

Discrete Transparent Boundary Conditions for Burgers' Equation

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Abstract

This paper is devoted to numerically solving the Burgers' equation in unbounded region, which describes the nonlinear wave propagation and diffusion effect. How to numerically and efficiently solve this problem remains a challenge due to the principal difficulties of not only the unboundedness but also nonlinearity. This paper aims to design the discrete transparent boundary conditions of Burgers' equation to overcome the unboundedness. The presence of a nonlinear term in the equation makes it challenging to derive suitable artificial boundary conditions. To deal with the nonlinear term, a linear parabolic equation is obtained by applying the well-known Cole-Hopf transformation to the Burgers' equation. Employing the \mathcal{Z} -transformation, we then establish the discrete transparent boundary conditions for the linearized problem. Subsequently, the original equation is reduced to an initial boundary value problem, that can be efficiently solved by the finite difference method. The numerical analysis is reported rigorously. Some numerical results are given to demonstrate the accuracy and feasibility of the proposed method.

Keywords

Burgers' Equation, Cole-Hopf Transformation, Discrete Transparent Boundary Conditions

1. Introduction

The nonlinear partial differential equations are widely applied to describe the complex phenomena in various fields. Burgers' equation is a classical nonlinear model that was studied by Burgers [1] to describe nonlinear wave propagation, particularly in the fluid mechanics. Burgers' equation plays a crucial role in capturing nonlinear propagation and diffusion effects, as it incorporates the nonlinear advection and dissipation terms to simulate the physical phenomena of wave

motion. Over the past several decades, Burgers' equation has been extensively investigated by numerous researchers across a broad spectrum of applications in science and engineering, including traffic flow, surface disturbances electromagnetic waves, density waves, cosmology and seismology [2] [3].

The aim of this paper is to numerically solve the following Burgers' equation on unbounded domain

$$u_t(x, t) = \mu u_{xx} - uu_x + f(x, t), \quad \forall (x, t) \in \mathbb{R} \times [0, T], \quad (1)$$

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}, \quad (2)$$

$$u(x, t) \rightarrow 0, \quad |x| \rightarrow \infty, \quad (3)$$

where $\mu > 0$ is kinematic viscosity parameter of the fluid, which corresponds to an inverse of Reynolds number. $f(x, t)$ and $u_0(x)$ are the source term and initial value with compact support, respectively, namely $\text{supp}\{f(x, t)\} \subset B_0^1$ and $\text{supp}\{u_0(x)\} \subset B_0^1$, where $B_0^1 = \{x \mid |x| \leq R\}$ and R is a constant.

Hopf [4] and Cole [5] independently introduced the Cole-Hopf transformation to convert the nonlinear Burgers' equation into a linear thermal equation, enabling the derivation of an exact solution for arbitrary initial conditions. However, the analytical solution involves an infinite series that converges very slowly, especially for small values of the viscosity parameter. Numerous efforts have been dedicated to exploring the numerical solution of the Burgers' equation on bounded domains, such as the finite difference method [6] [7], and the B-spline method [8] [9]. More recently, based on the integral formulation, Egidi *et al.* derived two numerical methods for solving the initial boundary value problem of the two-dimensional Burgers' equation on the plane in [10]. Li [11] investigated both the exact and numerical solutions of the time fractional Burgers' equation by applying the Cole-Hopf transformation. Zhang *et al.* developed two numerical schemes for generalized Burgers' equation, and analyzed the conservative invariants and convergence based on the cut-off function method in [12]. Based on the Haar wavelet collocation method coupled with a nonstandard finite difference scheme, Verma *et al.* [13] presented an efficient numerical method for a class of generalized Burgers' equation. For more work on numerical solution of Burgers' equation, the reader is referred to [14]-[16] and the references therein.

However, to the best of the authors' knowledge, limited research has been conducted in the literature on the numerical solution of the Burgers' equation on unbounded domains. This scarcity can be attributed to the challenges posed by the unbounded nature of the physical domain and the inherent nonlinearity of the equation. There are several methods to overcome the unboundedness, including the infinite element method, perfectly matched layer method and artificial boundary method. The artificial boundary method, recognized for its efficiency, has proven successfully in solving numerous partial differential equations across diverse fields. This success is achieved by devising appropriate artificial boundary conditions (ABCs), extending to equations such as Schrödinger-type equations, parabolic equation, wave equation, and time fractional partial differential equations.

The ABCs can be categorized into global (exact), local and discrete ABCs (also called discrete transparent boundary conditions). Based on the Fourier series expansion and the special functions techniques, Han and Huang [17] proposed a class of exact ABCs for the two-dimensional Schrödinger equation by introducing a circular artificial boundary. Ehrhardt and Zheng [18] derived exact ABCs for general problems with periodic structures at infinity, including the Schrödinger operator and a second-order hyperbolic equation in two dimensions. Zheng [19] designed the exact ABCs for the sine-Gordon equation based on nonlinear spectral analysis. Since the exact ABCs usually involve the integral operator, the local ABCs are developed to save the computation through the approximation of the integral operator. Zhang and his co-authors [20]-[22] designed the local ABCs for nonlinear Schrödinger equation and nonlinear wave equation through the unified approach, which spirit is the well-known operator splitting method. The fundamental concept behind discrete transparent boundary conditions (DTBCs) involves discretizing the equation in terms of both time and space, followed by the derivation of suitable ABCs for the fully discrete problem utilizing the \mathcal{Z} -transformation. Arnold and Ehrhardt [23] pioneered the development of transparent boundary conditions for wide angle parabolic equations, specifically in the context of underwater acoustics. Ehrhardt [24] extended this approach to address linear parabolic equations with discrete transparent boundary conditions. At the same time, the discrete transparent boundary conditions of Schrödinger equation were introduced in [25]. Interested readers are directed to relevant review papers for further exploration [26]-[28].

The exact ABCs with nonlinear forms were proposed for Burgers' equation by using the Cole-Hopf transformation in [29]. Based on the Cole-Hopf transformation and Fourier series expansion, Wu and Zhang [30] derived exact ABCs for two-dimensional Burgers' equation on a circular artificial boundary and a series of approximating boundary conditions. Given the limited exploration of ABCs for the Burgers' equation, this paper concentrates on formulating DTBCs for Burgers' equation. The objective is to address the challenges posed by the unboundedness of the physical domain and the inherent nonlinearity of the equation, utilizing the Cole-Hopf transformation.

The paper is organized as follows. In Section 2, we revisit the Cole-Hopf transformation of Burgers' equation, aimed at mitigating its nonlinearity, resulting in a linear parabolic equation. Based on the discretization of the parabolic equation and \mathcal{Z} -transformation, the DTBCs of Burgers' equation are constructed to obtain a reduced initial boundary value problem (IBVP) in Section 3. In Section 4, the stability of the reduced IBVP is rigorously analyzed. The feasibility and effectiveness of the DTBCs are illustrated by numerical examples in Section 5. Finally, a conclusion is presented in Section 6.

2. Cole-Hopf Transformation of Burgers' Equation

In order to design the ABCs for Burgers' equation, two boundaries for (1)-(3) are

introduced to truncate the unbounded domain as follows

$$\Gamma_l = \{(x, t) | x = x_l, 0 \leq t \leq T\}, \quad \Gamma_r = \{(x, t) | x = x_r, 0 \leq t \leq T\},$$

the unbounded domain $\mathbb{R} \times [0, T]$ is divided into three parts

$$\Omega_l = \{(x, t) | x \leq x_l, 0 \leq t \leq T\},$$

$$\Omega_r = \{(x, t) | x \geq x_r, 0 \leq t \leq T\},$$

$$\Omega_i = \{(x, t) | x_l < x < x_r, 0 \leq t \leq T\}.$$

To get the artificial boundary condition on Γ_r , the Burgers' equation on the right semi-infinite domain Ω_r is considered. Due to the compact support of the source term and initial value, we have

$$u_t(x, t) = \mu u_{xx} - uu_x, \quad \text{in } \Omega_r, \tag{4}$$

$$u(x, 0) = 0, \quad \text{in } \Omega_r, \tag{5}$$

$$u(x, t) \rightarrow 0, \quad |x| \rightarrow \infty. \tag{6}$$

In order to circumvent the nonlinear term, the Cole-Hopf transformation [29] is utilized to transform Burgers' equation into linear parabolic equation, let

$$\psi(x, t) = -\int_x^{+\infty} u(y, t) dy, \tag{7}$$

then

$$\psi_t(t) = -\int_x^{+\infty} u_t(y, t) dy = v u_x - \frac{1}{2} u^2, \tag{8}$$

$$\psi_x = u, \quad \psi_{xx} = u_x. \tag{9}$$

Substituting (7)-(9) into Equation (4), we get

$$\psi_t(t) + \frac{1}{2} \psi_x^2 - v \psi_{xx} = 0. \tag{10}$$

Denote $v = \kappa(\psi) - 1$ with $\kappa(\psi) = \exp\left(-\frac{\psi}{2\mu}\right)$, then we have the following linear problem

$$v_t = \mu v_{xx}, \quad \forall (x, t) \in \Omega_r \times [0, T], \tag{11}$$

$$v|_{t=0} = 0, \quad \forall x \in [x_r, +\infty), \tag{12}$$

$$v \rightarrow 0, \quad x \rightarrow +\infty, \tag{13}$$

$$v|_{x=x_r} = v(x_r, t), \quad t \in [0, T], \tag{14}$$

where $v(x_r, t)$ is unknown function. Our aim is to obtain the unknown function $v(x_r, t)$ to yield a well-posed problem.

3. Finite Difference Scheme for the Reduced IBVP with DTBCs

3.1. Discrete Transparent Boundary Conditions

In order to construct the DTBCs of the Burgers' equation, we revisit the DTBCs of the linear parabolic equation. We will only consider the right boundary

condition, due to the left boundary condition can be analogously obtained. We derive the DTBCs for the completely discrete problem based on the weighted average or θ -scheme ($0 \leq \theta \leq 1$). The parabolic equation (11) is discretized on the uniform grid points $x_j = jh$ ($j = 0, 1, \dots, J$), $t_n = n\tau$ ($n = 0, 1, 2, \dots, N$)

$$D_t^+ v_j^n - \mu D_x^2 v_j^{n+\theta} = 0, \tag{15}$$

with

$$v_j^n \approx v(x_j, t_n), \quad v_j^{n+\theta} = \theta v_j^{n+1} + (1-\theta)v_j^n, \quad 0 \leq \theta \leq 1,$$

where the difference quotients are defined

$$D_x^2 v_j^{n+\theta} = \frac{v_{j+1}^{n+\theta} - 2v_j^{n+\theta} + v_{j-1}^{n+\theta}}{h^2}, \quad D_t^+ v_j = \frac{v_j^{n+1} - v_j^n}{\tau}.$$

The aim of the constructed DTBCs is to completely avoid any numerical reflections at the boundary with no additional computational costs. In order to obtain the DTBCs, we consider the following discrete right exterior problem ($j \geq J-1$):

$$\Delta_t^+ v_j^n = r \Delta_x^2 v_j^{n+\theta}, \tag{16}$$

where $r = \frac{\mu\tau}{h^2}$ denotes the mesh ratio, the difference operators are $\Delta_t^+ = \tau D_t^+$, $\Delta_x^2 = h^2 D_x^2$.

Using \mathcal{Z} -transformation

$$\mathcal{Z}\{v_j^n\} = \hat{v}_j(z) = \sum_{n=0}^{\infty} v_j^n z^{-n},$$

and the initial data $v_j^0 = 0$ ($j \geq J-2$), we get

$$\frac{1}{r} \frac{z-1}{\theta z - \theta + 1} \hat{v}_j(z) = \Delta_x^2 \hat{v}_j(z), \quad j \geq J-1. \tag{17}$$

There are two linear independent solutions of the second-order difference Equation (17)

$$\hat{v}_j(z) = q_{1,2}^{j+1}(z), \quad j \geq J-1,$$

where $q_{1,2}(z)$ are the roots of the following equation

$$q^2 - 2 \left[1 + \frac{1}{r} \frac{z-1}{2(z\theta + 1 - \theta)} \right] q + 1 = 0.$$

In order to seek the decreasing mode (as $j \rightarrow \infty$), we have to require $|q_2(z)| > 1$ and obtain (using $q_1(z)q_2(z) = 1$) the \mathcal{Z} -transformation of right DTBCs as

$$\hat{v}_{J-1}(z) = q_2(z) \hat{v}_J(z) \tag{18}$$

Similarly, the \mathcal{Z} -transformation of left DTBCs reads

$$\hat{v}_1(z) = q_2(z) \hat{v}_0(z). \tag{19}$$

with $|q_1(z)| < 1$.

The $q_{1,2}(z)$ can be rewritten as

$$q_{1,2}(z) = 1 + \frac{1}{2(1-\theta)} \frac{1}{r} \left(\frac{1}{\theta} \frac{z}{z + \frac{1-\theta}{\theta}} - 1 \right) \pm \frac{\sqrt{Az^2 - 2Bz + C}}{2(\theta z + 1 - \theta)},$$

where

$$\begin{aligned} A &= 1 + 4r\theta, \\ B &= 1 - 2r(1 - 2\theta), \\ C &= 1 - 4r(1 - \theta). \end{aligned}$$

For the inverse \mathcal{Z} -transformation, we obtain

$$\begin{aligned} \frac{\sqrt{Az^2 - 2Bz + C}}{2(\theta z + 1 - \theta)} &= \frac{Az^2 - 2Bz + C}{2z(\theta z + 1 - \theta)} F(\lambda z, \sigma) \\ &= \frac{1}{\sqrt{A}} \left\{ \frac{A}{2\theta} + \frac{C}{2z(1-\theta)} - \frac{E}{2(\theta z + 1 - \theta)} \right\} F(\lambda z, \sigma), \end{aligned}$$

where

$$E = \frac{1-\theta}{\theta} A + 2B + \frac{\theta}{1-\theta},$$

and

$$F(z, \sigma) = \frac{z}{\sqrt{z^2 - 2\sigma z + 1}}, \quad \lambda = \frac{\sqrt{A}}{\sqrt{C}}, \quad \sigma = \frac{B}{\sqrt{AC}}.$$

Applying the inverse \mathcal{Z} -transformation, we have

$$\begin{aligned} &\mathcal{Z}^{-1} \left\{ \frac{\sqrt{Az^2 - 2Bz + C}}{2(\theta z + 1 - \theta)} \right\} \\ &= \frac{1}{\sqrt{A}} \left\{ \frac{A}{2\theta} \delta_n^0 + \frac{C}{2(1-\theta)} \delta_n^1 + \frac{E}{2(1-\theta)} \left[\left(-\frac{1-\theta}{\theta} \right)^n - \theta_n^0 \right] \right\} * \tilde{P}_n(\sigma) \\ &= \frac{1}{\sqrt{A}} \left\{ \frac{A}{2\theta} \tilde{P}_n(\sigma) + \frac{C}{2(1-\theta)} \tilde{P}_{n-1}(\sigma) + \frac{E}{2(1-\theta)} \sum_{k=0}^{n-1} \left(-\frac{1-\theta}{\theta} \right)^{n-k} \tilde{P}_k(\sigma) \right\}, \end{aligned}$$

where $*$ denotes the discrete convolution. Finally, we get

$$\begin{aligned} \mathcal{Z}^{-1} \{ q_{1,2}(z) \} &= \left[1 - \frac{1}{2r(1-\theta)} \right] \delta_n^0 + \frac{(-1)^n}{2r\theta(1-\theta)} \left(\frac{1-\theta}{\theta} \right)^n \pm \frac{1}{r} \left\{ \frac{A}{2\theta} \tilde{P}_n(\sigma) \right. \\ &\quad \left. + \frac{C}{2(1-\theta)} \tilde{P}_{n-1}(\sigma) + \frac{1}{2\theta(1-\theta)^2} \sum_{k=0}^{n-1} \left(-\frac{1-\theta}{\theta} \right)^{n-k} \tilde{P}_k(\sigma) \right\}. \end{aligned}$$

The obtained DTBCs can be rewritten as follows

$$v_1^n = l^{(n)} * v_0^n = \sum_{k=1}^n l^{(n-k)} v_0^k, \tag{20}$$

$$v_{j-1}^n = l^{(n)} * v_j^n = \sum_{k=1}^n l^{(n-k)} v_j^k, \tag{21}$$

with the convolution coefficients $l^{(n)}$ for $0 < \theta < 1$ given by,

$$l^{(0)} = 1 + \frac{1 + \sqrt{A}}{2\theta r},$$

$$l^{(n)} = \frac{(-1)^n}{2r\theta(1-\theta)} \left(\frac{1-\theta}{\theta} \right)^n + \frac{1}{r\sqrt{A}} \left(\frac{A}{2\theta} \tilde{P}_n(\sigma) + \frac{C}{2(1-\theta)} \tilde{P}_{n-1}(\sigma) \right. \\ \left. + \frac{1}{2\theta(1-\theta)^2} \sum_{k=0}^{n-1} \left(-\frac{1-\theta}{\theta} \right)^{n-k} \tilde{P}_k(\sigma) \right), \tag{22}$$

for $n \geq 1$. As $\theta = 1$, the convolution coefficients are given as follows

$$l^{(0)} = 1 + \frac{1 + \sqrt{A}}{2r},$$

$$l^{(1)} = -\frac{1}{2r} \left(1 + \frac{B}{\sqrt{A}} \right),$$

$$l^{(n)} = \frac{1}{2r\sqrt{A}} \left(A\tilde{P}_n(\sigma) - 2B\tilde{P}_{n-1}(\sigma) + C\tilde{P}_{n-2}(\sigma) \right), \quad n \geq 2,$$

where $\tilde{P}_n(\sigma) = \lambda^n P_n(\sigma)$ denotes the ‘‘damped’’ Legendre polynomials and δ_n^0 is the Kronecker symbol.

Substituting (20)-(21) back into Burgers’ equation, we have

$$v_x = \kappa'(\psi)u, \quad G(t) = \kappa'(\psi)|_{x=x_r} = -\frac{1}{2\mu} \exp\left(-\frac{\psi}{2\mu}\right) \Big|_{x=x_r}. \tag{23}$$

In order to discretize an infinite integral, we assume

$$g_r(t) = -\frac{1}{2\nu} \psi|_{x=x_r} = \frac{1}{2\nu} \int_{x_r}^{\infty} u(y,t) dy, \quad g_r(0) = 0,$$

we have

$$\frac{dg_r(t)}{dt} = -\frac{1}{2} u_x(x_r, t) + \frac{1}{4\mu} u(x_r, t)^2,$$

integrating the above equation, we get

$$g_r(t) = \int_0^t \left(-\frac{1}{2} u_x(x_r, \tau) + \frac{1}{4\mu} u(x_r, \tau)^2 \right) d\tau.$$

The discretization is obtained as follows

$$g_r^n = g_r^n + \frac{1}{2} (Q_r^n + Q_r^{n-1}) \tau, \tag{24}$$

$$Q_r^p = -\frac{u_M^p - u_{M-2}^p}{4h} + \frac{(u_{M-1}^p)^2}{4\mu}. \tag{25}$$

Similarly, we have the following discretization of left boundary

$$g_l^n = g_l^n + \frac{1}{2} (Q_l^n + Q_l^{n-1}) \tau, \tag{26}$$

$$Q_l^p = -\frac{u_2^p - u_0^p}{4h} + \frac{(u_1^p)^2}{4\mu}. \tag{27}$$

The discretization of (23) can be obtained on the right boundary

$$\left(v_j^n\right)_x = g_r^n u_j^n. \tag{28}$$

Similarly, we have the discretization on the left boundary

$$\left(v_j^n\right)_x = g_l^n u_j^n. \tag{29}$$

The equations (20)-(21) are treated as follows

$$\left(v_1^n\right)_x = \sum_{k=1}^n l^{(n-k)} \left(v_0^k\right)_x, \tag{30}$$

$$\left(v_{J-1}^n\right)_x = \sum_{k=1}^n l^{(n-k)} \left(v_J^k\right)_x. \tag{31}$$

Substituting equations (28)-(29) into (30)-(31)

$$g_l^n u_1^n = \sum_{k=1}^n l^{(n-k)} g_l^k u_0^k, \tag{32}$$

$$g_r^n u_{M-1}^n = \sum_{k=1}^n l^{(n-k)} g_r^k u_M^k. \tag{33}$$

The transparent boundary conditions (32)-(33) of Burgers' equation are obtained. Then, the original Burgers' equation defined on unbounded domain is reduced an initial boundary value problem on bounded domain with the initial value.

3.2. The Asymptotic Behaviour of the Convolution Coefficients

In this subsection, we use the progressive state of convolution coefficient to re-establish the DTBCs. The convolution coefficients for $n \rightarrow \infty$ has the following asymptotic form

$$l^{(n)} - \frac{(-1)^n}{2\theta(1-\theta)r} \left(\frac{1-\theta}{\theta}\right)^n \sim \frac{(-1)^n}{2r\theta(1-\theta)^2 \sqrt{A}} \left(\frac{1-\theta}{\theta}\right)^n \sum_{k=0}^{n-1} \left(-\frac{\theta}{1-\theta}\right)^k \tilde{P}_k(\sigma). \tag{34}$$

Using

$$\lim_{n \rightarrow \infty} \frac{1}{\sqrt{A}} \sum_{k=0}^{n-1} \left(-\frac{\theta}{1-\theta}\right)^k \tilde{P}_k(\sigma) = 1 - \theta,$$

The Equation (34) can be written as follows

$$l^{(n)} \sim \frac{(-1)^n}{\theta(1-\theta)r} \left(\frac{1-\theta}{\theta}\right)^n, \quad 0 < \theta < 1, \quad \text{as } n \rightarrow \infty.$$

We adopt the following formats in practice

$$s^{(n)} = \theta l^{(n)} + (1-\theta)l^{(n-1)}, \quad n \geq 1, \quad s^{(0)} = \theta l^{(0)}, \quad 0 < \theta \leq 1. \tag{35}$$

Lemma 1. (Formula of Laplace-Heine) [24] Let μ be an arbitrary real or complex number which does not belong to the closed segment $[-1,1]$. Then as $n \rightarrow \infty$,

$$P_n(\sigma) \cong \frac{1}{\sqrt{2\pi n}} (\sigma^2 - 1)^{\frac{1}{4}} \left\{ \sigma + \sqrt{\sigma^2 - 1} \right\}^{n + \frac{1}{2}}.$$

The “damped” legendre polynomials $\tilde{P}_n(\sigma) = \lambda^{-n} P_n(\sigma)$ can be derived from the following recursive formula

$$\tilde{P}_{n+1}(\sigma) = \frac{2n+1}{n+1} \mu \lambda^{-1} \tilde{P}_n(\sigma) - \frac{n}{n+1} \lambda^{-2} \tilde{P}_{n-1}(\sigma), \quad n \geq 0, \quad (36)$$

among $\tilde{P}_0 \equiv 1, \tilde{P}_{-1} \equiv 0$. Applying (36), we have

$$s^{(n)} = -\frac{\sqrt{A} \tilde{P}_n(\sigma) - \lambda^{-2} \tilde{P}_{n-2}(\sigma)}{2r}, \quad n \geq 2.$$

Thus, the discrete transparent boundary conditions become

$$\theta g_l^n u_1^n - \theta s^{(0)} g_l^n u_0^n = \sum_{k=1}^{n-1} s^{(n-k)} g_l^k u_0^k - (1-\theta) g_l^{n-1} u_1^{n-1}, \quad (37)$$

$$\theta g_r^n u_{M-1}^n - \theta s^{(0)} g_r^n u_M^n = \sum_{k=1}^{n-1} s^{(n-k)} g_r^k u_M^k - (1-\theta) g_r^{n-1} u_{M-1}^{n-1}, \quad n \geq 1. \quad (38)$$

We note that since the convolution coefficients ($\theta=1$) are consistent in (32)-(33). Then, the discrete transparent boundary conditions can be obtained.

4. Stability Analysis

The stability of the DTBCs was proved in this section. Divide it into M and N parts of equal length in space $[x_l, x_r]$ and time $[0, T]$, where M and N are positive integers. Therefore, we set the spatial step as $h = (x_r - x_l)/M$ and the time step as $\tau = T/N$, respectively. The mesh grid $x_i = x_l + ih$ ($i = 0, 1, \dots, M$) and $t_n = n\tau$ ($n = 0, 1, \dots, N$). Some notations are given

$$\delta_t u_i^{n+\frac{1}{2}} = \frac{u_i^{n+1} - u_i^n}{\tau}, \quad \delta_x^2 u_i^{n+\frac{1}{2}} = \frac{u_{i+1}^{n+\frac{1}{2}} - 2u_i^{n+\frac{1}{2}} + u_{i-1}^{n+\frac{1}{2}}}{h^2},$$

$$\delta_x u_i^{n+\frac{1}{2}} = \frac{u_{i+1}^{n+\frac{1}{2}} - u_{i-1}^{n+\frac{1}{2}}}{2h}, \quad u_i^{n+\frac{1}{2}} = \frac{u_i^n + u_i^{n+1}}{2}, \quad \|u^n\|_2^2 = h \sum_{i=1}^{M-1} |u_i^n|^2.$$

The Burgers’ equation in the interior domain can be discretized by applying the Crank-Nicolson scheme

$$\delta_t u_i^{n+\frac{1}{2}} + u_i^n \delta_x u_i^{n+\frac{1}{2}} - \mu \delta_x^2 u_i^{k+\frac{1}{2}} = 0, \quad (39)$$

with $i = 1, \dots, M - 1$.

Lemma 2. (Plancherel’s Theorem [24]) If $\hat{f}(z) = \mathcal{Z}\{f_n\}$ exists for $|z| > R_f \geq 0$ and $\hat{g}(z) = \mathcal{Z}\{g_n\}$ for $|z| > R_g \geq 0$ with $R_f R_g < 1$. Then there also exists $\mathcal{Z}\{f_n \bar{g}_n\}$ for $|z| > R_f R_g$ and the following relation holds:

$$\sum_{n=0}^{\infty} f_n \bar{g}_n = \mathcal{Z}\{f_n \bar{g}_n\}(z=1) = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(re^{i\varphi}) \overline{\hat{g}\left(\frac{e^{i\varphi}}{r}\right)} d\varphi.$$

The integration path is the circle \mathcal{C} defined by $R_f < r < \frac{1}{R_g}$ (if $R_g = 0: R_f < r < \infty$). Especially, if $R_f < 1, R_g < 1$ then $r = 1$ can be chosen to

obtain:

$$\sum_{n=0}^{\infty} f_n \bar{\delta}_n = \frac{1}{2\pi} \int_0^{2\pi} \hat{f}(e^{i\varphi}) \overline{\hat{g}(e^{i\varphi})} d\varphi.$$

Theorem 1. Let u^{n+1} be the numerical solution to the fully discrete system. The numerical scheme with constructed DTBCs is unconditionally stable with the following property

$$\|u^{n+1}\|^2 \leq \|u^0\|^2, \tag{40}$$

where $u^0 = u_0(x)$.

Proof. Multiplying both sides of Equation (39) by $hu_i^{n+\frac{1}{2}}$ and summing over $i = 1, \dots, M - 1$, we have

$$h \sum_{i=1}^{M-1} \left[(u_i^{n+1})^2 - (u_i^n)^2 \right] + 2h\tau \sum_{i=1}^{M-1} \left(u_i^{n+\frac{1}{2}} \right) u_i^n \delta_x^+ u_i^{n+\frac{1}{2}} - 2h\mu\tau \sum_{i=1}^{M-1} u_i^{n+\frac{1}{2}} \delta_x^2 u_i^{n+\frac{1}{2}} = 0.$$

Applying the summation by parts formula, we get

$$\begin{aligned} h \sum_{i=1}^{M-1} \left[(u_i^{n+1})^2 - (u_i^n)^2 \right] &= -2h\tau \sum_{i=1}^{M-1} \left(u_i^{n+\frac{1}{2}} \right) u_i^n \delta_x^+ u_i^{n+\frac{1}{2}} + 2h\mu\tau \sum_{i=1}^{M-1} u_i^{n+\frac{1}{2}} \delta_x^2 u_i^{n+\frac{1}{2}} \\ &= -2h\mu\tau \sum_{i=0}^{M-1} \left(D_x^+ u_i^{n+\frac{1}{2}} \right)^2 + 2v\tau u_M^{n+\frac{1}{2}} D_x^- u_M^{n+\frac{1}{2}} \\ &\quad - 2\mu\tau u_0^{n+\frac{1}{2}} D_x^+ u_0^{n+\frac{1}{2}} - 2\tau \sum_{i=1}^{M-1} u_i^n u_M^{n+\frac{1}{2}} u_{M-1}^{n+\frac{1}{2}} + 2\tau \sum_{i=1}^{M-1} u_i^n u_0^{n+\frac{1}{2}} u_1^{n+\frac{1}{2}} \\ &\leq -2\mu\tau u_M^{n+\frac{1}{2}} \left[\left(1 + \frac{1}{\mu} \sum_{i=1}^{M-1} u_i^n \right) u_{M-1}^{n+\frac{1}{2}} - u_M^{n+\frac{1}{2}} \right] \\ &\quad - 2\mu\tau u_0^{n+\frac{1}{2}} \left[\left(1 - \frac{1}{\mu} \sum_{i=1}^{M-1} u_i^n \right) u_1^{n+\frac{1}{2}} - u_0^{n+\frac{1}{2}} \right]. \end{aligned}$$

Therefore, summing over time, we get the following inequality

$$\begin{aligned} \|u^{n+1}\|_2^2 &\leq \|u^n\|_2^2 - 2\mu\tau \sum_{n=0}^N u_M^{n+\frac{1}{2}} \left[\left(1 + \frac{1}{\mu} \|u^n\|^2 \right) u_{M-1}^{n+\frac{1}{2}} - u_M^{n+\frac{1}{2}} \right] \\ &\quad - 2\mu\tau \sum_{n=0}^N u_0^{n+\frac{1}{2}} \left[\left(1 - \frac{1}{\mu} \|u^n\|^2 \right) u_1^{n+\frac{1}{2}} - u_0^{n+\frac{1}{2}} \right] \\ &\leq \|u^n\|_2^2 - 2\mu\tau \sum_{n=0}^N u_M^{n+\frac{1}{2}} \left(u_M^{n+\frac{1}{2}} * \tilde{l}^{(n)} \right) - 2\mu\tau \sum_{n=0}^N u_0^{n+\frac{1}{2}} \left(u_0^{n+\frac{1}{2}} * \tilde{l}^{(n)} \right), \end{aligned}$$

where $\tilde{l}^{(n)} = l^{(n)} - \delta_n^0$. Assuming $q_n = u_M^{n+\frac{1}{2}} * \tilde{l}^{(n)}$ and $k_n = u_0^{n+\frac{1}{2}}$ for $n = 0, 1, \dots, N$. Since we consider the linearized discrete Burgers' equation, then DTBCs can be abbreviated as

$$\begin{aligned} \hat{u}_1(z) &= \varphi_1(z) \hat{u}_0(z), \\ \hat{u}_{M-1}(z) &= \frac{1 + \omega^+}{1 + \omega^-} \varphi_1(z) \hat{u}_M(z), \end{aligned}$$

where $\varphi_1(z)$ is the solution of the discrete Burgers' equation, $\omega^+ = \zeta \frac{\|u_i^n\|^2 h}{\mu}$, $\omega^- = (1-\zeta) \frac{\|u_i^n\|^2 h}{\mu}$, $0 < \zeta \leq 1$. Applying the \mathcal{Z} -transformation for DTBCs, we have

$$\mathcal{Z}\{q_n\} = \hat{q}(z) = \frac{z+1}{2} u_M^N(z) \left[\left(1 - \frac{h\|u^n\|^2}{2\mu} \right) \varphi_1(z) - 1 \right]. \tag{41}$$

Similarly, for k_n we have

$$\mathcal{Z}\{k_n\} = \hat{k}(z) = \frac{z+1}{2} \hat{u}_M^N(z). \tag{42}$$

Using Lemma 2, we obtain

$$\begin{aligned} \sum_{n=0}^N q_n k_n &= \frac{1}{2\pi} \int_0^{2\pi} \hat{q}(z) \overline{\hat{k}(z)} \Big|_{z=e^{i\phi}} d\phi \\ &= \frac{1}{\pi} \int_0^\pi \text{Re} \left\{ \hat{f}(z) \overline{\hat{g}(z)} \right\} \Big|_{z=e^{i\phi}} d\phi. \end{aligned}$$

From equations (41)-(42), we get

$$\sum_{n=0}^N q_n k_n = \frac{1}{4\pi} \int_0^\pi |z+1|^2 |\hat{u}_M^N(z)|^2 \left[\left(1 - \frac{h\|u^n\|^2}{2\mu} \right) \text{Re} \{ \varphi_1(z) \} - 1 \right] \Big|_{z=e^{i\phi}} d\phi.$$

Since $z = i\phi$ ($0 \leq \phi \leq 2\pi$) is on the domain of the circle, we get

$$\left(1 - \frac{h\|u^n\|^2}{2\mu} \right) \text{Re} \{ \varphi_1(z) \} - 1 \geq 0.$$

Finally, we get the stability (40) of the reduced IBVP with the proposed DTBCs.

5. Numerical Results

In this section, some numerical results are presented to demonstrate the feasibility and the efficient of the DTBCs. **Example** Consider the Burgers' equation without a source term, the analytic solution

$$u(x,t) = \frac{\frac{x}{1+t}}{1 + \sqrt{\frac{1+t}{t_0}} \exp\left(\frac{x^2}{4\mu(1+t)}\right)}, \quad t_0 = \exp\left(\frac{1}{8\mu}\right).$$

The computational interval is selected as $[-3, 3]$. Tables 1-2 list the L_∞ and L_1 errors and convergence order for $T=2.5$ and $T=3$ with $\mu=0.1$. One can be seen from **Table 1** and **Table 2**, both cases achieve second-order accuracy as $\tau = h$. **Figure 1** illustrates numerical and exact solutions at different times. We can see that the numerical solution agrees well with the exact solution. No significant reflected wave is observed near the artificial boundary, demonstrating the

effectiveness of the proposed DTBCs.

Table 1. Error and accuracy under L_∞ norm and L_1 norm for $T = 2.5$ with $\mu = 0.1$.

h	L_∞ error	Order	L_1 error	Order
/2	6.60e-02	-	3.30e-02	-
/4	1.16e-02	2.51	7.31e-03	2.17
/8	3.20e-03	1.87	1.52e-03	2.27
/16	8.54e-04	1.89	3.73e-04	2.03

Table 2. Error and accuracy under L_∞ norm and L_1 norm for $T = 3$ with $\mu = 0.1$.

h	L_∞ error	Order	L_1 error	Order
/2	6.21e-02	-	3.05e-02	-
/4	1.43e-02	2.12	8.41e-03	1.86
/8	3.81e-03	1.90	2.12e-03	1.97
/16	9.63e-04	1.99	5.09e-04	2.07

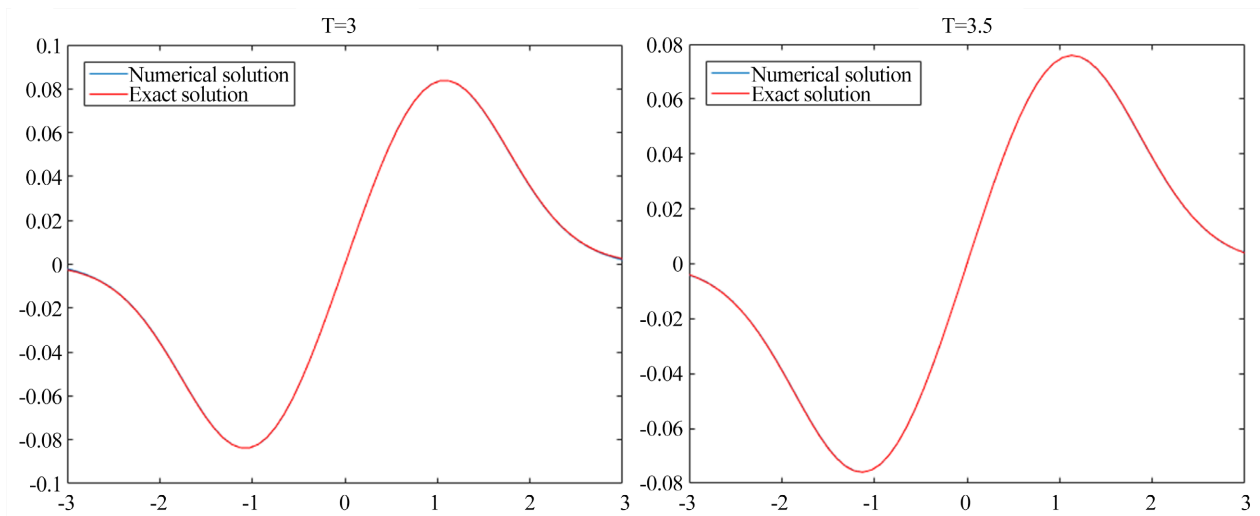


Figure 1. The numerical solution is compared with the exact solution for different T with $\mu = 0.1$.

6. Conclusion

In this paper, the DTBCs of Burgers' equation on an unbounded domain are discussed. A linear parabolic equation is obtained by applying the Cole-Hopf transformation. Based on the DTBCs of the linear parabolic equation, we designed the DTBCs of Burgers' equation. The original problem defined on unbounded domain is reduced into an IBVP on the bounded computational domain, which can be solved efficiently by finite difference method. The effectiveness of the method is validated through numerical results.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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