{2p} \xi^2(0, t) + \frac{p}{2} \int_0^1 \mu_{1,\vartheta\vartheta}^2(\vartheta, t) d\vartheta + \frac{2}{q\pi^2} \alpha_M \int_0^1 [-\mu_{1,\vartheta}(\vartheta, t)]^2 d\vartheta,
\end{aligned} \tag{A9}$$

where the last inequality is obtained by the Wirtinger's inequality. Then,

$$\begin{aligned}
\dot{\Upsilon}_1(\mu_1) &\leq a \int_0^1 \mu_1^2(\vartheta, t) d\vartheta - \left(1 - \frac{q}{2}\right) \int_0^1 \alpha(\vartheta)\mu_1^2(\vartheta, t) d\vartheta + \frac{1}{2p} \xi^2(0, t) \\
&\quad - \left(1 - \frac{p}{2} - \frac{2\alpha_M}{q\pi^2}\right) \int_0^1 \mu_{1,\vartheta\vartheta}^2(\vartheta, t) d\vartheta
\end{aligned} \tag{A10}$$

Choose  $q = 2$  and  $p$  satisfies  $0 < \frac{p}{2} < \min\left\{1 - \frac{\alpha_M}{\pi^2}, 1 - \frac{\alpha_M}{\pi^2} - \frac{4a}{\pi^2}\right\}$ . Then, we obtain  $1 - \frac{p}{2} - \frac{\alpha_M}{\pi^2} > 0$  and  $1 - \frac{p}{2} - \frac{\alpha_M}{\pi^2} - \frac{4a}{\pi^2} > 0$ .

Therefore, by the Wirtinger's inequality,

$$\begin{aligned}\dot{Y}_1(\mu_1) &\leq a \int_0^1 \mu_1^2(\vartheta, t) d\vartheta + \frac{1}{2p} \xi^2(0, t) - \frac{\pi^2}{4} \left(1 - \frac{p}{2} - \frac{\alpha_M}{\pi^2}\right) \int_0^1 \mu_1^2(\vartheta, t) d\vartheta \\ &= -\frac{\pi^2}{4} \left(1 - \frac{p}{2} - \frac{\alpha_M}{\pi^2} - \frac{4a}{\pi^2}\right) \int_0^1 \mu_1^2(\vartheta, t) d\vartheta + \frac{1}{2p} \xi^2(0, t)\end{aligned}\quad (\text{A11})$$

That is

$$\dot{Y}_1(\mu_1) \leq -\frac{\pi^2}{2} \left(1 - \frac{p}{2} - \frac{\alpha_M}{\pi^2} - \frac{4a}{\pi^2}\right) Y_1(\mu_1) + \frac{1}{2p} \xi^2(0, t) \quad (\text{A12})$$

By assumption, we know that  $\psi(\vartheta, t)$  is bounded, so as  $\xi(0, t)$ . Thus we can deduce that  $\int_0^1 \mu_1^2(\vartheta, t) d\vartheta$  is also bounded.

Finally, to prove the boundedness of  $\int_0^1 \|\mu_2(\vartheta, t)\|^2 d\vartheta$ , we define the following Lyapunov function

$$Y_2(\mu_2) = \frac{1}{2} \int_0^1 \mu_2(\vartheta, t) \mu_2^T(\vartheta, t) d\vartheta. \quad (\text{A13})$$

Similar to the proof for the boundedness of  $\int_0^1 \mu_1^2(\vartheta, t) d\vartheta$ , we can establish the boundedness of  $\int_0^1 \|\mu_2(\vartheta, t)\|^2 d\vartheta$ . The details are omitted here.