Scientific Research Publishing

Engineering, 2024, 16, 413-422
https://www.scirp.org/journal/eng
ISSN Online: 1947-394X
ISSN Print: 1947-3931

Cross-Migrativity of Continuous T-Conorms over I^h Implications

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How to cite this paper: Shi, X.R. (2024) Cross-Migrativity of Continuous T-Conorms over I^h Implications. *Engineering*, **16**, 413-422.
https://doi.org/10.4236/eng.2024.1612030

Received: November 20, 2024
Accepted: December 27, 2024
Published: December 30, 2024

Abstract

The cross-migrativity can be regarded as the weaker form of the commuting equation, which plays a crucial part in the framework of fuzzy connectives. This paper studies the cross-migrativity of continuous t-conorms over I^h implications. We obtain full characterizations for the cross-migrativity of continuous t-conorms over I^h implications.

Keywords

Cross-Migrativity, Continuous T-Conorms, I^h Implications

1. Introduction

Cross migrativity is a recently studied property of binary operations defined on the unit interval. The scholars had been extensively investigated the cross-migrativity between conjunctive operators, such as t-norms [1]-[3], overlap functions [4] [5] and uninorms [6]-[9]. Hence, the cross-migrativity of t-conorms over fuzzy implications [10] provides a new direction for the discussion of the relationships between disjunctive operators and fuzzy implications. Moreover, He and Fang [11] discussed the cross-migrativity of continuous t-conorms over generated implications, such as (f,g)-, k-, h- and (θ,t)-generated implications. However, there is a new fuzzy implication called the *h*-implications [12], which are generated by an additive generator of a representable uninorm in a similar way of Yager's *f*- and *g*-implications, that has not been discussed. Therefore, the α-cross-migrativity of continuous t-conorms over h-implications should be studied more thoroughly in this context to remedy that defect.

This paper is organized as follows. Section 2 briefly reviews several basic notions and results. Section 3 focuses on characterizations of α-cross-migrativity for continuous t-conorms over I^h implications. Section 4 concludes our research.

2. Preliminaries

Definition 2.1 ([13]) A t-conorm S is called [(i)]

i) *Archimedean*, if for all $u,w\in(0,1)$ there exists an $n\in\mathbb{N}$ such that $u_S^{[n]}>w$. If S is continuous, then S is Archimedean iff $S(u,u)>u$ for all $u\in\,]0,1[$.

ii) *strict*, if it is continuous and strictly monotone.

iii) *nilpotent*, if it is continuous and each $u\in(0,1)$ is a nilpotent element of S.

iv) *positive*, if $S(u,w)=1$, then either $u=1$ or $w=1$.

Theorem 2.2 ([13]) Let S be a t-conorm. [(i)]

i) S is continuous and Archimedean iff S is either strict or nilpotent.

ii) If S is continuous and Archimedean, then S is positive iff S is strict.

iii) S is strict iff there exists a strictly decreasing bijection $\psi:[0,1]\to[0,1]$ such that $S(u,w)=\psi^{-1}(\psi(u)\cdot\psi(w))$ for all $u,w\in[0,1]$.

iv) S is nilpotent iff there exists a strictly increasing bijection $\psi:[0,1]\to[0,1]$ such that $S(u,w)=\psi^{-1}(\min\{\psi(u)+\psi(w),1\})$ for all $u,w\in[0,1]$.

v) S is continuous iff there is a unique countable family $(]a_\lambda,e_\lambda[)_{\lambda\in A}$ of pairwise disjoint open subintervals of $[0,1]$ and a family of continuous Archimedean t-conorms $(S_\lambda)_{\lambda\in A}$ such that

$$S(u,w)=\begin{cases} a_\lambda+(e_\lambda-a_\lambda)S_\lambda\left(\dfrac{u-a_\lambda}{e_\lambda-a_\lambda},\dfrac{w-a_\lambda}{e_\lambda-a_\lambda}\right), & \text{if } (u,w)\in[a_\lambda,e_\lambda]^2;\\ \max(u,w), & \text{otherwise.}\end{cases}$$

In this case, we will write $S=(\langle a_\lambda,e_\lambda,S_\lambda\rangle)_{\lambda\in A}$.

Definition 2.3 ([12]) Let $e\in\,]0,1[$, $h:[0,1]\to[-\infty,+\infty]$ be a strictly increasing and continuous function with $h(0)=-\infty$, $h(e)=0$ and $h(1)=+\infty$. Then function $I:[0,1]^2\to[0,1]$ defined as

$$I(u,w)=\begin{cases} 1, & \text{if } u=0;\\ h^{-1}(u\cdot h(w)), & \text{if } u>0 \text{ and } w\le e;\\ h^{-1}\left(\dfrac{1}{u}\cdot h(w)\right), & \text{if } u>0 \text{ and } w>e;\end{cases}$$

is called an h-implication. The function h itself is called an h-generator (with respect to e) of the implication function I defined as above. We write it in this case I^h instead of I.

Definition 2.4 ([10]) Consider $\alpha\in[0,1]$, I be a fuzzy implication. A t-conorm S is said to be α*-cross-migrative* with respect to I, if for all $u,w\in[0,1]$,

$$I(u,S(\alpha,w))=S(w,I(u,\alpha)). \tag{1}$$

3. Cross-Migrativity of Continuous T-Conorms over I^h Implications

In this section, we characterize the α-cross-migrativity of continuous t-conorms over I^h implications.

Notice that Equation (1) is true for $\alpha = 1$. Thus, we only consider the case $\alpha \in [0,1[$. Firstly, we discuss $\alpha = 0$.

Theorem 3.1. Let S be a t-conorm and I^h be an h-generated implication. Then (S, I^h) is not 0-cross-migrative.

Proof. Suppose (S, I^h) is 0-cross-migrative, we have for all $u, w \in [0,1]$,

$$I(u,w) = I(u, S(0,w)) = S(w, I(u,0)).$$

The above equation is true for $u = 0$. Thus, we only discuss the case $u > 0$. By Definition 2.3, we obtain for all $u \neq 1$,

$$h^{-1}(u \cdot h(w)) = S(w, h^{-1}(u \cdot h(0))) = S(w, h^{-1}(-\infty)) = S(w,0) = w \quad \text{for all} \quad w \leq e;$$

$$h^{-1}\left(\frac{1}{u} \cdot h(w)\right) = S(w, h^{-1}(u \cdot h(0))) = S(w, h^{-1}(-\infty)) = S(w,0) = w \quad \text{for all} \quad w > e.$$

By the monotonicity of h^{-1}, we have $u = 1$, which is a contradiction.

In the equal, we discuss $\alpha \in]0, e[$.

Theorem 3.2. Take $\alpha \in]0, e[$. Then S be α-cross-migrative over I^h iff

$$S(u,w) = \begin{cases} h^{-1}\left(\dfrac{h(w)}{h(\alpha)} \cdot h(S(\alpha,u))\right), & \text{if } S(\alpha,u) \leq e \text{ and } w \in [\alpha, e[; \\ h^{-1}\left(\dfrac{h(\alpha)}{h(w)} \cdot h(S(\alpha,u))\right), & \text{if } S(\alpha,u) > e \text{ and } w \in [\alpha, e[. \end{cases} \tag{2}$$

Proof. ($\Rightarrow$). By the Definition 2.4, we have for all $u, v \in [0,1]$,

$$I(v, S(\alpha,u)) = S(u, I(v,\alpha)).$$

Consider $v = 0$. Then it is obvious. By Definition 2.3, one obtains

$$h^{-1}(v \cdot h(S(\alpha,u))) = S(u, h^{-1}(v \cdot h(\alpha))) \quad \text{for all} \quad v \in]0,1] \quad \text{and} \quad S(\alpha,u) \leq e;$$

$$h^{-1}\left(\frac{1}{v} \cdot h(S(\alpha,u))\right) = S(u, h^{-1}(v \cdot h(\alpha))) \quad \text{for all} \quad v \in]0,1] \quad \text{and} \quad S(\alpha,u) > e.$$

Denote $w = h^{-1}(v \cdot h(\alpha))$. Then one obtains

$$\alpha = h^{-1}(h(\alpha)) \leq w = h^{-1}(v \cdot h(\alpha)) < h^{-1}(0 \cdot h(\alpha)) = e.$$

That is $w \in [\alpha, e[$. Then we have

$$S(u,w) = \begin{cases} h^{-1}\left(\dfrac{h(w)}{h(\alpha)} \cdot h(S(\alpha,u))\right), & \text{if } S(\alpha,u) \leq e \text{ and } w \in [\alpha, e[; \\ h^{-1}\left(\dfrac{h(\alpha)}{h(w)} \cdot h(S(\alpha,u))\right), & \text{if } S(\alpha,u) > e \text{ and } w \in [\alpha, e[. \end{cases}$$

($\Leftarrow$). It is obvious.

Proposition 3.3. Take $\alpha \in]0,e[$. If S is α-cross-migrative over I^h, then $S(u,u) > u$ for all $u \in]\alpha,e[$.

Proof. We obtain for all $S(\alpha,u) \le e$ and $w \in [\alpha,e[$ by Theorem 3.2,

$$h(\alpha) \cdot h \circ S(u,w) = h(w) \cdot h \circ S(\alpha,u). \tag{3}$$

Suppose that there exists some $u_0 \in]\alpha,e[$ such that $S(u_0,u_0) = u_0$. Then we obtain for all $u \in [\alpha,u_0]$,

$$u_0 = \max(u,u_0) \le S(u,u_0) \le S(u_0,u_0) = u_0.$$

Hence we have $S(\alpha,u_0) = u_0 < e$. Let $u = u_0$ and $w \in]\alpha,u_0[$ in Equation (3). Then one obtains

$$h(\alpha) \cdot h(u_0) = h(\alpha) \cdot h \circ S(u_0,w) = h(w) \cdot h \circ S(\alpha,u_0) = h(w) \cdot h(u_0).$$

Thus we have $h(w) = h(\alpha)$, *i.e.*, $w = \alpha$, which is a contradiction. □

Lemma 3.4. Take $\alpha \in]0,e[$. If S be α-cross-migrative over I^h, then S is not nilpotent.

Proof. Assume that S is nilpotent, there exists a strictly increasing $\psi : [0,1] \to [0,1]$ such that $S(u,w) = \psi^{-1}(\min\{\psi(u)+\psi(w),1\})$ for all $u,w \in [0,1]$. Hence there must exist u_α such that $\psi(\alpha)+\psi(u_\alpha) = 1$. Choose $w_0 \in]\alpha,e[$ such that $u_0 = \psi^{-1}(1-\psi(w_0))$. Then we obtain $\psi(u_0)+\psi(w_0) = 1$ and

$$u_0 = \psi^{-1}(1-\psi(w_0)) < \psi^{-1}(1-\psi(\alpha)) = u_\alpha.$$

Thus $u_0 \in]0,u_\alpha[$. Hence we have $S(\alpha,u_0) < 1$. Let $u_0 \in]0,u_\alpha[$ and $w_0 \in]\alpha,e[$ in Equation (2). Then one obtains

$$1 = S(u_0,w_0) = \begin{cases} h^{-1}\left(\dfrac{h(w_0)}{h(\alpha)} \cdot h(S(\alpha,u_0))\right) \le h^{-1}\left(\dfrac{h(w_0)}{h(\alpha)} \cdot h(e)\right) = e < 1, & \text{if } S(\alpha,u_0) \le e; \\ h^{-1}\left(\dfrac{h(\alpha)}{h(w_0)} \cdot h(S(\alpha,u_0))\right) < h^{-1}\left(\dfrac{h(w_0)}{h(\alpha)} \cdot h(1)\right) = 1, & \text{if } S(\alpha,u_0) > e; \end{cases}$$

which is a contradiction. □

Theorem 3.5. Take $\alpha \in]0,e[$, S be a continuous t-conorm. Then S be α-cross-migrative over I^h iff one of the items holds. [(i)]

i) S is strict. There exists a strictly decreasing bijection $\psi : [0,1] \to [0,1]$ such that

$$\rho(\psi(\alpha)) \cdot \rho(\psi(u) \cdot \psi(w)) = \rho(\psi(w)) \cdot \rho(\psi(\alpha) \cdot \psi(u)) \text{ for all } S(\alpha,u) \le e \text{ and } w \in [\alpha,e[;$$

$$\rho(\psi(w)) \cdot \rho(\psi(u) \cdot \psi(w)) = \rho(\psi(\alpha)) \cdot \rho(\psi(\alpha) \cdot \psi(u)) \text{ for all } S(\alpha,u) > e \text{ and } w \in [\alpha,e[;$$

where $\rho : [0,\psi(\alpha)] \to [h(\alpha),+\infty]$ and $\rho(u) = h \circ \psi^{-1}(u)$.

ii) $\beta \in]0,\alpha[$ and $S = (\langle 0,\beta,S_a \rangle, \langle \beta,1,S_b \rangle)$, where

$\beta = \sup\{u \in [0,1[\mid S(u,u) = u\}$. There exists a strictly decreasing bijection $\xi : [\beta,1] \to [0,1]$ such that for all $u \in [\beta,1]$ and $w \in [\alpha,e[$,

$$\rho(\xi(\alpha)) \cdot \rho(\xi(u) \cdot \xi(w)) = \rho(\xi(w)) \cdot \rho(\xi(\alpha) \cdot \xi(u)) \text{ for all } S(\alpha,u) \le e;$$

$$\rho(\xi(w)) \cdot \rho(\xi(u) \cdot \xi(w)) = \rho(\xi(\alpha)) \cdot \rho(\xi(\alpha) \cdot \xi(u)) \text{ for all } S(\alpha,u) > e;$$

where $\rho : [0,\xi(\alpha)] \to [h(\alpha),+\infty]$ and $\rho(u) = h \circ \xi^{-1}(u)$.

iii) $\beta = \alpha$ and $S = (\langle 0,\beta,S_a \rangle, \langle \beta,1,S_b \rangle)$, where $\beta = \sup\{u \in [0,1[\mid S(u,u) = u\}$. Then

$$S(u,w) = \begin{cases} \alpha S_a\left(\dfrac{u}{\alpha},\dfrac{w}{\alpha}\right), & \text{if } (u,w) \in [0,\alpha]^2; \\ h^{-1}\left(\dfrac{h(u) \cdot h(w)}{h(\alpha)}\right), & \text{if } u \in [\alpha,1], S(\alpha,u) \le e \text{ and } w \in [\alpha,e[; \\ h^{-1}\left(\dfrac{h(u) \cdot h(\alpha)}{h(w)}\right), & \text{if } u \in [\alpha,1], S(\alpha,u) > e \text{ and } w \in [\alpha,e[; \\ \alpha + (1-\alpha) S_b\left(\dfrac{u-\alpha}{1-\alpha},\dfrac{w-\alpha}{1-\alpha}\right), & \text{if } (u,w) \in [\alpha,1] \times [e,1]; \\ \max(u,w), & \text{otherwise.} \end{cases}$$

Proof. Necessity. Denote $\beta = \sup\{u \in [0,1[\mid S(u,u) = u\}$. Then we have $u \in [0,\alpha]$ from Proposition 3.3. By the continuity of S, β satisfies $S(\beta,\beta) = \beta$. The conclusions are verified as follows.

Case (i) $\beta = 0$

We have $S(u,u) > u$ for all $u \in]0,1[$, that is, S is Archimedean by Definition 2.1(i). By Theorem 2.2(i) and Lemma 3.4, S is strict. Hence, there exists a strictly decreasing bijection $\psi : [0,1] \to [0,1]$ such that $S(u,w) = \psi^{-1}(\psi(u) \cdot \psi(w))$ for all $u,w \in [0,1]$. By Theorem 3.2, one obtains

$$h(\alpha) \cdot h \circ \psi^{-1}(\psi(u) \cdot \psi(w)) = h(w) \cdot h \circ \psi^{-1}(\psi(\alpha) \cdot \psi(u)) \text{ for all } S(\alpha,u) \le e \text{ and } w \in [\alpha,e[;$$

$$h(w) \cdot h \circ \psi^{-1}(\psi(u) \cdot \psi(w)) = h(\alpha) \cdot h \circ \psi^{-1}(\psi(\alpha) \cdot \psi(u)) \text{ for all } S(\alpha,u) > e \text{ and } w \in [\alpha,e[.$$

Let $\rho = h \circ \psi^{-1}$. Then one obtains

$$\rho(\psi(\alpha)) \cdot \rho(\psi(u) \cdot \psi(w)) = \rho(\psi(w)) \cdot \rho(\psi(\alpha) \cdot \psi(u)) \text{ for all } S(\alpha,u) \le e \text{ and } w \in [\alpha,e[;$$

$$\rho(\psi(w)) \cdot \rho(\psi(u) \cdot \psi(w)) = \rho(\psi(\alpha)) \cdot \rho(\psi(\alpha) \cdot \psi(u)) \text{ for all } S(\alpha,u) > e \text{ and } w \in [\alpha,e[.$$

Case (ii) $\beta \in]0,\alpha[$

By Theorem 2.2(v), S is an ordinal sum t-conorm. Because S_b is obviously

Archimedean by Definition 2.1(i) and Proposition 3.3. Hence S_b is strict by Theorem 2.2(i) and Lemma 3.4. There exists a strictly decreasing bijection $\psi:[0,1]\to[0,1]$ such that $S_b(u,w)=\psi^{-1}(\psi(u)\cdot\psi(w))$ for all $u,w\in[0,1]$. Denote $\xi:[\beta,1]\to[0,1]$ with $\xi(u)=\psi\left(\frac{u-\beta}{1-\beta}\right)$ for all $u\in[\beta,1]$. Then ξ is a strictly increasing bijection and $S(u,w)=\xi^{-1}(\xi(u)\circ\xi(w))$ for all $u,w\in[\beta,1]$. Hence, we have for all $u\in[\beta,1]$,

$$h(\alpha)\cdot h\left(\xi^{-1}\left(\xi(u)\circ\xi(w)\right)\right)=h(w)\cdot h\left(\xi^{-1}\left(\xi(u)\circ\xi(\alpha)\right)\right)\text{ for all } S(\alpha,u)\le e \text{ and } w\in[\alpha,e[;$$

$$h(w)\cdot h\left(\xi^{-1}\left(\xi(u)\circ\xi(w)\right)\right)=h(\alpha)\cdot h\left(\xi^{-1}\left(\xi(u)\circ\xi(\alpha)\right)\right)\text{ for all } S(\alpha,u)> e \text{ and } w\in[\alpha,e[.$$

Let $\rho(u)=h\circ\xi^{-1}(u)$. Then we obtain

$$\rho(\xi(\alpha))\cdot\rho(\xi(u)\cdot\xi(w))=\rho(\xi(w))\cdot\rho(\xi(\alpha)\cdot\xi(u))\text{ for all } u\in[\beta,1],\ S(\alpha,u)\le e \text{ and } w\in[\alpha,e[;$$

$$\rho(\xi(w))\cdot\rho(\xi(u)\cdot\xi(w))=\rho(\xi(\alpha))\cdot\rho(\xi(\alpha)\cdot\xi(u))\text{ for all } u\in[\beta,1],\ S(\alpha,u)> e \text{ and } w\in[\alpha,e[.$$

Case (iii) $\beta=\alpha$.

By Theorem 2.2(v), S is an ordinal sum t-conorm. One has S_b which is strict and Archimedean in a similar way as for Case (ii). By the continuity of S and $S(\alpha,\alpha)=\alpha$, we have $S(u,\alpha)=\max(u,\alpha)=u$ for all $u\in[\alpha,1]$. We have for all $u\in[\alpha,1]$ by Theorem 3.2,

$$S(u,w)=\begin{cases} h^{-1}\left(\dfrac{h(w)}{h(\alpha)}\cdot h(S(\alpha,u))\right)=h^{-1}\left(\dfrac{h(u)\cdot h(w)}{h(\alpha)}\right), & \text{if } S(\alpha,u)\le e \text{ and } w\in[\alpha,e[;\\ h^{-1}\left(\dfrac{h(\alpha)}{h(w)}\cdot h(S(\alpha,u))\right)=h^{-1}\left(\dfrac{h(u)\cdot h(\alpha)}{h(w)}\right), & \text{if } S(\alpha,u)> e \text{ and } w\in[\alpha,e[. \end{cases}$$

Thus we obtain the form of S.

Sufficiency. (i) $\beta=0$.

By item (i), we have

$$\rho(\psi(\alpha))\cdot\rho(\psi(u)\cdot\psi(w))=\rho(\psi(w))\cdot\rho(\psi(\alpha)\cdot\psi(u))\text{ for all } S(\alpha,u)\le e \text{ and } w\in[\alpha,e[;$$

$$\rho(\psi(w))\cdot\rho(\psi(u)\cdot\psi(w))=\rho(\psi(\alpha))\cdot\rho(\psi(\alpha)\cdot\psi(u))\text{ for all } S(\alpha,u)> e \text{ and } w\in[\alpha,e[;$$

where $\rho=h\circ\psi^{-1}$. Hence one obtains for all $w\in[\alpha,e[$,

$$h\left(\psi^{-1}(\psi(\alpha))\right)\cdot h\left(\psi^{-1}(\psi(u)\cdot\psi(w))\right)=h\left(\psi^{-1}(\psi(w))\right)\cdot h\left(\psi^{-1}(\psi(\alpha)\cdot\psi(u))\right)\text{ for all } S(\alpha,u)\le e;$$

$$h\left(\psi^{-1}\left(\psi\left(w\right)\right)\right)\cdot h\left(\psi^{-1}\left(\psi\left(u\right)\cdot\psi\left(w\right)\right)\right)=h\left(\psi^{-1}\left(\psi\left(\alpha\right)\right)\right)\cdot h\left(\psi^{-1}\left(\psi\left(\alpha\right)\cdot\psi\left(u\right)\right)\right)$$ for all $S\left(\alpha,u\right)>e$.

That is,

$$h\left(\alpha\right)\cdot h\circ S\left(u,w\right)=h\left(w\right)\cdot h\circ S\left(\alpha,u\right) \text{ for all } S\left(\alpha,u\right)\le e \text{ and } w\in\left[\alpha,e\right[;$$

$$h\left(w\right)\cdot h\circ S\left(u,w\right)=h\left(\alpha\right)\cdot h\circ S\left(\alpha,u\right) \text{ for all } S\left(\alpha,u\right)>e \text{ and } w\in\left[\alpha,e\right[.$$

Therefore, $\left(S,I^{h}\right)$ is α-cross-migrative by Theorem 3.2.

In the sequel, we only verify the sufficiency for item (ii), *i.e.*, $\beta\in\left]0,\alpha\right[$. The item (iii) can be proven in a similar way. We prove Equation (2) in the following cases.

Case (a) $u\in\left[0,\beta\right]$ and $w\in\left[\alpha,e\right[$.

By the continuity of S and $S\left(\alpha,\alpha\right)=\alpha$, we have $S\left(u,w\right)=\max\left(u,w\right)=w$ and $S\left(\alpha,u\right)=\max\left(u,\alpha\right)=\alpha<e$. Hence we obtain

$$h\left(\alpha\right)\cdot h\left(w\right)=h\left(\alpha\right)\cdot h\left(S\left(u,w\right)\right)=h\left(w\right)\cdot h\left(S\left(\alpha,u\right)\right)=h\left(\alpha\right)\cdot h\left(w\right) \text{ for all } \left(u,w\right)\in\left[0,\beta\right]\times\left[\alpha,e\right[.$$

Case (b) $u\in\left[\beta,1\right]$ and $w\in\left[\alpha,e\right[$.

The following can be proven in a similar way as for the sufficiency of item (i),

$$h\left(\alpha\right)\cdot h\left(S\left(u,w\right)\right)=h\left(w\right)\cdot h\left(S\left(\alpha,u\right)\right) \text{ for all } u\in\left[\beta,1\right],\ S\left(\alpha,u\right)\le e \text{ and } w\in\left[\alpha,e\right[;$$

$$h\left(w\right)\cdot h\left(S\left(u,w\right)\right)=h\left(\alpha\right)\cdot h\left(S\left(\alpha,u\right)\right) \text{ for all } u\in\left[\beta,1\right],\ S\left(\alpha,u\right)>e \text{ and } w\in\left[\alpha,e\right[.$$

Therefore, $\left(S,I^{h}\right)$ is α-cross-migrative by Theorem 3.2. □

In the equal, we characterize $\alpha=e$.

Theorem 3.6. Let S be a t-conorm and I^{h} be an h-generated implication. Then $\left(S,I^{h}\right)$ is not e-cross-migrative.

Proof. Suppose $\left(S,I^{h}\right)$ is e-cross-migrative, we have for all $u\in\left]0,1\right[$ and $S\left(\alpha,w\right)\le e$,

$$h^{-1}\left(u\cdot h\left(S\left(e,w\right)\right)\right)=S\left(w,h^{-1}\left(u\cdot h\left(e\right)\right)\right)=S\left(w,h^{-1}\left(0\right)\right)=S\left(w,e\right).$$

Hence one obtains for all $u\in\left]0,1\right[$ and $S\left(\alpha,w\right)\le e$,

$$h\left(S\left(w,e\right)\right)=u\cdot h\left(S\left(e,w\right)\right).$$

Then we obtain $u=1$, which is a contradiction. □

Finally, we discuss $\alpha\in\left]e,1\right[$.

Theorem 3.7. Take $\alpha\in\left]e,1\right[$. Then S be α-cross-migrative over I^{h} iff

$$S\left(u,w\right)=h^{-1}\left(\frac{h\left(w\right)}{h\left(\alpha\right)}\cdot h\left(S\left(\alpha,u\right)\right)\right) \text{ for all } \left(u,w\right)\in\left[0,1\right]\times\left[\alpha,1\right]. \quad (4)$$

Proof. ($\Rightarrow$). By definition 2.4, we have for all $u,v\in\left[0,1\right]$,

$$I(v, S(\alpha, u)) = S(u, I(v, \alpha)).$$

Consider $v = 0$. Then it is obvious. By the property of S, one obtains $S(\alpha, u) \geq \max(\alpha, u) \geq \alpha > e$ for all $u \in [0,1]$. One has for all $(v, u) \in]0,1] \times [0,1]$

$$h^{-1}\left(\frac{1}{v} \cdot h(S(\alpha, u))\right) = S\left(u, h^{-1}\left(\frac{1}{v} \cdot h(\alpha)\right)\right).$$

Denote $w = h^{-1}\left(\frac{1}{v} \cdot h(\alpha)\right)$. Then one obtains

$$\alpha = h^{-1}(h(\alpha)) \leq w = h^{-1}\left(\frac{1}{v} \cdot h(\alpha)\right) = I^h(v, \alpha) < I^h(0, \alpha) = 1.$$

It is $w \in [\alpha, 1[$. Hence we obtain for all $(u, w) \in [0,1[\times [\alpha, 1]$

$$S(u, w) = h^{-1}\left(\frac{h(w)}{h(\alpha)} \cdot h(S(\alpha, u))\right).$$

Consider $w = 1$. Then we have $1 = S(w, 1) = h^{-1}\left(\frac{h(1)}{h(\alpha)} \cdot h(S(\alpha, u))\right) = h^{-1}(+\infty) = 1$. Thus we obtain the conclusion.

($\Leftarrow$). It is obvious. □

Proposition 3.8. Take $\alpha \in]e, 1[$. If S be α-cross-migrative over I^h, then $S(u, u) > u$ for all $u \in]\alpha, 1[$.

Proof. By Theorem 3.7, we have for all $u \in [0,1]$ and $w \in [\alpha, 1]$,

$$h(\alpha) \cdot h(S(u, w)) = h(w) \cdot h(S(\alpha, u)). \tag{5}$$

Suppose that there exists some $u_0 \in]\alpha, 1[$ such that $S(u_0, u_0) = u_0$. Then we have for all $u \in [\alpha, u_0]$,

$$u_0 = \max(u, u_0) \leq S(u, u_0) \leq S(u_0, u_0) = u_0.$$

Let $u = u_0$ and $w \in]\alpha, u_0[$ in Equation (5). Then one has

$$h(u_0) \cdot h(\alpha) = h(\alpha) \cdot h(S(u_0, w)) = h(w) \cdot h(S(\alpha, u_0)) = h(w) \cdot h(u_0).$$

It implies $h(\alpha) = h(w)$, *i.e.*, $\alpha = w$, which is a contradiction. □

Lemma 3.9. Take $\alpha \in]e, 1[$. If S be α-cross-migrative over I^h, then S is not nilpotent.

Proof. Assume that S is nilpotent, there exists a strictly increasing bijection $\psi : [0,1] \to [0,1]$ such that $S(u, w) = \psi^{-1}(\min\{\psi(u) + \psi(w), 1\})$ for all $u, w \in [0,1]$. Hence there must exist u_α such that $\psi(\alpha) + \psi(u_\alpha) = 1$. Choose $w_0 \in]\alpha, 1[$ such that $u_0 = \psi^{-1}(1 - \psi(w_0))$. Then one obtains $\psi(u_0) + \psi(w_0) = 1$ and

$$u_0 = \psi^{-1}(1 - \psi(w_0)) < \psi^{-1}(1 - \psi(\alpha)) = u_\alpha.$$

That is $u_0 \in]0, u_\alpha[$. Hence we have $S(\alpha, u_0) < 1$. Let $u_0 \in]0, u_\alpha[$ and $w_0 \in]\alpha, 1[$ in Equation (4). Then one obtains

$$1 = S(u_0, w_0) = h^{-1}\left(\frac{h(w_0)}{h(\alpha)} \cdot h(S(\alpha, u_0))\right) < h^{-1}\left(\frac{h(w_0)}{h(\alpha)} \cdot h(1)\right) = h^{-1}(+\infty) = 1,$$

which is a contradiction.

Theorem 3.10. Take $\alpha \in]e, 1[$, S be a continuous t-conorm. Then S be α-cross-migrative over I^h iff one of the items holds. [(i)]

i) S is strict. There exists a strictly decreasing bijection $\psi : [0,1] \to [0,1]$ such that

$$\rho(\psi(\alpha)) \cdot \rho(\psi(u) \cdot \psi(w)) = \rho(\psi(w)) \cdot \rho(\psi(\alpha) \cdot \psi(u)) \text{ for all } (u, w) \in [0,1] \times [\alpha, 1].$$

where $\rho : [0, \psi(\alpha)] \to [h(\alpha), +\infty]$ and $\rho(u) = h \circ \psi^{-1}(u)$.

ii) $\beta \in]0, \alpha[$ and $S = (\langle 0, \beta, S_a \rangle, \langle \beta, 1, S_b \rangle)$, where $\beta = \sup\{u \in [0,1[\mid S(u,u) = u\}$. There exists a strictly decreasing bijection $\xi : [\beta, 1] \to [0,1]$ such that

$$\rho(\xi(\alpha)) \cdot \rho(\xi(u) \cdot \xi(w)) = \rho(\xi(w)) \cdot \rho(\xi(\alpha) \cdot \xi(u)) \text{ for all } (u, w) \in [\beta, 1] \times [\alpha, 1],$$

where $\rho : [0, \xi(\alpha)] \to [h(\alpha), +\infty]$ and $\rho(u) = h \circ \xi^{-1}(u)$.

iii) $\beta = \alpha$ and $S = (\langle 0, \beta, S_a \rangle, \langle \beta, 1, S_b \rangle)$, where $\beta = \sup\{u \in [0,1[\mid S(u,u) = u\}$. Then

$$S(u, w) = \begin{cases} \alpha S_a\left(\frac{u}{\alpha}, \frac{w}{\alpha}\right), & \text{if } (u, w) \in [0, \alpha]^2; \\ h^{-1}\left(\frac{h(u) \cdot h(w)}{h(\alpha)}\right), & \text{if } (u, w) \in [\alpha, 1]^2; \\ \max(u, w), & \text{otherwise.} \end{cases}$$

Proof. It can be proven in a similar way as for Theorem 3.5.

4. Conclusion

In this paper, full characterizations for the α-cross-migrativity of continuous t-conorms over I^h implications are obtained. Moreover, we investigate all solutions of the cross-migrativity equation for all possible combinations of continuous t-conorms and I^h implications.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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