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2-Factors with a Few Components in Balanced Bipartite Graphs

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Abstract

In this paper, a sufficient condition for a balanced bipartite graph to contain a 2-factor F is given. We show that every balanced bipartite graph of order $2n$ $(n \geq 6)$ and $e(G) > n^2 - 2n + 4$ contains a 2-factor with k components, $2d_1$-cycle, $\cdots$, $2d_k$-cycle, if one of the following is satisfied: (1) $k = 2$, $\delta(G) \geq 2$ and $d_1 - 2 \geq d_2 \geq 2$; (2) $k = 3$, $\delta(G) \geq d_3 + 2$ and $d_1 - 2 \geq d_2 \geq d_3 \geq 4$. In particular, this extends one result of Moon and Moser in 1963 under condition (1).

Keywords

2-Factor, Bipartite Graph, Degree Condition

1. Introduction

All graphs considered in this paper are simple undirected graphs. Let $G = (V, E)$ be a graph, where $V = V(G)$ and $E = E(G)$ denote the vertex set and the edge set of G, respectively. We call a graph $G = (X, Y; E)$ is a bipartite graph if $V(G) = X \cup Y$, $X \cap Y = \varnothing$ and each edge $e \in E$ if and only if e has one vertex in X and the other one in Y. If $|X| = |Y|$, $G = (X, Y; E)$ is called a balanced bipartite graph. For a bipartite graph $G = (X, Y; E)$, if for any $x \in X$, $y \in Y$, there is $xy \in E$, then call G a complete bipartite graph. In particular, denote a complete bipartite graph by $K_{m,n}$ when $|X| = m$, $|Y| = n$. For a vertex x of G, the degree of x is the number of incidence edges of x in G, and it is denoted by $d_G(x)$. A 2-factor of a graph G is a spanning subgraph of G in which all vertices have a degree exactly 2. Call $|V(G)|$, $e(G) = |E(G)|$, $\delta(G) = \min\{d_G(x), x \in V(G)\}$ and $\bar{d}(G) = \frac{2e(G)}{|V(G)|}$, the order of G, the number of edges of G, the minimum degree of G and the average degree of G, respectively. When each vertex of G has degree

k, call G k-regular. The missing degree of $v \in X$ in a bipartite graph $G = (X, Y; E)$ is $|Y| - d_G(v)$.

An h-cycle is a graph $C = (V, E)$ with $V = \{x_1, x_2, \cdots, x_h\}$, $E = \{x_1x_2, x_2x_3, \cdots, x_{h-1}x_h, x_hx_1\}$. We often denote an h-cycle as $x_1x_2\cdots x_hx_1$. A Hamilton cycle of G is a cycle that visits every vertex of G exactly once. If G has a Hamilton cycle, then it is called Hamiltonian. For $U \subseteq V(G)$, $G[U]$ denotes the subgraph of G induced by U. A component of G is a maximal connected induced subgraph. For any graph G, F is a 2-factor of G if and only if F is a 2-regular subgraph of G and $V(G) = V(F)$. Clearly, a 2-factor F of G is a collection of vertex disjoint cycles that cover all vertices of G.

In the next section, a statement of the problem is introduced. Afterwards, the proof of the main results is established.

2. Statement of the Problem

The problem of the existence of 2-factors in a graph has been extensively studied. The special case that a 2-factor has one component, that is, Hamilton cycle problem has received wide attention.

Many sufficient conditions for a graph to be Hamiltonian have been obtained. In 1952, Dirac [1] gave a minimum degree condition for Hamiltonian graphs. The following theorem is a classic result about sufficient conditions on Hamiltonian graphs.

Theorem 1. (Dirac [1]) Let G be a graph of order $n \ge 3$. If $\delta(G) \ge \frac{n}{2}$, then G contains a Hamilton cycle.

In 1959, Erdős and Gallai [2] gave a condition on the number of edges to guarantee that a graph has a Hamilton cycle.

Theorem 2. [2] Let G be a graph of order $n \ge 3$. If $e(G) \ge \binom{n-1}{2} + 2$, that is, $\bar{d}(G) \ge \frac{(n-1)(n-2)+4}{n}$, then G contains a Hamilton cycle.

In 1960, Ore [3] obtained a well-known result on Hamiltonian graphs, which is stronger than Theorem 1.

Theorem 3. (Ore [3]) Let G be a graph of order $n \ge 3$. If $d_G(u) + d_G(v) \ge n$ for every pair of non-adjacent vertices u and v in $V(G)$, then G contains a Hamilton cycle.

If a graph satisfies the condition of Theorem 2, then it also satisfies the one of Theorem 3. So Theorem 3 is stronger than Theorem 2.

Both Theorem 1 and Theorem 3 imply that G has a 2-factor with exactly one component. The research on the existence of 2-factors in a graph is mostly motivated by results from Hamilton cycles. In 1997, Chen *et al.* [4] proved that for $k \le \frac{n}{4}$, the condition of Theorem 3 for a graph to be Hamiltonian $(k = 1)$ implies that the graph contains a 2-factor with exactly k components.

Theorem 4. [4] Let k be a positive integer and let G be a graph of order $n \ge 4k$.

If $d_G(u)+d_G(v)\geq n$ for every pair of non-adjacent vertices u and v in $V(G)$, then G contains a 2-factor with exactly k vertex disjoint cycles.

Chen *et al.* [4] also obtained a corollary of Theorem 4, which generalizes the classic Hamiltonian result of Theorem 1.

Corollary 1. [4] Let k be a positive integer and let G be a graph of order $n\geq 4k$. If $\delta(G)\geq \frac{n}{2}$, then G contains a 2-factor with exactly k vertex disjoint cycles.

When G is a complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$, we can see that the conclusions of Theorem 4 and Corollary 1 are the best possible that any 2-factor can contain at most $\left\lfloor \frac{n}{4} \right\rfloor$ components.

Moon and Moser [5] considered a bipartite version of Theorem 3 and obtained the following result for balanced bipartite graphs.

Theorem 5. [5] Let $G=(X,Y;E)$ be a balanced bipartite graph of order $2n$ $(n\geq 2)$, with $d_G(u)+d_G(v)\geq n+1$ for every pair of non-adjacent vertices $u\in X$ and $v\in Y$, then G contains a Hamilton cycle.

The problem of determining whether a given graph has vertex disjoint cycles or not, is NP-complete. Many scholars have researched such a problem as the existence of a 2-factor with exactly k vertex disjoint cycles in balanced bipartite graphs. They have investigated degree conditions for partitioning in terms of, for example, minimum degree, average degree, degree sum of independent vertices and so on.

In 1999, Wang [6] gave a Dirac-Type degree condition for a balanced bipartite graph to contain a 2-factor with exactly k components and obtained the following result.

Theorem 6. [6] Let k be a positive integer and let $G=(X,Y;E)$ be a bipartite graph with $|X|=|Y|=n\geq 2k+1$, if $\delta(G)\geq \left\lceil \frac{n}{2} \right\rceil+1$, then G contains a 2-factor with exactly k vertex disjoint cycles.

In 2000, Chen *et al.* [7] obtained the following Ore-Type result, which generalizes Theorem 5.

Theorem 7. [7] Let k be a positive integer and let $G=(X,Y;E)$ be a balanced bipartite graph of order $2n$ where $n\geq \max\left\{51,\frac{k^2}{2}+1\right\}$. If $d_G(u)+d_G(v)\geq n+1$ for every $u\in X$ and $v\in Y$, then G contains a 2-factor with exactly k vertex disjoint cycles.

Later, in 2001, Li *et al.* [8] improved Theorem 6 with Ore-Type degree condition. They also showed that the degree condition in Theorem 8 is sharp when $n=2k+1$.

Theorem 8. [8] Let k be a positive integer and let $G=(X,Y;E)$ be a bipartite graph with $|X|=|Y|=n\geq 2k+1$. If $d_G(u)+d_G(v)\geq n+2$ for every pair of non-adjacent vertices $u,v\in V(G)$, then G has a 2-factor with exactly k vertex disjoint cycles.

In 2017, Chiba and Yamashita [9] proved that the degree condition in Theorem

5 also guarantees the existence of a 2-factor with exactly k vertex disjoint cycles when $n \geq 12k+2$. The following result generalizes Theorem 7 and Theorem 8.

Theorem 9. [9] Let k be a positive integer and let $G=(X,Y;E)$ be a bipartite graph with $|X|=|Y|=n \geq 12k+2$. If $d_G(u)+d_G(v) \geq n+1$ for every pair of non-adjacent vertices $u \in X$ and $v \in Y$, then G has a 2-factor with exactly k vertex disjoint cycles.

As early as 1963, Moon and Moser [5] considered a bipartite version of Ore's Theorem (Theorem 3) and they obtained Theorem 5. They also discussed the problem of how many edges are necessary to ensure that a balanced bipartite graph contains a Hamilton cycle. Moon and Moser [5] gave the conditions on the minimum degree and the number of edges for the existence of a Hamilton cycle in balanced bipartite graphs.

Theorem 10. [5] Let G be a balanced bipartite graph of order $2n$ $(n \geq 2)$ and $\delta(G) \geq r$, where $1 \leq r \leq \frac{n}{2}$. If $e(G) > n(n-r)+r^2$, then G contains a Hamilton cycle.

Theorem 10 implies that if there is no restriction on the minimum degree (*i.e.* $r=1$), then the sufficient condition for a balanced bipartite graph to be Hamiltonian is $e(G) > n^2-n+1$.

Let H and G be two graphs. We call G H-free if G contains no subgraphs isomorphic to H. For a graph H and positive integers m, n, the bipartite Turán number of H, denoted by $\mathrm{ex}(m,n,H)$, is the maximum number of edges among H-free bipartite graphs with two parts of sizes m and n, respectively. Recently, Zhang *et al.* [10] determined that the bipartite Turán number of F for any 2-factor F in $K_{n,n}$.

Theorem 11. [10] $\mathrm{ex}(n,n,F)=n^2-n+1$.

Theorem 11 implies the following corollary.

Corollary 2. [10] Let G be a balanced bipartite graph of order $2n$ $(n \geq 2)$. If $e(G) > n^2-n+1$, then G contains all non-isomorphic 2-factors of $K_{n,n}$.

3. Main Results

In this paper, we consider conditions on the minimum degree and number of edges for a balanced bipartite graph to contain some class of 2-factors and obtain the following result. This result extends Theorem 10 under condition (1). Since Theorem 10 implies that if G is a balanced bipartite graph of order $2n$, $\delta(G) \geq 2$ and $e(G) > n^2-2n+4$, then G contains a Hamilton cycle. Theorem 10 show that if G is a balanced bipartite graph of order $2n$, $\delta(G) \geq 2$ and $e(G) > n^2-2n+4$, then G contains a 2-factor with 2 components.

Theorem 12. Let G be a balanced bipartite graph of order $2n$ $(n \geq 6)$ and $e(G) > n^2-2n+4$. Then G contains a 2-factor with k components, $2d_1$-cycle, $\cdots$, $2d_k$-cycle, if one of the following is satisfied:

(1) $k=2$, $\delta(G) \geq 2$ and $d_1-2 \geq d_2 \geq 2$;

(2) $k=3$, $\delta(G) \geq d_3+2$ and $d_1-2 \geq d_2 \geq d_3 \geq 4$.

4. The Proof of Theorem 12

Proof. Let $K_{n,n}=(X,Y;E)$ be a balanced complete bipartite graph, where $X=\{x_1,x_2,\cdots,x_n\}$, $Y=\{y_1,y_2,\cdots,y_n\}$. Let k be a positive integer with $2\le k\le 3$. Let $\mathcal{F}(k)$ be the collection of non-isomorphic 2-factors of $K_{n,n}$ with k components, $2d_1$-cycle, $\cdots$, $2d_k$-cycle, when $k=2$, $d_1-2\ge d_2\ge 2$ and when $k=3$, $d_1-2\ge d_2\ge d_3\ge 4$.

Let G be a spanning subgraph of $K_{n,n}$ with $e(G)>n^2-2n+4$. When $\delta(G)\ge 2$, by Theorem 10, G contains a Hamilton cycle.

(1) $k=2$, $\delta(G)\ge 2$.

For any $F\in\mathcal{F}(2)$, let two components of F be $2d_1$-cycle and $2d_2$-cycle. Note that $d_1-2\ge d_2\ge 2$. Let $C=v_1v_2\cdots v_{2n}v_1$ (arranged in clockwise) be a Hamilton cycle of G. There are exactly $2n$ different ways to take a pair of vertices v_i,v_j with distance $2d_2-1$ on the Hamilton cycle C in the clockwise direction. We know that the sequence of vertices $v_iv_{i+1}\cdots v_jv_i$ is a $2d_2$-cycle and $v_{j+1}v_{j+2}\cdots v_{2n}v_1\cdots v_{i-1}v_{j+1}$ is a $2d_1$-cycle in $K_{n,n}$. Then these two vertex disjoint cycles form a 2-factor isomorphic to F (when the subscript is greater than $2n$, do operation module $2n$). Noting that $2\le d_2<\frac{n-1}{2}$, the number of two-edge sets $\{v_iv_j,v_{j+1}v_{i-1}\}$ in $K_{n,n}$ is exactly $2n$ and such $2n$ two-edge sets are pairwise non-intersecting. Compared with $K_{n,n}$, G has at most $2n-5$ missing edges. So G contains at least five such two-edge sets. Taking such a two-edge set, we can obtain vertex disjoint $2d_1$-cycle and $2d_2$-cycle in G, based on the Hamilton cycle C. *I.e.*, G contains a 2-factor isomorphic to F.

(2) $k=3$, $\delta(G)\ge d_3+2$.

For any $F\in\mathcal{F}(3)$, let three components of F be $2d_1$-cycle, $2d_2$-cycle, $2d_3$-cycle. Note that $d_1-2\ge d_2\ge d_3\ge 4$. Since $\delta(G)\ge d_3+2>2$ and $e(G)>n^2-2n+4$, for any 2-factor $F'\in\mathcal{F}(2)$, G contains a 2-factor isomorphic to F'. Let $\hat{F}$ be a 2-factor with two components: a $2(d_1+d_2)$-cycle and a $2d_3$-cycle. Since $d_1+d_2-2>d_3\ge 4$, we can see that $\hat{F}\in\mathcal{F}(2)$. Then G contains a 2-factor isomorphic to $\hat{F}$ with two components: a $2(d_1+d_2)$-cycle C_1 and a $2d_3$-cycle C_2. Let

$$X_1=\{x_1,\cdots,x_{n-d_3}\},X_2=\{x_{n-d_3+1},\cdots,x_n\},$$
$$Y_1=\{y_1,\cdots,y_{n-d_3}\},Y_2=\{y_{n-d_3+1},\cdots,y_n\}.$$

For the sake of convenience, we assume

$$V(C_1)=X_1\cup Y_1,$$
$$V(C_2)=X_2\cup Y_2.$$

Write $G_1=G[X_1\cup Y_1]$.

To the contrary, we assume that G contains no subgraphs isomorphic to F. Then G_1 does not contain vertex disjoint $2d_1$-cycle and $2d_2$-cycle. (Otherwise, combined with $2d_3$-cycle C_2, G contains a 2-factor isomorphic to F, a contradiction.) Noting that G_1 is a spanning subgraph of $K_{n-d_3,n-d_3}$ and

$\delta(G_1) \ge d_3 + 2 - d_3 = 2$, we know that $e(G_1) \le (n-d_3)^2 - 2(n-d_3) + 4$ by (1). This means that G_1 has at least $2(n-d_3)-4$ missing edges compared with $K_{n-d_3,n-d_3}$, which leads to the following claim.

Claim 1. $G_1 = G[X_1 \cup Y_1]$ has at least $2(n-d_3)-4$ missing edges between X_1 and Y_1.

According to the number of edges between two vertex sets $V(C_1)$ and $V(C_2)$ in *G*, there are three cases to be discussed.

Case 1. For any d_3-set $A \subset X_1$, $G[A \cup Y_2]$ contains a $2d_3$-cycle.

By Claim 1, *G* has at least $2(n-d_3)-4$ missing edges between X_1 and Y_1 with respect to $K_{n,n}$. This means that there are at least two missing incidence edges at each vertex in X_1 (except at most four vertices) on average. Since $n \ge 3d_3 + 2$, $n - d_3 - 4 \ge d_3 + 2$. We can take a d_3-set $A' \subset X_1$ such that there are at least $2d_3$ missing edges between A' and Y_1. For the sake of convenience, the partition of *X* is adjusted as follows: $X = (X \setminus A') \cup A'$. By the assumption, $G[A' \cup Y_2]$ contains a $2d_3$-cycle. Then $G_2 = G[(X \setminus A') \cup Y_1]$ contains no vertex disjoint $2d_1$-cycle, $2d_2$-cycle. (Otherwise, *G* contains a 2-factor isomorphic to F, a contradiction.) Noting that G_2 is a spanning subgraph of $K_{n-d_3,n-d_3}$ and $\delta(G_2) \ge d_3 + 2 - d_3 = 2$, we know that $e(G_2) \le (n-d_3)^2 - 2(n-d_3) + 4$ by (1). So G_2 has at least $2(n-d_3)-4$ missing edges with respect to $K_{n-d_3,n-d_3}$. According to above discussion, *G* has at least $2d_3$ missing edges between A' and Y_1, and at least $2(n-d_3)-4$ missing edges between $X \setminus A'$ and Y_1 with respect to $K_{n,n}$. So there are at least $2d_3 + 2(n-d_3) - 4 = 2n-4$ missing edges in total. It follows that $e(G) \le n^2 - 2n + 4$, a contradiction.

Case 2. For any d_3-set $B \subset Y_1$, $G[B \cup X_2]$ contains a $2d_3$-cycle.

By symmetry, the proof in Case 2 is same as the one in Case 1.

Case 3. There exist two d_3-sets $A \subset X_1$, $B \subset Y_1$ such that both $G[A \cup Y_2]$ and $G[B \cup X_2]$ do not contain $2d_3$-cycle.

Let $G_3 = G[A \cup Y_2]$, $G_4 = G[B \cup X_2]$. If $\delta(G_i) = 0$, $i = 3,4$, G_i has at least d_3 missing edges. If $\delta(G_i) = 1$, $i = 3,4$, G_i has at least $d_3 - 1$ missing edges. If $\delta(G_i) \ge 2$, $i = 3,4$, G_i has at least $2d_3 - 4 \ge d_3$ missing edges by Theorem 10. If both $G[X_1 \cup Y_2]$ and $G[X_2 \cup Y_1]$ have at least d_3 missing edges, by Claim 1, *G* has at least $2(n-d_3) - 4 + d_3 + d_3 = 2n-4$ missing edges, a contraction. So at least one of $G[X_1 \cup Y_2]$ and $G[X_2 \cup Y_1]$, say $G[X_1 \cup Y_2] \supset G_3$, has exactly $d_3 - 1$ missing edges. According to the above discussion, the unique possibility is that G_3 has exactly $d_3 - 1$ missing edges and $\delta(G_3) = 1$. This means that the minimum degree vertex, say *w*, is unique in G_3, and $G_3 - \{w\}$ is a complete bipartite graph, where $G_3 - \{w\}$ is the subgraph of G_3 by removed of *w*. Next, we discuss two cases according to the minimum degree vertex belonging to A $(\subset X_1)$ or Y_2.

Case 3.1. If the minimum degree vertex *w* of G_3 in *A*, then by $\delta(G) \ge d_3 + 2$, *w* has at least $d_3 + 1$ neighbors in Y_1. So there are at most
$n - d_3 - (d_3 + 1) = n - 2d_3 - 1$ missing edges between $\{w\}$ and Y_1. Noting that $d_3 \ge 4$, there are at least $2(n-d_3) - 4 - (n - 2d_3 - 1) = n - 3 \ge n - d_3 + 1$ missing

edges between $X_1\setminus\{w\}$ and Y_1. By the pigeonhole principle, there exists at least one vertex $x\in X_1\setminus\{w\}$ such that there are at least two missing edges between $\{x\}$ and Y_1. So we can take a d_3-set $S\subset X_1\setminus\{w\}$ such that there are at least d_3+1 missing edges between S and Y_1. (We can choose the first d_3 vertices with missing degree as large as possible of $X_1\setminus\{w\}$ in G_1) Noting that $G[(X_1\setminus\{w\})\cup Y_2]$ is a complete bipartite graph and $S\subset X_1\setminus\{w\}$, $G[S\cup Y_2]$ contains a $2d_3$-cycle. (See Figure 1)

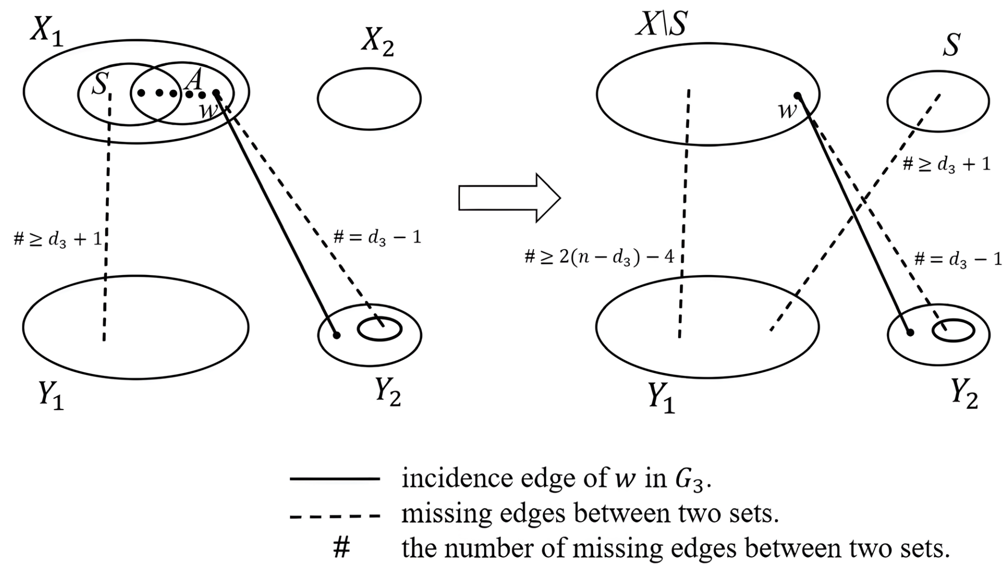

Figure 1. Choosing a d_3-set $S\subset X_1\setminus\{w\}$ to replace X_2 in Case 3.1, where $\delta(G_3)=d_{G_3}(w)=1$ and $w\in A$.

Case 3.2. If the minimum degree vertex w of G_3 in Y_2, then w has exactly d_3-1 non-neighbors in X_1. Let x_1 be a vertex of such d_3-1 vertices and x_1 has the smallest missing degree in G_1. Recalling that both $G[X_1\cup Y_2]$ and $G[X_2\cup Y_1]$ have at least d_3-1 missing edges and $e(G)>n^2-2n+4$, G_1 has at most $2n-5-(d_3-1)-(d_3-1)=2n-2d_3-3$ missing edges. By the pigeonhole principle, there are at most $\frac{1}{d_3-1}(2n-2d_3-3)$ missing edges between $\{x_1\}$ and Y_1. Recalling that G_1 has at least $2(n-d_3)-4$ missing edges and $d_1-2\geq d_2\geq d_3\geq 4$, the number h of missing edges between $X_1\setminus\{x_1\}$ and Y_1 satisfies that

$$
\begin{aligned}
h &\geq 2(n-d_3)-4-\frac{1}{d_3-1}(2n-2d_3-3)\\
&=\left(1-\frac{1}{d_3-1}\right)(2n-2d_3-3)-1\\
&\geq \frac{2}{3}(2n-2d_3-3)-1\\
&=\frac{4}{3}n-\frac{4}{3}d_3-3
\end{aligned}
$$

$$= n - d_3 + \frac{1}{3}n - \frac{1}{3}d_3 - 3$$
$$\geq n - d_3 + \frac{1}{3}(2d_3 + 2) - 3$$
$$> n - d_3.$$

This means that there are at least $n - d_3 + 1$ missing edges between $X_1 \setminus \{x_1\}$ and Y_1. Therefore, we can pick a d_3-set $S \subset X_1 \setminus \{x_1\}$ such that the number of missing edges between S and Y_1 is at least $d_3 + 1$. (We can choose the first d_3 vertices with missing degree as large as possible of $X_1 \setminus \{x_1\}$ in G_1.) Noting that $x_1 w \notin E(G)$ and there are exactly $d_3 - 1$ missing edges between X_1 and Y_2, there are at most $d_3 - 2$ missing edges between S and Y_2. By Corollary 2, $G[S \cup Y_2]$ contains a $2d_3$-cycle. (See **Figure 2**)

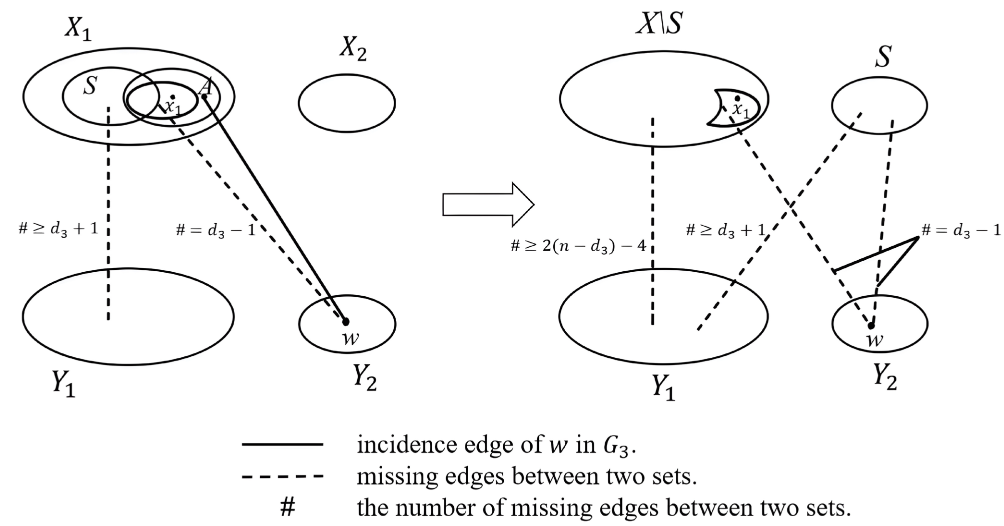

Figure 2. Choosing a d_3-set $S \subset X_1 \setminus \{x_1\}$ to replace X_2 in Case 3.2, where $\delta(G_3) = d_{G_3}(w) = 1$ and $w \in Y_2$.

Totally, in Case 3.1 or Case 3.2, we can always pick a d_3-set $S \subset X_1$ such that $G[S \cup Y_2]$ contains a $2d_3$-cycle and there are at least $d_3 + 1$ missing edges between S and Y_1. Now, the partition of X is adjusted as follows:
$X = (X \setminus S) \cup S$. Noting that $G[S \cup Y_2]$ contains a $2d_3$-cycle,
$G_5 = G[(X \setminus S) \cup Y_1]$ contains no vertex disjoint $2d_1$-cycle, $2d_2$-cycle. (Otherwise, G contains a 2-factor isomorphic to F, a contradiction.) Noting that G_5 is a subgraph of $K_{n-d_3,n-d_3}$ and $\delta(G_5) \geq d_3 + 2 - d_3 = 2$, we know that
$e(G_5) \leq (n - d_3)^2 - 2(n - d_3) + 4$ by (1). So G_5 has at least $2(n - d_3) - 4$ missing edges with respect to $K_{n-d_3,n-d_3}$. According to above discussion, G has at least $d_3 - 1$ missing edges between X and Y_2, at least $d_3 + 1$ missing edges between S and Y_1, and at least $2(n - d_3) - 4$ missing edges between $X \setminus S$ and Y_1 with respect to $K_{n,n}$. In total, there are at least
$2(n - d_3) - 4 + (d_3 - 1) + (d_3 + 1) = 2n - 4$ missing edges with respect to $K_{n,n}$. It follows that $e(G) \leq n^2 - 2n + 4$, a contradiction.

In conclusion, for any balanced bipartite graph G of order $2n$ $(n \geq 6)$ and $e(G) > n^2 - 2n + 4$, if $\delta(G) \geq 2$, then G contains a 2-factor isomorphic to F for every $F \in \mathcal{F}(2)$; if $\delta(G) \geq d_3 + 2$, then G contains a 2-factor isomorphic to F for every $F \in \mathcal{F}(3)$.

□

5. Conclusions and Suggestions

Theorem 12 extends one result of Theorem 10 under condition (1). Furthermore, it is possible to relax the condition $d_3 \geq 4$ into $d_3 \geq 2$, but there are more cases to be discussed.

The condition $\delta(G) \geq 2$ is necessary, because G contains no 2-factors when $\delta(G) \leq 1$. **Figure 3** is a counterexample graph.

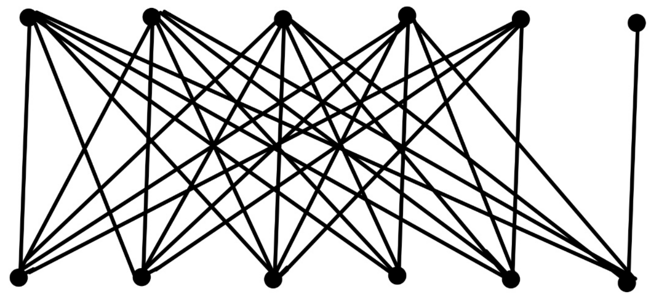

Figure 3. $e(G) = 29 > 6^2 - 2 \times 6 + 4$ and $\delta(G) < 2$, G contains no 2-factors.

It would be interesting to discuss whether the condition $\delta(G) \geq d_3 + 2$ can be replaced with $\delta(G) \geq 2$ when $k = 3$.

There are some limitations to our results. Theorem 12 only considers conditions on the minimum degree and number of edges for a balanced bipartite graph to contain 2-factors with k components, where $k \leq 3$. Actually, it is significant to consider the more general problem for the cases $k \geq 4$.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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