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On the Global Attractivity of a Delayed Boom Model with Diffusion

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Abstract

Some mathematical boom equation of SEIR type with diffusion, which appears as a model of social and economical boom for the spread of trend-causing, is treated. The asymptotic properties of the diffusive equation are studied by applying the strong maximum principle, strong fading memory property and luxury Liapunov functions.

Keywords

Delayed SEIR Type Model of Boom with Diffusion, Global Attractor, Strong Maximum Principle

1. Introduction

The space and time dependent before boom, booming, boom established and boom not established (SEIR type) model was proposed and applied to fit and then predict the space and time series of trend diffusive evolution observed in the few years till 2021 in various provinces and metropolises in Japan [1]-[4]. These SEIR-type models have responded differently to monitoring of trend so far, although these predictions contain uncertainty due to the intrinsic change of the maximum boom population and boom established/boom not established rates within the different areas. Mathematical models are among the necessary tools to quantify the social boom dynamics and are the primary objective motivating this study. To address the questions mentioned above, this study is organized as follows. Section 2 proposes an updated epidemic SEIR type model for, where " S ", " E ", " I " and " R " stand for before boom, booming, established boom and not established boom people, respectively (cf. [5]). Our model is then applied to fit and predict the boom spread in various provinces and some major cities, resulting in abundant datasets to derive the core characteristics of the boom dynamics of transmission/fashion and removal.

In this paper, we shall consider the following diffusive system with boundary condition

$$
\begin{aligned}
\frac{\partial S}{\partial t}(t,x) &= d\Delta S(t,x) - \beta S(t,x)E(t-h_1,x) - \mu S(t,x) \\
&\quad + \gamma I(t-h_2,x) - \zeta + b \quad t>0, x\in\Omega, \\
\frac{\partial E}{\partial t}(t,x) &= d\Delta E(t,x) + \beta S(t,x)E(t-h_1,x) - (\kappa+\lambda)E(t,x) \\
&\quad + \mu S(t,x) \quad t>0, x\in\Omega, \\
\frac{\partial I}{\partial t}(t,x) &= d\Delta I(t,x) + \kappa E(t,x) - \gamma I(t-h_2,x) + \zeta \quad t>0, x\in\Omega, \\
\frac{\partial R}{\partial t}(t,x) &= d\Delta R(t,x) + \lambda E(t,x) - \eta R(t-h_3,x) \quad t>0, x\in\Omega, \\
\frac{\partial S}{\partial n}(t,x) &= \frac{\partial E}{\partial n}(t,x) = \frac{\partial I}{\partial n}(t,x) = \frac{\partial R}{\partial n}(t,x) = 0 \quad t>0, x\in\partial\Omega,
\end{aligned}
\tag{1}
$$

where $S(t,x)+E(t,x)+I(t,x)+R(t,x)\equiv N_1(t,x)$ denotes the total number of a population at time t and space $x\in\mathbf{R}^4$, $\mathbf{R}^4$ is a four-dimensional Euclidean space. Here Δ is the Laplacian in $\mathbf{R}^4$, $\Omega\subset\mathbf{R}^4$ is a bounded domain with smooth boundary $\partial\Omega$ and $\partial/\partial n$ is the outward normal derivative to $\partial\Omega$. For Equation (1), we assume that d is a common diffusion coefficient for $\{S,E,I,R\}$, and $1>d=$ mean value of $\{d_S,d_E,d_I,d_R\}>0$, that the diffusion coefficients of each $\{d_S,d_E,d_I,d_R\}$ for $\{S,E,I,R\}$ is average constants, because $1>d_S>d_E>d_I>d_R\gg 0$. In this paper, we will deal only with the simplified case with a common diffusion coefficient d for $\{d_S,d_E,d_I,d_R\}$.

$S:=S(t,x)$ denotes the number of the population of before boom to the disease, $E:=E(t,x)$ denotes the number of booming individual, $I:=I(t,x)$ denotes the number of boom established individual and $R:=R(t,x)$ denotes the number who have been removed from the possibility of boom established through full immunity. It assumes that all newborns are before boom. The positive number μ is a nonnegative constant, representing the death rates of before boom. In addition, the positive constants b,ζ,κ,γ and λ represent the birth rates of the population of the before boom, the number of people forced to produce from pre-booming conditions, the rate at which people in product booming transition to trend entrenchment, the rate at which people in the boom return to their pre-boom state and the rate at which people in the booming transition to the unestablished state, respectively. The positive constant β is the average number of contacts per booming per day. It is natural to assume that the assumption $\beta<\sigma$ is little bit analytically condition and it is natural in the sense of booming from some data in Japan (cf. [1]-[4]).

To summarize the above coefficients of Equation (1), it is as follows:

β: The rate at which pre-boom people transition to an on-boom state by contact.

κ: The rate at which people during an on-boom transition to a state of boom establishment.

λ: The rate of people who are in an on-boom move to an on-boom state.

μ : The rate of people before the boom transition to an on-boom state without contact.

γ : The rate of people who have become boom and return to their pre-boom state.

ζ : The number of people who are (forced) produced by the boom from the pre-boom state.

b : the number of people supplied in the pre-boom state at a certain rate to the pre-boom people.

The nonnegative constant $h_i, i = 1, 2$ is the time delay of presenting the symptoms of a boom. It is natural to assume that $h_1 < h_2$. The reason for this is that booms occur in a short period of time, but on the other hand, it seems that it takes a certain amount of time for the established boom to take hold. The term $\beta S(t,x)E(t-h_1,x)$ can be considered as the force of booming at time t and space x, respectively. For the detailed social trend meanings, refer to [1]-[4].

In particular by Ohta and Mizutani [2] [3], the rate at which people go from a per-boom state to an on-boom state is proportional to the number of people in a pre-boom state and the number of people who changed to an on-boom state before h_1 in the second terms of the first Equation in (1), and we assume that people in a pre-boom state naturally adopt booms at a fixed rate in the third term of the first Equation in (1). Moreover, it expresses the resurgence of the boom by transferring the rate of people who became on-boom prior to h_2 in the fourth term of the first Equation in (1). In addition, ζ expresses the ratio of people in a rooted boom state who were forced to enter that state in other Japanese words, “Sakura”. Furthermore, the given ratio of people in an on-boom state enters a rooted or unrooted state in the second and third terms of the second Equation in (1).

Historical Motivation. So far, the research using mathematical models of epidemics, such as influenza has been conducted. The research foundation was laid by the differential equation model by Kermack and Mckendrick from the 1920s [6] and with the spread of AIDS which became a threat in developed countries in the 1980s [7]-[9], and the spread of emerging infectious diseases such as COVID-19 from 2023 [5] [10] [11], it has attracted the attention of many researchers and continues to develop even now.

In the case of epidemical model, in 1979, for the ordinary differential equation (without time delay), Anderson and May [12] studied the asymptotic stability of the following SEIR epidemic differential equation

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= -\beta S(t)I(t) - \mu S(t) + \mu, \\ \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \beta S(t)E(t) - \kappa E(t), \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \kappa E(t) - \mu I(t) - \lambda I(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \lambda I(t) - \mu R(t), \quad t \ge 0, \end{aligned} \tag{2}$$

where, b, β, κ, μ and λ are positive constants. In (2), it assumes that the birth and the death rates of population are the same value. The original SIR model of (2) is in [6].

On the other hand, as the ordinary differential equation of SIR type with time delay, Takeuchi and Ma [13] have shown the global asymptotic stability of the solution $(S(t), I(t), R(t))$ of

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= -\beta S(t) I(t-h) - \mu_1 S(t) + b, \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \beta S(t) I(t-h) - \mu_2 I(t) - \lambda I(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \lambda I(t) - \mu_3 R(t), \quad t \geq 0, \end{aligned} \tag{3}$$

which describes the spread within a population of an infectious disease.

Recently, Hamaya and Arai [14] have studied the permanence of solution $(S(t,x), I(t,x), R(t,x))$ of the partial integrodifferential equation with diffusion for Equation (3), and also Saito [15] has studied the global asymptotic stability of a discrete SIR model.

The classical SEIR model containing four population (S, E, I and R) takes the form [16].

To allow for possible sensitive rate for COVID-19 evolution [11], we revise model (2)

$$\begin{aligned} \frac{\mathrm{d}S(t)}{\mathrm{d}t} &= -\beta_1 \frac{S(t)}{N(t)} I(t) - \beta_2 \frac{S(t)}{N(t)} E(t), \\ \frac{\mathrm{d}E(t)}{\mathrm{d}t} &= \beta_1 \frac{S(t)}{N(t)} I(t) + \beta_2 \frac{S(t)}{N(t)} E(t) - \kappa E(t), \\ \frac{\mathrm{d}I(t)}{\mathrm{d}t} &= \kappa E(t) - \lambda I(t), \\ \frac{\mathrm{d}R(t)}{\mathrm{d}t} &= \lambda I(t) - \mu R(t), \\ \frac{\mathrm{d}^\beta D(t)}{\mathrm{d}t^\beta} &= \lambda_\beta I(t) \quad t \geq 0, \end{aligned}$$

where D represents the number of deaths which is one component in I, β_2 is the number of healthy susceptible people that are contacted by the exposed people daily and μ is the rate of the removed individuals returning to the susceptible status. We add the fractional-order differential equation containing the death probability of λ while the other patients are cured,

$\frac{\mathrm{d}^\beta D}{\mathrm{d}t^\beta} = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial D(s)}{\partial s} (t-s)^{-\beta} \mathrm{d}s$, which is the Caputo fractional derivative [11] with order $\beta (0 < \beta \leq 1)$. When the order $\beta = 1$, equation reduces to the classical integer-order differential equation or the death evolution.

On the other hand, there are many research results on trends in society from the viewpoints of sociology and psychology, and there is almost no research from a mathematical perspective using models [17]-[20]. However, in recent years, companies have been focusing on the development of hit products that emphasize customer preferences and trend analysis using SNS such as Twitter and Facebook, which have become important in the marketing field.

As a result of research using mathematical models, Nakagiri and Krita [1] have mathematically modeled fashionable problems using a system of linear differential equations and fitted them to real data. This model, although mathematically linear and simple, is very versatile. In addition, Ueda and Asahi [4] extended Nakagiri's model and analyzed the construction of a model for the transfer of interest of Twitter users and the verification of the model using actual data. Recently, Ohta and Mizutani [2] have proposed a nonlinear model with a time delay that takes into account the SIR model and the innovator theory proposed by Rogers [21], extending the Nakagiri and Ueda models, analyzing the data, and evaluating and determining the parameters. However, although the model fits reality due to the nonlinear and time delay factored in, mathematical rigor is not questioned in this paper.

We study is based on the model of Ohta *et al.* [2] [3] considering the SIR model, which is a pioneering study in mathematical ecologically epidemics such as infectious diseases, and the purpose of this study is to extend the SEIR model from SIR model of Ohta *et al.* [2] [3] and propose a new social trend model with the diffusion term for spreading boom.

We consider the following key points which were discussed in our model. The first point is the contact between a susceptible person and infected person. Infectious disease epidemics such as influenza are thought to occur when a virus invades and infects a healthy person's body from contact with an infected person. In our model, we define "interesting information" to be a virus, which is transmitted from people in an on-boom state to those whom the boom has not reached yet (pre-boom). Thus, our model incorporates the perspective of the contact, which was not considered in [1].

The second point is the time delay. The Diffusion of Innovation theory, developed by E. M. Rogers [21], separates consumers into five categories based on the speed at which people are likely to adopt innovation. Based on this theory, we seem that time lags exist in the adoption of booms by people in a social system, and thus developed a model that considers the effects of a time delay, by ([2], Section 2 of 19) and see above historical motivation. Similar to [3], this model (1) views booms to transmitted ("infected") through contact as follows: people in a pre-boom state can become infected by coming into contact with information of interest, and they can enter an on-trend state after a given time. Then, when the person enters an established-trend state, he will be able to "infected" transmit the trend to others in a per-boom state.

In this article, we consider the global asymptotic properties of the solution of diffusive Equation (1) with finite delay which based on [5] (and for SIR, [8] [10] [14] [22]) and particular [2] [3].

For Equation (1), (S,E,I,R) functions ($S,E,I,R,\in C\left(\left[0,\infty\right)\times\bar{\Omega},R\right)$, is called a (classical) solution of (1) if $\partial S/\partial t$, $\partial S/\partial x$, $\partial^2 S/\partial x^2$, $\partial E/\partial t$, $\partial E/\partial x$, $\partial^2 E/\partial x^2$, $\partial I/\partial t$, $\partial I/\partial x$, $\partial^2 I/\partial x^2$, $\partial R/\partial t$, $\partial R/\partial x$ and $\partial^2 R/\partial x^2$, belong to the space $C\left(\left(0,\infty\right)\times\Omega\right)$, $\partial S/\partial n$, $\partial E/\partial n$, $\partial I/\partial n$ and $\partial R/\partial n$ exist on $\left(0,\infty\right)\times\partial\Omega$ and (1) is identically satisfied. From ([23], Chapter 6) and (cf. [24]), we can show that the existence of global classical solution is guaranteed for (1) whenever the initial function

$$\begin{aligned}
&S(\theta,x)=\phi_1(0,x)>0, x\in\bar{\Omega},\\
&E(\theta,x)=\phi_2(\theta,x)\ge 0,\theta\in C\left[-h_1,0\right], x\in\bar{\Omega},\\
&I(\theta,x)=\phi_3(\theta,x)\ge 0,\theta\in C\left[-h_2,0\right], x\in\bar{\Omega},\\
&R(\theta,x)=\phi_4(\theta,x)\ge 0,\theta\in C\left[-h_3,0\right], x\in\bar{\Omega},\\
&\text{where } \phi_i:=\phi_i(\theta,x),\\
&\phi_i(0,x)>0, x\in C^1\left(\bar{\Omega}\right),(i=1)\\
&\text{and } \phi_i:=\phi_i(\theta,x)\ge 0\quad \theta\in C\left[-h_i,0\right], x\in C^1\left(\bar{\Omega}\right),(i=2,3,4).
\end{aligned}\tag{4}$$

For any parameters $h_i(i=1,2,3),\beta,b,\kappa,\lambda$ and μ, it is easy to check that the equilibrium solution $\left(S(t,x),E(t,x),I(t,x),R(t,x)\right)$ of (1) with the initial condition (4) exists and is unique for all $t\ge 0$.

For Equation (1), it has a unique positive established boom equilibrium $\mathbf{E}^+=\left(S^*,E^*,I^*,R^*\right)$, where

$$S^*=\frac{\kappa b+\lambda b}{\beta b+\lambda\mu},\ E^*=\frac{b}{\lambda},$$

$$I^*=\frac{\kappa b+\lambda\zeta}{\gamma\lambda}\text{ and } R^*=\frac{b}{\eta}.$$

We next observe that $R(t,x)$ can be immediately obtained once $E(t,x)$ are known, so the system (1) can be reduced to

$$\begin{aligned}
&\frac{\partial S}{\partial t}(t,x)=d\Delta S(t,x)-\beta S(t,x)E(t-h_1,x)-\mu S(t,x)\\
&\qquad\qquad +\gamma I(t-h_2,x)-\zeta+b\quad t>0, x\in\Omega,\\
&\frac{\partial E}{\partial t}(t,x)=d\Delta E(t,x)+\beta S(t,x)E(t-h_1,x)-(\kappa+\lambda)E(t,x)\\
&\qquad\qquad +\mu S(t,x)\quad t>0, x\in\Omega,\\
&\frac{\partial I}{\partial t}(t,x)=d\Delta I(t,x)+\kappa E(t,x)-\gamma I(t-h_2,x)+\zeta\quad t>0, x\in\Omega,\\
&\frac{\partial S}{\partial n}(t,x)=\frac{\partial E}{\partial n}(t,x)=\frac{\partial I}{\partial n}(t,x)=0\quad t>0, x\in\partial\Omega,
\end{aligned}\tag{5}$$

where $N\equiv N(t,x)=S(t,x)+E(t,x)+I(t,x)$.

Remark 1. It is clear for Equation (5) that

If $b>\zeta>0$, then Equation (5) always has a unique positive established boom equilibrium $\mathbf{E}^+=\left(S^*,E^*,I^*\right)$, where

$$S^*=\frac{\kappa b+\lambda b}{\beta b+\lambda\mu},\ E^*=\frac{b}{\lambda}\text{ and } I^*=\frac{\kappa b+\lambda\zeta}{\gamma\lambda}.\tag{5*}$$

In particular, for parameter b and ζ, we can only set from view of mathematical conditions as following:

If $b = \zeta = 0$, then Equation (1) always has a trivial equilibrium $\mathbf{E}_0 = (0,0,0)$.

In this paper, we do not need to treat this condition since our assumption is $b > \zeta > 0$.

We discuss the large time behavior of the solution of Equation (1) (cf. [14]).

2. Preliminary Lemmas

Before main theorem, we mention the following theorem (the strong maximum principle in [25]), and then the main results of our paper are stated as follows.

Theorem A. Let $w \in C^{1,2}(D_T)$ and that

$$w_t - d\nabla^2 w + cw \geq 0 \ \text{ in } D_T = (0,T] \times \Omega,$$

$$Bw = 0 \ \text{ on } S_T = (0,T] \times \partial\Omega,$$

$$w(0,x) \geq 0 \ \text{ in } \bar{\Omega},$$

where B is Neumann type boundary condition and $c \equiv c(t,x)$ is a bounded function in D_T. If w attains a maximum value M at some point in D_T, then $w \equiv M$ throughout D_T.

In order to prove of after lemmas and theorem, we need the following assumptions that

(H_0) $\max\{\beta,\mu\} < \sigma, 0 < \zeta < b, \beta < \lambda$ and $\gamma < \dfrac{\beta\zeta}{\kappa}$.

The condition of this (H_0) is a completely mathematical relationship between the large and small coefficients of Equation (1), but it is a somewhat natural assumption considering the boom meaning of the before coefficients.

And moreover,

(H_1) $(E - E^*)(I - I^*) > 0$ if $\dfrac{\kappa}{\beta} < I$, and

(H_2) if $E^* < E$ implies $\mu S < \sigma E$, that is $1 < \dfrac{S}{E} < \dfrac{\sigma}{\mu}$, where $\sigma = \kappa + \lambda$, or otherwise, if $E < E^*$ implies $\sigma E < \mu S$, that is $\dfrac{E}{S} < \dfrac{\mu}{\sigma} < 1$.

The condition of (H_1) is generally a condition that guarantees that if we assume $\beta > \kappa$, that in a sense it means $I > 1$, that the state in which the trend is established is greater (or less) than the equilibrium state, and the state in which the trend is greater (or smaller) occurs at the same time.

And, the condition of (H_2) means that the size of the state of being in an on-boom and its equilibrium state is actually size of the number of people who move to the boom state without contact before the boom and the number of people who are in the boom who have both become established and unestablished.

If $E(t,x)$ and $I(t,x)$ have no booming and established boom from past time and that we say the strong fading memory property for time t and h as follows (cf. [5] [10]).

(H$_3$) $E(t+s,x) \le E(t,x)$ for $t>0$, $s \in [-h_1, 0]$, $x \in \bar{\Omega}$, for the finite time delay $h_1 \ge 0$ and

$I(t+s,x) \le I(t,x)$ for $t>0$, $s \in [-h_2, 0]$, $x \in \bar{\Omega}$, for the finite time delay $h_2 \ge 0$, and moreover $h_1 < h_2$.

In mathematics, assumption (H$_3$) is like the Razumikhin type condition (cf. [26]) for delay term, and in other words, in order for an on-boom adopter to influence the boom adopter though contact, a certain amount of information and knowledge about the boom is required, and it is believed that it is possible to influence the unadopted person only after a certain period of time to obtain them. Moreover, this influence is believed to be that past information and knowledge will gradually fade and weaken memory.

In the rest of this paper, we will report results only for system (5). Before the proof of Theorem 1, we prepare lemmas.

Lemma 1. Under the assumption (H$_0$), the solution $(S(t,x), E(t,x), I(t,x))$ of Equation (5) with (4) except for $R(0,x)$ satisfies for $t \ge 0$, the following inequality

$$0 < N(t,x), \tag{6}$$

where $N(t,x) = S(t,x) + E(t,x) + I(t,x)$ and $N(0,x) = S_0(x) + E_0(x) + I_0(x)$.

Proof. For the first inequality of (6), it is sufficient to prove that if for any small $\epsilon > 0$, $N(t_0,x) > \epsilon$ for some $t_0 > 0$ and $x \in \bar{\Omega}$, then $N(t,x) > \epsilon/2$ for $t > t_0, x \in \bar{\Omega}$. If it is not true, then

$$N(t,x) < \frac{\epsilon}{2} \text{ for } t > t_1, x \in \bar{\Omega} \text{ and } N(t_1,x_1) = \frac{\epsilon}{2} \text{ for some } t_1 > t_0, x_1 \in \bar{\Omega}$$

with t_1 being the smallest among all such points (t_1, x_1). If we set $w_0(t,x) = N(t,x) - \epsilon/2$, then $w_0(t,x) < 0$ ($t > t_1$, $x \in \bar{\Omega}$), $w_0(t_1,x_1) = 0$ and $\sup_{x \in \bar{\Omega}} w_0(t_0,x) > 0$, hence the function $w_0(t,x)$ takes a nonnegative minimum on $[t_0, t_1] \times \bar{\Omega}$. On the other hand, we have

$$\begin{aligned} \partial w_0/\partial t &= \partial N/\partial t \\ &= d\Delta N - \lambda E(t,x) + b \\ &\ge d\Delta N - \lambda N(t,x) + b \\ &= d\Delta w_0 - \lambda\left(w_0 + \frac{\epsilon}{2}\right) + b \end{aligned}$$

and consequently

$$d\Delta w_0 - \partial w_0/\partial t - \lambda w_0 \le \lambda \frac{\epsilon}{2} - b < 0$$

on $(t_1, \infty) \times \Omega$. Then there arises a contradiction by the strong maximum principle (cf. [5] [8] [10] [14] [22] [25] [27] [28]). Indeed, if $x_1 \in \Omega$, then $d\Delta w_0 - \partial w_0/\partial t - \lambda w_0$ must be nonnegative at (t_1, x_1). This is a contradiction. We thus obtain that $x_1 \in \partial\Omega$ and $w_0(t,x) > w_0(t_1,x_1)$ for all $(t,x) \in [t_0, t_1] \times \Omega$, and hence $\partial w_0/\partial n \le 0$ at (t_1, x_1). This is a contradiction, again (cf. [25]). It is clear that, by the initial point $N(0,x)\ge 0$ and the reduction of the above, $N(t,x)>0$ for $(t,x)\in(0,t_0)\times\bar{\Omega}$. Therefore, we must have (6).

Lemma 2. Under the assumptions through (H_0) to (H_3), the solution $(S(t,x),E(t,x),I(t,x))$ of Equation (5) with (4) except for $R(0,x)$ satisfies the following inequality

$$\liminf_{t\to\infty}\left[\inf_{x\in\bar{\Omega}} I(t,x)\right]\ge\frac{\zeta}{\gamma}>0. \tag{7}$$

where $\frac{\zeta}{\gamma}>\frac{\kappa}{\beta}$ by condition $\gamma<\beta\zeta/\kappa$ in (H_0).

Proof. By the same argument in the proof of Lemma 1, For some $t_2>t_0$, we can show that

$$\hat{I}(t)\le I(t,x),\ t>t_2, x\in\bar{\Omega},$$

where $\hat{I}(t)$ is the solution of ordinary differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{I}(t)=-\gamma\hat{I}(t)+\zeta-\epsilon,\quad t>t_2, \tag{8}$$

$$\hat{I}(t_2)=\hat{I}_2 \text{ and } \hat{I}_2\ge\inf_{x\in\bar{\Omega}} I(t_2,x)\ge\frac{\zeta}{\gamma}.$$

To see this, we consider the function $w_1(t,x):=I(t,x)-\hat{I}(t)$ on $[t_2,\infty)\times\bar{\Omega}$. Then $w_1(0,x)=I(0,x)-\hat{I}(0)\le 0$ for $x\in\bar{\Omega}$, and since

$$\partial I/\partial t=d\Delta I+\kappa E-\gamma I+\zeta\ge d\Delta I-\gamma I+\zeta,$$

We have

$$\begin{aligned}\partial w_1/\partial t&=\partial I/\partial t-\mathrm{d}\hat{I}(t)/\mathrm{d}t\\&\ge d\Delta I-\gamma I+\zeta-\left(-\gamma\hat{I}+\zeta-\epsilon\right)\\&\ge d\Delta w_1-\gamma\left(w_1+\hat{I}\right)+\zeta-\left(-\gamma\hat{I}+\zeta-\epsilon\right)\\&\ge d\Delta w_1-\gamma w_1+\epsilon.\end{aligned}$$

Hence,

$$d\Delta w_1-\partial w_1/\partial t-\gamma w_1\le-\epsilon\le 0\quad\text{on } [t_2,\infty)\times\bar{\Omega}.$$

Thus, by the strong maximum principle, we have a contradiction. Therefore, by the same reasoning as the one for $w_0(t,x)$ of Lemma 1, one can see that $w_1(t,x)\ge 0$ on $[t_2,\infty)\times\bar{\Omega}$. Thus, we must have (8).

Moreover, we consider

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{I}=-\gamma\hat{I}+\zeta-\epsilon. \tag{9}$$

By solving Equation (9), we obtain that

$$\hat{I}=\frac{\zeta-\epsilon}{\gamma}+\hat{C}\mathrm{e}^{-\gamma t}$$

and

$$\hat{C} = \mathrm{e}^{\gamma t_2}\left(\hat{I}_2 - \frac{\zeta - \epsilon}{\gamma}\right).$$

Therefore, we have

$$\frac{\zeta - \epsilon}{\gamma} \le \hat{I}, \quad t \ge t_2$$

for small $\epsilon > 0$. Thus, we obtain

$$\frac{\zeta - \epsilon}{\gamma} \le I$$

on $t \ge t_2, x \in \Omega$. By taking the infimum, $t \to \infty$ and later letting $\epsilon \to 0$ in the above inequality, we obtain (7):

$$\frac{\zeta}{\gamma} \le \liminf_{t \to \infty}\left[\inf_{x \in \bar{\Omega}} I(t, x)\right].$$

This completes the proof of Lemma 2.

Remark 2. In the SIR model of infectious diseases, a time delay term is necessary for the "I" term of the duration of infection (especially COVID-19, HIV, influenza, etc.), but for the SEIR model of infectious diseases, the "E" term of the incubation period is not only particularly necessary, but also only complicates the system. However, in our SEIR type booms model, we need to consider the time delay terms. Because, a boom is actually completely different phenomenon with social behavior from an infectious disease epidemic.

3. Global Attractor

We show the following theorem.

Theorem 1. If the assumptions through (H_0) to (H_3) and $\phi_1 > 0$, $\phi_2 \not\equiv 0$ and $\phi_3 \not\equiv 0$, then, for each nonnegative continuous initial function, there is a unique positive equilibrium (S^*, E^*, I^*) of (5) satisfies

$$\lim_{t \to \infty}\left[\sup_{x \in \bar{\Omega}}\left|E(t, x) - E^*\right|\right] = 0,$$

$$\lim_{t \to \infty}\left[\sup_{x \in \bar{\Omega}}\left|I(t, x) - I^*\right|\right] = 0$$

and

$$\lim_{t \to \infty}\left[\sup_{x \in \bar{\Omega}}\left|S(t, x) - S^*\right|\right] = 0.$$

Proof of Theorem 1. Now, from $N = S + E + I$, the system (5) drives to

$$\frac{\partial N}{\partial t} = d\Delta N - \lambda E + b \tag{10}$$

$$\frac{\partial E}{\partial t} \le d\Delta E + \beta E\left[(N - E - I) - \frac{\sigma}{\beta}\right] + \mu S, \tag{11}$$

where $\sigma = \kappa + \lambda$. The system (5) has the positive equilibrium (S^*, E^*, I^*) where $N^* = S^* + E^* + I^*$. We can rewrite (10) in the form

$$\frac{\partial N}{\partial t}=d\Delta N-\lambda\left(E-E^{*}\right). \tag{12}$$

because $-\lambda E^{*}+b=0$. Moreover, for the first Equation of S in (5), since

$$-\beta S^{*}E^{*}-\mu S^{*}+\gamma I^{*}+b-\zeta=0,\ E^{*}=\frac{b}{\lambda}\ \text{and}\ \gamma I^{*}=\kappa E^{*}+\zeta$$

we have

$$\left(\beta E^{*}+\mu\right)S^{*}=\kappa E^{*}+b=\frac{(\kappa+\lambda)b}{\lambda},$$

by I^{*} and $E^{*}>0$ in (5^{*}). On the other hand,

$$\beta S^{*}\left(E^{*}+\frac{\mu}{\beta}\right)=\beta S^{*}\left(\frac{\beta(b/\lambda)+\mu}{\beta}\right)=\beta S^{*}\frac{(\beta b+\lambda\mu)}{\beta\lambda}.$$

Thus,

$$\beta S^{*}\frac{(\beta b+\lambda\mu)}{\beta\lambda}=\frac{(\kappa+\lambda)b}{\lambda},\ \text{then},\ \beta S^{*}\frac{(\beta b+\lambda\mu)}{\beta b}=\kappa+\lambda.$$

Therefore,

$$\beta a\left(N^{*}-\left(E^{*}+I^{*}\right)\right)=\beta a S^{*}=\kappa+\lambda=\sigma,\ \text{where}\ a=\frac{\beta b+\lambda\mu}{\beta b}>1.$$

For also (11), by assumption (H_3),

$$\begin{aligned}\frac{\partial E}{\partial t}&\le d\Delta E+\beta E\left[N-(E+I)-aS^{*}\right]+\mu S\\&=d\Delta E+\beta E\left[N-(E+I)\right]-\sigma E+\mu S\\&\le d\Delta E+\beta E\left[G(N)-\left(E-E^{*}\right)-\left(I-I^{*}\right)\right]-\sigma E+\mu S,\end{aligned} \tag{13}$$

where

$$G(N)=\frac{N-\tilde{N}^{*}}{N},$$

where $\tilde{N}^{*}=N^{*}-S^{*}=E^{*}+I^{*}\left(<N^{*}\right)$. Then, $G(N)>0$ for $N>\tilde{N}^{*}$, and $G(N)<0$ for $N<\tilde{N}^{*}$. We now define a function $V(t)$ by

$$\begin{aligned}V(t)&=V(N,E,I)(t)\\&=\int_{\Omega}\left\{\int_{\tilde{N}^{*}}^{N}G(s)\,\mathrm{d}s+\left(E-E^{*}\right)-E^{*}\log\frac{E}{E^{*}}+\left(I-I^{*}\right)-I^{*}\log\frac{I}{I^{*}}\right\}\mathrm{d}x,\end{aligned}$$

where $\beta<\lambda$. Then $V\left(\tilde{N}^{*},E^{*},I^{*}\right)(t)=0$ and $V(N,E,I)(t)>0$ for other admissible (N,E,I). Furthermore, we calculate $\mathrm{d}V/\mathrm{d}t$ along the solution of (12) and (13).

$$\begin{aligned}\frac{\mathrm{d}V(t)}{\mathrm{d}t}&=\int_{\Omega}\left\{G(N)\frac{\partial N}{\partial t}+\frac{\partial E}{\partial t}-E^{*}\frac{\partial E/\partial t}{E}+\frac{\partial I}{\partial t}-I^{*}\frac{\partial I/\partial t}{I}\right\}\mathrm{d}x\\&\le\int_{\Omega}\Big[G(N)\left(d\Delta N-\lambda\left(E-E^{*}\right)\right)+d\Delta E\\&\quad+\beta E\left(G(N)-\left(E-E^{*}\right)-\left(I-I^{*}\right)\right)-\sigma E+\mu S\end{aligned}$$

$$
\begin{aligned}
&-E^{*}\left\{d\frac{\Delta E}{E}+\beta\left(G(N)-\left(E-E^{*}\right)-\left(I-I^{*}\right)\right)\right.\\
&-\sigma+\mu\frac{S}{E}+d\Delta I+\kappa\left(E-E^{*}\right)-\gamma\left(I-I^{*}\right)\\
&\left.\left.-I^{*}\left\{d\frac{\Delta I}{I}+\frac{1}{I}\left(\kappa\left(E-E^{*}\right)-\gamma\left(I-I^{*}\right)\right)\right\}\right]\mathrm{d}x\\
={}&d\int_{\Omega}\Delta NG(N)\mathrm{d}x-\lambda\int_{\Omega}G(N)\left(E-E^{*}\right)\mathrm{d}x+d\int_{\Omega}\Delta E\mathrm{d}x\\
&+\beta\int_{\Omega}EG(N)\mathrm{d}x-\beta\int_{\Omega}E\left(E-E^{*}\right)\mathrm{d}x\\
&-\beta\int_{\Omega}E\left(I-I^{*}\right)\mathrm{d}x-\sigma\int_{\Omega}E\mathrm{d}x+\mu\int_{\Omega}S\mathrm{d}x\\
&-dE^{*}\int_{\Omega}\frac{\Delta E}{E}\mathrm{d}x-E^{*}\beta\int_{\Omega}G(N)\mathrm{d}x+E^{*}\beta\int_{\Omega}\left(E-E^{*}\right)\mathrm{d}x\\
&+E^{*}\beta\int_{\Omega}\left(I-I^{*}\right)\mathrm{d}x+E^{*}\sigma\int_{\Omega}\mathrm{d}x-E^{*}\mu\int_{\Omega}\frac{S}{E}\mathrm{d}x\\
&+d\int_{\Omega}\Delta I\mathrm{d}x+\kappa\int_{\Omega}\left(E-E^{*}\right)\mathrm{d}x-\gamma\int_{\Omega}\left(I-I^{*}\right)\mathrm{d}x\\
&-d\int_{\Omega}\frac{I^{*}\Delta I}{I}\mathrm{d}x-\int_{\Omega}\frac{I^{*}\kappa\left(E-E^{*}\right)}{I}\mathrm{d}x+I^{*}\gamma\int_{\Omega}\frac{I-I^{*}}{I}\mathrm{d}x\\
={}&d\int_{\Omega}\Delta NG(N)\mathrm{d}x-\lambda\int_{\Omega}G(N)\left(E-E^{*}\right)\mathrm{d}x\\
&+d\int_{\Omega}\Delta E\frac{E-E^{*}}{E}\mathrm{d}x+\beta\int_{\Omega}G(N)\left(E-E^{*}\right)\mathrm{d}x\\
&-\beta\int_{\Omega}\left(E-E^{*}\right)^{2}\mathrm{d}x-\beta\int_{\Omega}\left(E-E^{*}\right)\left(I-I^{*}\right)\mathrm{d}x\\
&+\sigma\int_{\Omega}\left(E-E^{*}\right)\mathrm{d}x+\mu\int_{\Omega}\frac{S\left(E-E^{*}\right)}{E}\mathrm{d}x+d\int_{\Omega}\Delta I\frac{I-I^{*}}{I}\mathrm{d}x\\
&+\kappa\int_{\Omega}\frac{\left(E-E^{*}\right)\left(I-I^{*}\right)}{I}\mathrm{d}x-\gamma\int_{\Omega}\frac{\left(I-I^{*}\right)^{2}}{I}\mathrm{d}x\\
<{}&0
\end{aligned}
\tag{14}
$$

whenever $(N,E,I)\neq\left(N^{*},E^{*},I^{*}\right)$ and $E<E^{*}$, and also in the case where if $E>E^{*}$, we can get the similar result of $\frac{\mathrm{d}V(t)}{\mathrm{d}t}<0$ from after (17) and (H$_2$). To drive this, we continue to estimate for (14) in more detail.

$$
\begin{aligned}
d\xi\int_{\Omega}\Delta NG(N)\mathrm{d}x&=d\xi\left\{\left[\frac{\partial N}{\partial x}G(N)\right]_{\Omega}-\int_{\Omega}\frac{\partial N}{\partial x}\frac{\partial G(N)}{\partial x}\mathrm{d}x\right\}\\
&=-d\xi\int_{\Omega}\frac{\partial N}{\partial x}\frac{\partial G(N)}{\partial x}\mathrm{d}x.
\end{aligned}
\tag{15}
$$

Here

$$\frac{\partial G(N)}{\partial x}=\frac{\partial}{\partial x}\left\{\left(1-\frac{N^{*}}{N}\right)\right\}=\frac{N^{*}}{N^{2}}\frac{\partial N}{\partial x}.$$

Thus, expression (15) is

$$d\int_{\Omega}\Delta NG(N)\mathrm{d}x=-dN^{*}\int_{\Omega}\left(\frac{1}{N}\frac{\partial N}{\partial x}\right)^{2}\mathrm{d}x<0.$$

Moreover, we have

$$\begin{aligned}
&-\lambda\int_\Omega G(N)\left(E-E^*\right)\mathrm{d}x+\beta\int_\Omega G(N)\left(E-E^*\right)\mathrm{d}x \\
&=-(\lambda-\beta)\int_\Omega G(N)\left(E-E^*\right)\mathrm{d}x \\
&\le-(\lambda-\beta)\int_\Omega \frac{\left(N-\tilde{N}^*\right)\left(E-E^*\right)}{N}\mathrm{d}x \\
&\le-(\lambda-\beta)\int_\Omega \frac{\left(E-\tilde{N}^*\right)\left(E-E^*\right)}{N}\mathrm{d}x \\
&=-(\lambda-\beta)\int_\Omega \frac{\left(E-N^*\right)^2}{N}\mathrm{d}x<0 \text{ for } \lambda>\beta
\end{aligned}$$

and

$$\begin{aligned}
&-\beta\int_\Omega\left(E-E^*\right)\left(I-I^*\right)\mathrm{d}x+\kappa\int_\Omega\frac{\left(E-E^*\right)\left(I-I^*\right)}{I}\mathrm{d}x \\
&=\int_\Omega\frac{(-\beta I+\kappa)\left(E-E^*\right)\left(I-I^*\right)}{I}\mathrm{d}x<0
\end{aligned} \tag{16}$$

and

$$\begin{aligned}
&-\sigma\int_\Omega\left(E-E^*\right)\mathrm{d}x+\mu\int_\Omega\frac{S\left(E-E^*\right)}{E}\mathrm{d}x \\
&=\int_\Omega\frac{\left(-(\kappa+\lambda)E+\mu S\right)\left(E-E^*\right)}{E}\mathrm{d}x<0,
\end{aligned} \tag{17}$$

because (16) is indicated from assumption (H_1) and also (17) is indicated from assumption (H_2). It is clear from (H_2) that (17) is correct by $\kappa+\lambda=\sigma$, in the case where if $E>E^*$.

Similarly of (15), we can check

$$\begin{aligned}
d\int_\Omega \Delta E\frac{E-E^*}{E}\mathrm{d}x&=d\int_\Omega \Delta E\left(1-\frac{E^*}{E}\right)\mathrm{d}x \\
&=d\left[\frac{\partial E}{\partial x}\left(1-\frac{E^*}{E}\right)\right]_\Omega-dE^*\int_\Omega\frac{\left(\partial E/\partial x\right)^2}{E^2}\mathrm{d}x<0,
\end{aligned}$$

and

$$\begin{aligned}
d\int_\Omega \Delta I\frac{I-I^*}{I}\mathrm{d}x&=d\int_\Omega \Delta I\left(1-\frac{I^*}{I}\right)\mathrm{d}x \\
&=d\left[\frac{\partial I}{\partial x}\left(1-\frac{I^*}{I}\right)\right]_\Omega-dI^*\int_\Omega\frac{\left(\partial I/\partial x\right)^2}{I^2}\mathrm{d}x<0.
\end{aligned}$$

It is clear that

$$-\gamma\int_\Omega\frac{\left(I-I^*\right)^2}{I}\mathrm{d}x<0 \text{ for } I>0.$$

Therefore, $V(t)$ is non-increasing in t that is there exists a constant $c_1\ge0$ such that $V(t)\to c_1$ as $t\to\infty$.

$E(t,x)$ can see uniformly bounded on $[0,\infty)\times\bar{\Omega}$ and $I(t,x)$ can see uniformly bounded on $[0,\infty)\times\bar{\Omega}$. Thus, we see that for any $h>0$, there exists $C(h)>0$ such that $|E(t+h,\cdot)-E(t,\cdot)|\le C(h)$ for $t\ge 0$, and $|I(t+h,\cdot)-I(t,\cdot)|\le C(h)$ for $t\ge 0$. From (14), we have $\dot{V}(t)\le -W(N,E,I)(t)\le 0$ (included equilibrium point case), where $W(N,E,I)(t)$ is the function of the right-hand side in (14). Suppose that $\dot{V}(t)\ne 0$. For any sequence $\{t_k\},t_k\to\infty$ as $k\to\infty$ and some positive number γ, there exists $\delta>0$ such that

$$\dot{V}(t)<-\gamma \tag{18}$$

if $|E(t+t_k,\cdot)-E(t,\cdot)|\le\delta, 0\le t\le\delta$, $|I(t+t_k,\cdot)-I(t,\cdot)|\le\delta, 0\le t\le\delta$ and k is sufficient large. For regions $[t_k,t_k+\delta]$, we can see that

$$V(t_k+\delta)\le V(t_k)-\gamma\delta \tag{18*}$$

to integral on $[t_k,t_k+\delta]$ for the both sides of (18). Since (18*) is true for all large number k and $\lim_{t\to\infty}V(t)=c_1\ge 0$, it contradicts by $\gamma\delta$ is positive. This shows that $\dot{V}(t)=0$. Then, we have $W(N,E,I)(t)=0$. We thus obtain $N\to N^*$, $E\to E^*$ and $I\to I^*$ by continuity of V and W. The asymptotic behavior of S now follows from the above result on the behavior of N, E and I. Thus, it is clear from $S=N-E-I$ that $S\to S^*$. This completes the proof.

Remark 3. It is a valid that the assumptions (H_0), (H_1) and (H_2) are not only little bit analytically conditions, but also the natural in the sense of trend from some data (cf. [1]-[4]).

4. Example

We first note that the following example is a numerical model. Needless to say, there are a great many factors involved in the actual boom of trend, and a simple mathematical model captures only a part of them. Nevertheless, it should be noted that it creates a perception that cannot be reached by natural language and simple data.

We consider the following equation of

$$\begin{aligned}
\frac{\partial S}{\partial t}(t,x) &= 0.03\Delta S(t,x)-0.00369S(t,x)E(t-16.0,x)\\
&\quad -0.01259S(t,x)+0.00201I(t-155.0,x)+7.0 \quad t>0, x\in\Omega,\\
\frac{\partial E}{\partial t}(t,x) &= 0.03\Delta E(t,x)+0.00369S(t,x)E(t-16.0,x)\\
&\quad -0.04939E(t,x)+0.01259S(t,x) \quad t>0, x\in\Omega,\\
\frac{\partial I}{\partial t}(t,x) &= 0.03\Delta I(t,x)+0.00348E(t,x)\\
&\quad -0.00201I(t-155.0,x)+1.0 \quad t>0, x\in\Omega,
\end{aligned} \tag{19}$$

where in Equation (19), these data are quoted from ([3], see 6.2.1. Pokemon Go, pp.37) in references.

$d=0.03$,

$h_1 = 16.0$,
$h_2 = 155.0$,
$\beta = 0.00369$,
$\mu = 0.01259$,
$\kappa = 0.00348$,
$\lambda = 0.04591$,
$\gamma = 0.00201 < \beta\zeta/\kappa \approx 1.06034$,
$b = 8.0$,
$\zeta = 1.0$, $-\zeta + b = 7.0$ and
$\sigma = \kappa + \lambda = 0.04939 > \max\{\beta, \mu\} = 0.01259$.

Thus, it is satisfied with the assumption (H_0), and

$$E^+ = \left(S^*, E^*, I^*\right) \approx (13.13, 174.25, 819.44),$$

where

$$S^* = \frac{(\kappa+\lambda)b}{b\beta+\lambda\mu} = \frac{0.39512}{0.03010} \approx 13.13,$$

$$E^* = \frac{b}{\lambda} = \frac{8}{0.04591} \approx 174.25,$$

$$I^* = \frac{b\kappa+\lambda\zeta}{\gamma\lambda} = \frac{0.07375}{0.00009} \approx 819.44.$$

The initial functions are

$$S(\theta, x) = \phi_1(0, x) \equiv 1 > 0, x \in \bar{\Omega},$$

$$E(\theta, x) = \phi_2(\theta, x) \equiv 1 > 0, \theta \in [-h_1, 0], x \in \bar{\Omega} \text{ and}$$

$$I(\theta, x) = \phi_3(\theta, x) \equiv 1 > 0, \theta \in [-h_2, 0], x \in \bar{\Omega}$$

belong to the $(\theta, x) \in [-h_i, 0] \times C^1(\bar{\Omega}), (i = 1, 2)$.

5. Conclusions

We obtain the results of Theorem 1 that the global attractivity of the boom equilibrium point E^+ and the property of Equation (5), by using the method of the strong maximum principle, the technique of Liapunov functions and others. This phenomenon mathematically indicates that the phased state of booms, such as anchoring or not anchoring, eventually asymptotes to the equilibrium point. Moreover, we have given the simple example for Theorem 1 that the equilibrium point E^+ of Equation (19), that is Equation (5), is the global attractor by assumptions through (H_0) to (H_3), the strong maximum principle and the Liapunov function.

At this time, although we have not simulated with specific real data, there are many good examples in the papers of Nakagiri [1], Ueda [4], Ohta and Mizutani [2] [3], and others, so it is important, it is omitted here.

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Ethical Recognition

It does not constitute ethical approval, and does not touch on any moral conduct.

Authors' Contributions

Y. Hamaya wrote lemmas and main theorem, K. Saito wrote the example and references, calibration and also we proofread and checked all of our paper.

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