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Two-Dimensional Lattice-Ordered Algebras over $\mathbb{R}$-Revisit

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Abstract

In this paper, it is first pointed out that the enumeration of two-dimensional lattice-ordered real algebras provided by Birkhoff and Pierce is complete. Then we classify lattice-ordered algebra $F \times F$ over a totally ordered field F. Finally in section 4, we provide more information on the partially ordered rings in which every square is positive, especially a characterization of partially ordered fields in which every square is positive is given.

Keywords

Direct Product, ℓ-Algebra, Squares Positive, $v\ell$-Basis

1. Introduction

In [1], Birkhoff and Pierce provided the enumeration of all two-dimensional lattice-ordered algebras (under the isomorphism) over the real field $\mathbb{R}$ to show how pathological lattice-ordered algebras can be ([1], Example 9, p. 48). In his book "Lattice-Ordered Rings and Modules" published in 2010, Steinberg supplied more details to Birkhoff and Pierce's constructions ([2], Exercise 24, p. 140).

In [3], the authors corrected a typo of Birkhoff and Pierce's classification ([3], Footnote 2) and claimed that they have added new cases to Birkhoff and Pierce's classification of two-dimensional lattice-ordered real algebras. In Section 2, we show that the additional cases presented in [3] are actually isomorphic to the two-dimensional lattice-ordered real algebras classified by Birkhoff and Pierce [1], so the additional cases given in [3] are not new and Birkhoff and Pierce's classification is complete. In Section 3, we classify lattice-ordered algebras $F \times F$ over a totally ordered field F. When $F = \mathbb{R}$, we recover the Birkhoff and Pirece's results for this case. It provides another proof that the additional cases presented in [3] are already covered by Birkhoff and Pierce's classification. Section 4 is separated from the previous sections. It provides more information related to a statement in [3] on lattice-ordered division ring in which every square is positive.

The readers are referred to the references for undefined terminologies and definitions.

2. Two-Dimensional Lattice-Ordered Real Algebras

In this section, we show that the additional cases of two-dimensional lattice-ordered real algebras presented in [3] are not new and isomorphic to those presented in [1]. We consider following cases from [3].

(I) (**Case 2** ([3], p. 2551)) In this case, the multiplication on $A = \mathbb{R} \times \mathbb{R}$ is given by $e_1^2 = e_2$, $e_1e_2 = e_2e_1 = e_2^2 = 0$. In ([3], Case 2), it states that in the non-Archimedean case, A^+ must be taken as $-\pi/2 < \theta \le \pi/2$ or $\pi/2 \le \theta < 3\pi/2$.

However in ([2], 24(a), p. 140), the only non-Archimedean lattice order was the one determined by $-\pi/2 < \theta \le \pi/2$. In fact, two lattice orders determined by $-\pi/2 < \theta \le \pi/2$ and $\pi/2 \le \theta < 3\pi/2$ are isomorphic as shown below. Let φ be the reflection on y-axis, that is, $\varphi : A \to A$, for any ordered pair $(x, y) \in A$, $\varphi(x, y) = (-x, y)$. It is straightforward to check that φ is an algebra automorphism of A with the multiplication defined above. The positive cone determined by $-\pi/2 < \theta \le \pi/2$ is mapped by φ to the positive cone determined by $\pi/2 \le \theta < 3\pi/2$. Thus, the lattice-ordered algebra with the positive cone determined by $-\pi/2 < \theta \le \pi/2$ is isomorphic to the lattice-ordered algebra with the positive cone determined by $\pi/2 \le \theta < 3\pi/2$.

(II) (**Case 6** ([3], p. 2551)) In this case, the multiplication on $A = \mathbb{R} \times \mathbb{R}$ is given by $e_1^2 = e_1$, $e_2^2 = e_2$, $e_1e_2 = e_2e_1 = 0$, that is, A is the direct product of two copies of $\mathbb{R}$. In ([3], Case 6), it states that in the Archimedean case, the positive cone A^+ is determined by following five possibilities.

(1) $\alpha = 0$ and $\beta = \pi/2$,

(2) $-\pi/4 \le \alpha \le 0 \le \arctan(\tan^2 \alpha) \le \beta \le \pi/4 (\alpha \ne \beta)$,

(3) $0 = \alpha < \beta \le \pi/4$,

(4) $\pi/4 \le \alpha < \beta = \pi/2$,

(5) $3\pi/4 \ge \beta \ge \pi/2 \ge \arctan(\tan^2 \beta) \ge \alpha \ge \pi/4 (\alpha \ne \beta)$.

The authors in [3] claimed that Birkhoff and Pierce only listed (1) and (2) in [1].

The (3) seems a special case of (2) when taking $\alpha = 0$ in (2). Let ϕ be the reflection on the straight line $y = x$, that is, $\phi : A \to A$, for any ordered pair $(x, y) \in A$, $\phi(x, y) = (y, x)$. It is straightforward to check that ϕ is an algebra automorphism of A with the multiplication defined above.

Moreover, ϕ sends the positive cone determined by (3) to the positive cone determined by (4), so the lattice-ordered algebra A with the positive cone determined by (3) is isomorphic to the lattice-ordered algebra A with the positive cone determined by (4). Similarly, by ϕ, the lattice-ordered algebra A with the positive cone determined by (2) is isomorphic to the lattice-ordered algebra A with the positive cone determined by (5).

3. Lattice-Ordered Algebras $F \times F$

Let F be a totally ordered field and $A = F \times F = \{(a,b) \mid a,b \in F\}$ be the direct product of two copies of F. Then A is a two-dimensional algebra over F with the coordinate-wise operations. In this section, we determine all isomorphic lattice-ordered algebras A over F.

We use 1 to denote the identity element of A, that is, $1 = (1,1)$ and consider two cases: 1 is positive and 1 is not positive. In the following, $\geq$ denotes the lattice order on A and $\geq_F$ denotes the total order on F. A $v\ell$ *-basis* $\{u,v\}$ for the lattice-ordered algebra A over F is a vector space basis of A over F, and for any $z \in A$ such that $z = au + bv$, $z \geq 0$ if and only if there are $a,b \geq_F 0$. If $u,v \in A$ are nonzero elements such that $u \wedge v = 0$, then $\{u,v\}$ is a $v\ell$-basis for A ([4], Theorem 1.13).

Theorem 1. Let $A = F \times F$ be a lattice-ordered algebra over F with $1 > 0$. Then A is isomorphic to the lattice-ordered algebra (A,P) with the positive cone:

1. $P = \{(a,b) \in A \mid a,b \geq_F 0\}$, or
2. $P = \{a1 + bu \mid a,b \geq_F 0, \text{ where } u = c1 + de \neq 0, 0 \leq_F c \leq_F d, e = (1,-1)\}$.

Proof. It is clear that the positive cone in (1) is a lattice order. For the positive cone in (2), a direct calculation shows that 1 and u are linearly independent over $\mathbb{R}$, and $u^2 = (d^2 - c^2)1 + 2cu$. Since $d^2 - c^2 \geq 0$, $c \geq 0$, by ([1], Example 6), it is a lattice order. We show that every lattice order $\geq$ on A with $1 > 0$ is isomorphic to the lattice orders given in (1) or (2).

Since A contains nontrivial idempotent elements $(1,0),(0,1)$, $\geq$ cannot be a totally order. Thus there exist $u,v \in A$ such that $u > 0$, $v > 0$ and $u \wedge v = 0$, so $(1 \wedge u) \wedge (1 \wedge v) = 0$. Consider following two cases.

(1) $u_1 = 1 \wedge u \neq 0$ and $v_1 = 1 \wedge v \neq 0$. Then $u_1 \wedge v_1 = 0$ implies that u_1, v_1 form a $v\ell$-basis. Hence $1 = \delta u_1 + \gamma v_1$, and $\delta, \gamma \geq_F 0$. Let $z_1 = \delta u_1$ and $z_2 = \gamma v_1$. Then $z_1 \neq 0, z_2 \neq 0$ form a $v\ell$-basis and $1 = z_1 + z_2$. Thus $z_1^2 = z_1$, $z_2^2 = z_2$, $z_1 z_2 = z_2 z_1 = 0$. Then the mapping that maps each element $xz_1 + yz_2$ to the ordered pair (x,y) is an isomorphism from the lattice-ordered algebra A to the lattice-ordered algebra (A,P) with the P given in (1).

(2) $1 \wedge u = 0$ or $1 \wedge v = 0$. Without loss of generality, we may assume that $1 \wedge u = 0$. Then $\{1,u\}$ is a $v\ell$-basis of A over F. Since $\{1, e = (1,-1)\}$ is a vector space basis of A over F, $u = c1 + de$ for some $c,d \in F$. Then

$$d^2 1 = (de)^2 = (u - c1)^2 = u^2 - (2c)u + c^2 1,$$

so $u^2 = (d^2 - c^2)1 + (2c)u \geq 0$ implies that $d^2 - c^2 \geq_F 0$ and $c \geq_F 0$. Since $(d-c)(d+c) \geq 0$, $0 \leq_F c \leq_F d$ and $d \leq_F -c \leq_F 0$. For the first case, the positive cone of A is

$$A^+ = \{a1 + bu \mid a,b \geq_F 0\} \text{ with } u = c1 + de \neq 0, 0 \leq_F c \leq_F d.$$

Thus A^+ is the positive cone given in Theorem 1(2). For the second case, the positive cone of A is

$$\begin{aligned} A^+ &= \{a1 + bu \mid a, b \geq_F 0, u = c1 + de \neq 0, d \leq_F -c \leq_F 0, e = (1, -1)\} \\ &= \{a1 + bu \mid a, b \geq_F 0, u = c1 + (-d)(-e) \neq 0, 0 \leq_F c \leq_F -d\}. \end{aligned}$$

Then under the automorphism $\phi(x, y) = (y, x)$ for any ordered pair $(x, y) \in A$, (A, A^+) is isomorphic to the lattice-ordered algebras A with the positive cone given in Theorem 1(2), since $\phi(a1 + bu) = a1 + b(c1 + (-d)e)$.

Remark Let $F = \mathbb{R}$ in Theorem 1. The positive cone in Theorem 1(1) is the positive cone determined in Section 2 (II) (1): $\alpha = 0$ and $\beta = \pi/2$; and the positive cone in Theorem 1(2) is the positive cone determined in Section 2 (II) (2) with $-\pi/4 \leq \alpha \leq 0$ and $\beta = \pi/4$.

Let R be a lattice-ordered ring with 1 and positive cone P. Suppose $u \in P$ is a unit of R. Then uP is the positive cone of a lattice order on R ([4], Theorem 1.19). Let $1 \in P$. If u^{-1} is also in P, then $uP = P$. If u^{-1} is not in P, then uP is a lattice order with $1 \notin uP$. This method provides a way to produce lattice orders with $1 \not> 0$ from lattice orders with $1 > 0$. For instance, for the matrix algebras $M_n(F)$ over a subfield F of $\mathbb{R}$, each lattice order is isomorphic to $AM_n(F^+)$, where $A \in M_n(F^+)$ is an invertible matrix and $M_n(F^+)$ is the entry-wise lattice order on $M_n(F)$ [5].

Theorem 2. Let $A = F \times F$ be a lattice-ordered algebra over F with $1 \not> 0$. Then A is isomorphic to the lattice-ordered algebra A with the positive cone wP, where P is the positive cone given in Theorem 1(2) and $w \in P$ is an invertible element of A.

Proof. Since A cannot be a totally ordered algebra, there exist $u, v \in A$ such that $u \wedge v = 0$ and $u > 0$, $v > 0$. So $\{u, v\}$ is a $v\ell$-basis of A over F. Let $eu = a_1u + b_1v$ and $ev = a_2u + b_2v$ for some $a_1, a_2, b_1, b_2 \in F$, where $e = (1, -1)$. Since $e^2 = 1$, we have

$$\begin{aligned} u &= e(eu) \\ &= e(a_1u + b_1v) \\ &= a_1(eu) + b_1(ev) \\ &= a_1(a_1u + b_1v) + b_1(a_2u + b_2v) \\ &= (a_1^2 + b_1a_2)u + (a_1b_1 + b_1b_2)v, \end{aligned}$$

so $a_1^2 + b_1a_2 = 1$ and $a_1b_1 + b_1b_2 = b_1(a_1 + b_2) = 0$. Similarly,

$$v = (a_2a_1 + b_2a_2)u + (a_2b_1 + b_2^2)v,$$

implies $a_2b_1 + b_2^2 = 1$ and $a_2a_1 + b_2a_2 = a_2(a_1 + b_2) = 0$.

We first notice that $a_1 + b_2 = 0$. If $a_1 + b_2 \neq 0$. Then $b_1 = a_2 = 0$, so $eu = a_1u$, $ev = b_2v$, and $a_1^2 = b_2^2 = 1$. If $a_1 = b_2 = 1$ or $a_1 = b_2 = -1$, then u and v are linearly dependent, a contradiction. Let $a_1 = 1$ and $b_2 = -1$. Then $eu = u$ and $ev = -v$. Suppose that $1 = au + bv$ for some $a, b \in F$. Then $e = au + (-b)v$. Since 1 is not positive, one of a, b is positive and another one is negative, and hence e is positive or negative. It follows that $e^2 = 1$ is positive, a contradiction. Similarly, $a_1 = -1$ and $b_2 = 1$ will cause a contradiction as well. Therefore, we must have $a_1 + b_2 = 0$. We consider following cases.

(I) $a_1 + b_2 = 0$, and $a_1 = b_2 = 0$. Then $eu = b_1 v$, $ev = a_2 u$, and $a_2 b_1 = 1$. Let us consider following cases.

(1) $a_2 >_F 0$. Since $u \wedge v = 0$ implies that $a_2 u \wedge v = 0$ ([4], (17), p.~45), $v \wedge ev = 0$, so $\{v, ev\}$ is a $v\ell$-basis. Then the positive cone is given by:

$$A^+ = \left\{ av + b(ev) \mid a, b \in F^+ \right\} = vP,$$

where $P = \{a + be \mid a, b \geq_F 0\}$ is the positive cone given in Theorem 1(2) with $c = 0$ and $d = 1$. Let $v = s1 + te$ for some $s, t \in F$. We have $v^2 = sv + t(ev) \geq_{A^+} 0$, so $s, t \geq_F 0$ and hence $v \in P$.

(2) $a_2 <_F 0$. Then $-ev = (-a_2)u > 0$ implies $v \wedge -ev = 0$, so $\{v, -ev\}$ is a $v\ell$-basis. Then the positive cone is given by:

$$A^+ = \left\{ av + b(-ev) \mid a, b \in F^+ \right\} = vP',$$

where $P' = \{a1 + b(-e) \mid a, b \geq_F 0\}$. Since $\phi: A \to A$, $\phi(x, y) = (y, x)$ is an algebra automorphism of A and $\phi(a1 + b(-e)) = a1 + b(\phi(-e)) = a1 + be$, $\phi(A^+) = \phi(v)P$, where $P = \{a1 + be \mid a, b \in F^+\}$ is a positive cone given in Theorem 1(2). Let $v = s1 + t(-e)$ for some $s, t \in F$. Then $v^2 = sv + t(-ev) \geq 0$ implies that $s, t \geq_F$, so $\phi(v) = s1 + te \in P$. Therefore, (A, A^+) is isomorphic to $(A, \phi(v)P)$.

(II) $a_1 + b_2 = 0$, and $a_1 >_F 0$. Then $b_2 = -a_1 <_F 0$.

(1) $a_2 >_F 0$. Let $z = -b_2 1 + e = a_1 1 + e$. Then $zv = ev - b_2 v = a_2 u > 0$, so $v \wedge zv = 0$ and hence $\{v, zv\}$ is a $v\ell$-basis. Thus, the positive cone:

$$A^+ = \left\{ av + b(zv) \mid a, b \geq_F 0 \right\} = vP,$$

where $P = \{a1 + bz \mid a, b \geq_F 0\}$. We have:

$$\begin{aligned} z^2 &= (a_1 1 + e)^2 \\ &= (1 + a_1^2)1 + 2a_1 e \\ &= (1 + a_1^2)1 + 2a_1(z - a_1 1) \\ &= (1 - a_1^2)1 + 2a_1 z. \end{aligned}$$

Let $v^2 = f_1 v + f_2(zv)$ and $zv^2 = (zv)v = g_1 v + g_2(zv)$ with $f_1, f_2, g_1, g_2 \geq_F 0$. Then

$$\begin{aligned} zv^2 &= z(f_1 v + f_2(zv)) \\ &= f_1(zv) + f_2(z^2 v) \\ &= f_1(zv) + f_2(1 - a_1^2)v + 2f_2 a_1(zv) \\ &= f_2(1 - a_1^2)v + (f_1 + 2f_2 a_1)(zv). \end{aligned}$$

Thus $f_2(1 - a_1^2) = g_1$ and $f_1 + 2f_2 a_1 = g_2$.

If $g_1 >_F 0$, then $f_2 >_F 0$ and $1 - a_1^2 >_F 0$. It follows that $z^2 = (1 - a_1^2)1 + 2a_1 z \in P$, so P is a lattice order with $1 > 0$ [1]. Since $z = a_1 1 + e$ and $0 \leq_F a_1 \leq_F 1$, P is the lattice order given in Theorem 1(2). Let $v = s + tz$ for some $s, t \in F$. Then $v^2 = sv + t(zv)$ implies that $s, t \geq_F 0$, and hence $v \in P$.

If $g_1 = 0$, then $f_2 = 0$ or $1 - a_1^2 = 0$. Suppose $f_2 = 0$. Then $v^2 = f_1 v$ and $zv^2 = g_2(zv)$. Since A does not contain nonzero nilpotent element, $f_1 >_F 0$ and $f_1 = g_2$. We have

$$(zv)^2 = z^2v^2 = f_1 z^2 v = f_1(1 - a_1^2)v + 2 f_1 a_1 (zv) \ge 0,$$

and hence $1 - a_1^2 \ge_F 0$. Thus P is a lattice order with $1 > 0$, $z = a_1 1 + e$, and $0 \le_F a_1 \le_F 1$, so P is given in Theorem 1(2) and $v \in P$.

(2) $a_2 \le_F 0$. If $b_1 \ge_F 0$, then $eu \ge 0$ and $ev \le 0$, and hence $euv = 0$, so $uv = 0$. Let $1 = su + tv$ for some $s, t \in F$. Then

$$1 = 1^2 = (su + tv)^2 = s^2u^2 + t^2v^2 \ge 0,$$

a contradiction. Hence, we must have $b_1 <_F 0$. Let $w = -e + a_1 1$. Then $wu = -eu + a_1 u = -b_1 v > 0$, so $\{u, wu\}$ is a $v\ell$-basis and the positive cone is:

$$A^+ = \{au + b(wu) \mid a, b \ge_F 0\} = uP',$$

where $P' = \{a + bz \mid a, b \ge_F 0\}$. Since

$$w^2 = (-e + a_1 1)^2 = 1 - 2a_1 e + a_1^2 1 = (1 + a_1^2)1 - 2a_1 e = (1 - a_1^2)1 + 2a_1 w,$$

and $a_1^2 + b_1 a_2 = 1$ and $b_1 a_2 \ge_F 0$ implies $1 - a_1^2 \ge_F 0$, $w^2 \in P'$. Thus P' is a lattice order with $1 > 0$. Let $u = s1 + tw$. Then $u^2 = su + t(wu) \ge 0$ implies $s, t \ge_F 0$, so $u \in P'$.

Let $\phi : A \to A$, $\phi(x, y) = (y, x)$. One may verify that for any $a, b \in F$, $\varphi(a1 + bw) = a1 + b(a_1 1 + e)$, so $\phi(A^+) = \phi(u)P$, where

$$P = \phi(P') = \{a + b(a_1 1 + e) \mid a, b \ge_F 0\}$$

is a lattice order given in Theorem 1(2) and $\phi(u) \in P$. Thus (A, A^+) is isomorphic to $(A, \phi(u)P)$.

(III) $a_1 + b_2 = 0$, and $a_1 < 0$. Then $b_2 = -a_1 > 0$. The proof for this case is similar to the case (II) and is omitted.

In summary, any lattice-ordered algebra A over F with $1 \not> 0$ is isomorphic to the lattice-ordered algebra A with the positive cone wP, where P is a lattice order with $1 > 0$ given in Theorem 1(2) and $w \in P$. This completes the proof.

Remark. Consider the lattice-ordered real algebra $A = \mathbb{R} \times \mathbb{R}$ with $1 \not> 0$. Then by Theorem 2, A^+ is isomorphic to wP, where P is given in Theorem 1(2) and $w \in P$ is an invertible element of A such that $w^{-1} \notin P$. Let $w = (x, y)$. Then $x >_{\mathbb{R}} 0$. From Theorem 1(2),

$$P = \{a1 + bu \mid a, b \in \mathbb{R}^+, u = c1 + de \ne 0, 0 \le_{\mathbb{R}^+} c \le_{\mathbb{R}^+} d, c, d \in \mathbb{R}\}.$$

Then $wP = \{aw + b(wu) \mid a, b \in \mathbb{R}^+\}$. Since $w \in P$ is invertible and $w^{-1} \notin P$, we need to consider two cases.

(1) $0 <_{\mathbb{R}^+} y <_{\mathbb{R}^+} x$.

Let $\alpha = \arg(wu)$ and $\beta = \arg w$. Then

$$wP = \{z \in A \mid \alpha \le \arg z \le \beta\}.$$

Since $u=(c+d,c-d)$, $wu=(x(c+d),y(c-d))$ and hence

$\tan^2\alpha=\left(\frac{y(c-d)}{x(c+d)}\right)^2<_{\mathbb{R}^+}\frac{y}{x}=\tan\beta$. Thus $-\pi/4<\alpha\le 0$ and

$\text{arc}(\tan^2\alpha)\le\beta<\pi/4$, so wP is the positive cone given in the section 2 (II)(2).

(2) $\frac{c-d}{c+d}\le_{\mathbb{R}^+}\frac{y}{x}<_{\mathbb{R}^+}0$.

Let $\beta=\arg(wu)$ and $\alpha=\arg w$. Then

$$wP=\{z\in A\,|\,\alpha\le\arg z\le\beta\}.$$

Since $\tan\alpha=\frac{y}{x}$ and $\tan\beta=\frac{y(c-d)}{x(c+d)}$, $\frac{c-d}{c+d}\le_{\mathbb{R}^+}\frac{y}{x}<_{\mathbb{R}^+}0$ implies that $\tan^2\alpha=\left(\frac{y}{x}\right)^2\le_{\mathbb{R}^+}\frac{y(c-d)}{x(c+d)}=\tan\beta$. Thus $-\pi/4<\alpha\le 0$ and

$\text{arc}(\tan^2\alpha)\le\beta<\pi/4$, so wP is the positive cone given in the section 2 (II) (2).

Therefore, a lattice-ordered algebra $A=\mathbb{R}\times\mathbb{R}$ in which $1\not>0$ is isomorphic to a lattice-ordered algebra A with the positive cone given in the section 2 (II) (2) with $-\pi/4<\alpha$ and $\beta<\pi/4$.

4. Partially Ordered Rings with Squares Positive

The topics in this section are separated from the previous sections. We provide here more information on partially ordered rings in which every square is positive. In ([3], Remark 4), it stated that in 2006, Yang first proved that a lattice-ordered skew-field (skew-fields are also called division rings) in which any square is positive must be totally ordered.

In 1956, Birkhoff and Pierce proved that a lattice-ordered field in which every square is positive must be totally ordered ([1], Corollary 2, p. 59). In the proof, they didn't use the commutative condition for the multiplication. Therefore, as pointed out by Steinberg in 1970, Birkhoff and Pierce actually proved that a lattice-ordered division ring in which every square is positive is totally ordered [6], although they didn't precisely state the result. Steinberg also generalized the result to the following result [6].

Theorem 3. Let R be a lattice-ordered ring with the identity element in which every square is positive. If R has the minimal condition on right ideals, then R is an f-ring.

Since a division ring has the minimal condition on right ideals and it is well known that a division ring that is f-ring must be totally ordered, a lattice-ordered division ring in which every square is positive must be totally ordered by Steinberg's result.

For the readers' convenience, we present an elementary direct proof that a lattice-ordered division ring R in which each square is positive must be totally ordered. Let $x\in R$. Define $a=x^+ +1$ and $b=x^- +1$. Then $a^{-1}=a\left(a^{-1}\right)^2>0$ and similarly $b^{-1}>0$. We have

$$0 \le a^{-1}\left(ax^{+}b \wedge ax^{-}b\right)b^{-1} \le a^{-1}ax^{+}bb^{-1} \wedge a^{-1}ax^{-}bb^{-1} = x^{+} \wedge x^{-} = 0.$$

It follows that $a^{-1}\left(ax^{+}b \wedge ax^{-}b\right)b^{-1} = 0$, so $ax^{+}b \wedge ax^{-}b = 0$. Thus

$$x^{+}x^{-} = x^{+}x^{-} \wedge x^{+}x^{-} \le ax^{+}b \wedge ax^{-}b = 0,$$

so $x^{+}x^{-} = 0$ and hence $x^{+} = 0$ or $x^{-} = 0$. Therefore, R is totally ordered.

A related question is how to characterize partially ordered field in which every square is positive. Recall that a partially ordered ring $(R,\ge)$ is called *division closed* if for any $a,b \in R$, $ab > 0$ and $a > 0$ (or $b > 0$) implies $b > 0$ (or $a > 0$).

Theorem 4. Let R be an integral domain and P be a partial order on R. Then P is division closed and $\forall x \in R, x^2 \in P$ if and only if P is the intersection of all the total orders containing P.

Proof. It is clear that if P is the intersection of all the total orders containing P, then P is division closed and for each element $x \in R$, $x^2 \in P$.

Now suppose that P is division closed and for each element $a \in R$, $a^2 \in P$. Let P_1 be the intersection of all maximal partial orders containing P. If $P \ne P_1$, take $x \in P_1 \setminus P$ and define $P' = P + P(-x)$. Since $(-x)^2 \in P$, $P' + P' \subseteq P'$ and $P'P' \subseteq P'$. If $P' \cap -P' \ne \{0\}$, then there are $0 \ne a$, $0 \ne b \in P$ such that $a + b(-x) = 0$, so $bx = a$ and P is division closed implies that $x \in P$, a contradiction. Thus we must have $P' \cap -P' = \{0\}$, that is, P' is a partial order on R. By Zorn's lemma, $P' \subseteq P_m$, where P_m is a maximal partial order. Since $P \subseteq P' \subseteq P_m$, $x \in P_1$ implies $x \in P_m$, also $-x \in P' \subseteq P_m$, a contradiction. Thus $P = P_1$.

Let P_m be a maximal partial order containing P. Then $\forall x \in R$, $x^2 \in P_m$, so P_m is a total order on R ([7]). Therefore, P is the intersection of all the total orders containing P.

When Theorem 4 is applied to a partially ordered field, we have following corollary.

Corollary 1. (Dubios) Let $(F,\ge)$ be a partially ordered field. Then every square is positive if and only if the partial order is an intersection of total orders.

Proof. Let $a,b \in F$ such that $ab > 0$ and $a > 0$. Then $a^{-1} = a(a^{-1})^2 > 0$, so $b = a^{-1}(ab) > 0$. Thus, the partial order is division closed, the result follows from Theorem 4.

Summary. 70 years ago, Birkhoff and Pierce published the paper "Lattice-ordered Rings". This was the first paper that provided a systematic study of lattice-ordered rings. The main purpose of the current paper is to show the readers that the classification of two-dimensional real lattice-ordered algebras presented in Birkhoff and Pierce's paper is correct and complete. The method to produce the lattice orders in which $1 \not> 0$ in Section 3 can be used for any two-dimensional ℓ-algebras over a totally ordered field.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

References

[1] Birkhoff, G. and Pierce, R.S. (1956) Lattice-Ordered Rings. *Anais da Academia Brasileira de Ciências*, **28**, 41-69.

[2] Steinberg, S. (2010) Lattice-Ordered Rings and Modules. Springer.

[3] Yang, Y. and Zhang, X. (2017) Note on Classification of Two-Dimensional Associative Lattice-Ordered Real Algebras. *Soft Computing*, **21**, 2549-2552. https://doi.org/10.1007/s00500-017-2580-0

[4] Ma, J. (2014) Lecture Notes on Algebraic Structure of Lattice-Ordered Rings. World Scientific. https://doi.org/10.1142/9009

[5] Ma, J. and Wojciechowski, P.J. (2002) Lattice Orders on Matrix Algebras. *Algebra Universalis*, **47**, 435-441. https://doi.org/10.1007/s00012-002-8198-8

[6] Steinberg, S. (1970) Lattice-Ordered Rings and Modules. Ph.D. Thesis, University of Illinois at Urbana-Champaign.

[7] Ma, J. (2023) Some Questions on Partially Ordered Rings—A Survey. *Quaestiones Mathematicae*, **46**, 2611-2624. https://doi.org/10.2989/16073606.2023.2177206