

Some Parameterized Simpson-Mercer Type Inequalities for General Fractional Operators

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How to cite this paper: Lo, J.C. (2026)

Some Parameterized Simpson-Mercer Type Inequalities for General Fractional Operators. *Advances in Pure Mathematics*, **16**, 195-225.

<https://doi.org/10.4236/apm.2026.163010>

Received: January 6, 2026

Accepted: March 20, 2026

Published: March 23, 2026

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Abstract

This paper establishes new parameterized Simpson Mercer-type inequalities for functions of two variables within the framework of generalized fractional integral operators. By deriving novel integral identities for twice partially differentiable mappings, we provide several generalized forms for free parameters. Our results extend and unify a wide range of existing fractional integral inequalities, including those based on the Riemann-Liouville, k -Riemann-Liouville, and other fractional operators. These findings enrich the theory of integral inequalities and offer new tools for applications in convex analysis, numerical integration, and fractional calculus.

Keywords

Simpson-Mercer-Type Inequalities, Integral Inequalities, Fractional Calculus, Convex Functions

1. Introduction

Body Math Simpson's inequality is a classical result in real analysis and plays a central role in approximation theory, numerical integration, and convex analysis. It provides an explicit bound for the error of Simpson's quadrature rule in terms of the fourth derivative of the integrand and has motivated a large body of research on quadrature-type inequalities and their refinements.

Body Math Over the last decades, many authors have obtained extensions and improvements of Simpson's inequality under various structural assumptions on the underlying functions, including convexity, boundedness, bounded variation, s -convexity, preinvexity, and co-ordinated convexity on rectangles. For functions of two variables, Simpson-type inequalities on the co-ordinates have proved to be a powerful tool in studying integral inequalities and in analyzing numerical schemes on rectangular domains.

Body Math In parallel with these developments, fractional calculus has provided an efficient framework for generalizing classical inequalities. Generalized fractional integral operators, defined via suitable kernel functions, subsume as special cases a number of well-known fractional integrals such as the Riemann-Liouville and k -Riemann-Liouville fractional integrals, the Katugampola fractional integral, conformable fractional integrals, and the Hadamard fractional integral. This unified viewpoint allows one to derive families of integral inequalities that simultaneously cover many previously known results and yield sharper bounds in several directions.

Body Math Another important refinement in convex analysis is due to Mercer, who introduced in 2003 a sharpened version of Jensen's inequality, now commonly referred to as the Jensen-Mercer inequality. This result provides improved estimates for convex mappings and has been successfully applied to obtain refined forms of various integral inequalities. Combining Simpson-type quadrature formulas with Mercer-type refinements naturally leads to what we call Simpson-Mercer-type inequalities, which retain the structure of Simpson's rule while incorporating additional flexibility through Mercer-type weights.

Body Math Motivated by these ideas, and by recent contributions on Simpson-type inequalities for generalized fractional integrals, we investigate parameterized Simpson-Mercer-type inequalities for functions of two variables via generalized fractional integral operators. We first establish new integral identities for twice partially differentiable functions defined on rectangular domains, involving a pair of free parameters and double generalized fractional integrals. Under appropriate co-ordinated convexity assumptions on the mixed partial derivatives, these identities yield several families of Simpson-Mercer-type inequalities in both L^p and L^∞ forms. By choosing specific kernels and parameter values, our results reduce to many known Simpson-type and fractional Simpson-type inequalities for co-ordinated convex functions and, at the same time, provide new generalizations in the planar setting.

Simpson's inequality plays an important role in many areas of mathematics. The classical Simpson's inequality is expressed as follows:

Theorem 1.1

Suppose that $f : [a, b] \rightarrow \mathbb{R}$ is a four-times continuously differentiable mapping on (a, b) , and let $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then one has the inequality

$$\left| \frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$

Many researchers have studied various Simpson's inequalities, such as convex functions, bounded functions, functions of bounded variation, etc. Specifically, since convexity theory is an effective and powerful way to solve a large number of problems from different branches of pure and applied mathematics, many papers have been dedicated to Simpson's inequality for convex functions.

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.2

A function $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is called co-ordinated convex on $[a, b] \times [c, d]$ for all $(x, u), (y, v) \in [a, b] \times [c, d]$ and $t, s \in [0, 1]$ if it satisfies the following inequality:

$$f(tx + (1-t)y, su + (1-s)v) \leq tsf(x, u) + t(1-s)f(x, v) + s(1-t)f(y, u) + (1-t)(1-s)f(y, v).$$

In [1], Dragomir *et al.* proved the following some Simpson’s inequality for which the remainder is expressed in terms of lower derivatives than the fourth.

Theorem 1.3

Suppose $f : [a, b] \rightarrow \mathbb{R}$ is an absolutely continuous mapping on $[a, b]$ whose derivative belongs to $L_p[a, b]$. Then, the following inequality holds:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{b-a} \int_a^b f(x) dx \right| \leq \frac{1}{6} \left[\frac{2^{q+1} + 1}{3(q+1)} \right]^{\frac{1}{q}} (b-a)^{\frac{1}{q}} \|f'\|_p,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In 2003, Mercer [2] introduced a refined version of the classical Jensen inequality, now known as the Jensen-Mercer inequality, stated as follows.

For a convex mapping $f : [a, b] \rightarrow \mathbb{R}$, the following inequality holds for each $x_j \in [a, b]$:

$$f\left(a + b - \sum_{j=1}^n u_j x_j\right) \leq f(a) + f(b) - \sum_{j=1}^n u_j f(x_j),$$

where $u_j \in [0, 1]$ and $\sum_{j=1}^n u_j = 1$.

2. Preliminaries

In this section, we present some definitions and results which will be used in our main section.

Let’s define a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ satisfying the follow conditions:

$$\int_0^1 \frac{\varphi(t)}{t} dt < \infty.$$

We define the following left-sided and right-side generalized fractional integral operators, respectively, as follows:

$${}_a^+ I_\varphi f(x) = \int_a^x f(t) \frac{\varphi(t-a)}{t-a} dt, \quad t > a,$$

$${}_b^- I_\varphi f(x) = \int_x^b f(t) \frac{\varphi(b-t)}{b-t} dx, \quad t < b,$$

$${}_c^+ I_\varphi f(y) = \int_c^y f(s) \frac{\varphi(s-c)}{(s-c)} ds, \quad s > c,$$

$${}_d^- I_\varphi f(y) = \int_y^d f(s) \frac{\varphi(d-s)}{d-s} ds, \quad s < d.$$

The most important feature of generalized fractional integrals is that they generalize some type of fractional integrals such as the Riemann-Liouville fractional integral, k-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional, and Hadamard fractional integrals. These important special cases of integral operator are mentioned below.

1) If we choose $\varphi(x) = x$, the operators ${}_a^+ I_\varphi f(x)$ and ${}_b^- I_\varphi f(x)$ are reduce to the Riemann integral.

2) Considering $\varphi(x) = \frac{x^\alpha}{\Gamma(\alpha)}$ and $\alpha > 0$, the operators ${}_a^+ I_\varphi f(x)$ and ${}_b^- I_\varphi f(x)$ are reduce to the Riemann-Liouville fractional integrals $I_{a^+}^\alpha f(x)$ and $I_{b^-}^\alpha f(x)$, respectively. Here, Γ is a gamma function.

3) For $\varphi(x) = \frac{1}{k\Gamma_k(\alpha)} x^{\frac{\alpha}{k}}$ and $\alpha, k > 0$, the operators ${}_a^+ I_\varphi f(x)$ and ${}_b^- I_\varphi f(x)$ are reduce to the k-Riemann-Liouville fractional integrals $I_{a^+,k}^\alpha f(x)$ and $I_{b^-,k}^\alpha f(x)$, respectively. Here, Γ_k is a k-gamma function.

In [3], Sarikaya and Ertuğral established the following Hermite-Hadamard inequality for the generalized fractional integral operator.

Theorem 2.1

Let $f : [a, b] \rightarrow \mathbb{R}$ be a convex function on $[a, b]$ with $a < b$. Then we have the following inequalities for fractional integral operators:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2\Delta(1)} \left[{}_a^+ I_\varphi f(b) + {}_b^- I_\varphi f(a) \right] \leq \frac{f(a) + f(b)}{2},$$

where the mapping $\Delta : [0, 1] \rightarrow \mathbb{R}$ is defined by

$$\Delta(t) = \int_0^t \frac{\varphi((b-a)u)}{u} du.$$

In [4], Ali et al. established the following Simpson's-type inequality for the generalized fractional integral operator.

Theorem 2.2

Assume $f : [a, b] \rightarrow \mathbb{R}$ be a twice-differentiable mapping in (a, b) such that $f'' \in L_1([a, b])$ and the mapping $|f''|$ is convex on $[a, b]$. Then we have the following inequalities for fractional integral operators:

$$\left| \frac{1}{6} \left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b) \right] - \frac{1}{2T\left(\frac{1}{2}\right)} \left[{}_{\frac{a+b}{2}}^+ I_\varphi f(b) + {}_{\frac{a+b}{2}}^- I_\varphi f(a) \right] \right| \leq \frac{(b-a)^2}{6} \left(\int_0^{\frac{1}{2}} \left| t - \frac{3A(t)}{T\left(\frac{1}{2}\right)} \right| dt \right) \left[|f''(a)| + |f''(b)| \right],$$

where

$$A(t) = \int_0^t T(s) ds$$

and

$$T(s) = \int_0^s \frac{\varphi((b-a)u)}{u} du.$$

In [5], Budak *et al.* established the following the parametered Simpson's-type inequality for the generalized fractional integral operator.

Theorem 2.3

Assume $f : [a, b] \rightarrow \mathbb{R}$ be a differentiable function on (a, b) . If f' is continuous and $|f'|$ is convex on $[a, b]$, then for $\rho, \sigma \geq 0$, we have the following inequalities for fractional integral operators:

$$\begin{aligned} & \left| (1-\sigma)f(a) + (\sigma+\rho)f\left(\frac{a+b}{2}\right) + (1-\rho)f(b) - \frac{1}{\Delta(1)} \left[{}_{a^+}I_\varphi\left(\frac{a+b}{2}\right) + {}_{b^-}I_\varphi\left(\frac{a+b}{2}\right) \right] \right| \\ & \leq \frac{(b-a)}{4\Delta(1)} \left\{ |f'(b)| \left[\Pi_1^\varphi(\rho) + \Pi_2^\varphi(\sigma) \right] + |f'(a)| \left[\Pi_1^\varphi(\sigma) + \Pi_2^\varphi(\rho) \right] \right\}, \end{aligned}$$

where

$$\Pi_1^\varphi(\tau) = \int_0^1 (1-t) |\Delta(t) - \Delta(1)\tau| dt, \quad \Pi_2^\varphi(\tau) = \int_0^1 (1+t) |\Delta(t) - \Delta(1)\tau| dt.$$

In [6], Khan and Budak established the following the parametered Simpson's-type inequality for the generalized fractional integral operator.

Theorem 2.4

Assume $f : \Delta : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a twice partially differentiable function on Δ° (Δ° is the interior of Δ). If $\frac{\partial^2 f}{\partial t \partial a} \in L(\Delta)$ and $\left| \frac{\partial^2 f}{\partial t \partial s} \right|$ is convex on Δ , then we have the following inequalities for fractional integral operators:

$$\begin{aligned} & \left| \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{36} \right. \\ & + \frac{1}{9} \left[f\left(\frac{a+b}{2}, c\right) + f\left(\frac{a+b}{2}, d\right) + f\left(a, \frac{c+d}{2}\right) + f\left(b, \frac{c+d}{2}\right) + 4f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] \\ & + \frac{1}{4\Lambda(1)\nabla(1)} \left[{}_{a^+,c^+}I_{\varphi,\psi} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_{a^+,d^-}I_{\varphi,\psi} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right. \\ & \left. + {}_{b^-,c^+}I_{\varphi,\psi} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + {}_{b^-,d^-}I_{\varphi,\psi} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] \\ & - \frac{1}{12\Lambda(1)} \left[{}_{a^+}I_\varphi f\left(\frac{a+b}{2}, c\right) + {}_{a^+}I_\varphi f\left(\frac{a+b}{2}, d\right) + {}_{b^-}I_\varphi f\left(\frac{a+b}{2}, c\right) \right. \\ & \left. + {}_{b^-}I_\varphi f\left(\frac{a+b}{2}, d\right) + 4 {}_{a^+}I_\varphi f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + 4 {}_{b^-}I_\varphi f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \right] \\ & - \frac{1}{12\nabla(1)} \left[{}_{c^+}I_\psi f\left(a, \frac{c+d}{2}\right) + {}_{c^+}I_\psi f\left(b, \frac{c+d}{2}\right) + {}_{d^-}I_\psi f\left(a, \frac{c+d}{2}\right) \right. \\ & \left. + {}_{d^-}I_\psi f\left(b, \frac{c+d}{2}\right) \right] \end{aligned}$$

$$\begin{aligned}
 &+ {}_d^- I_{\psi} f\left(b, \frac{c+d}{2}\right) + 4 {}_c^+ I_{\psi} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) + 4 {}_d^- I_{\psi} f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \Bigg] \\
 &\leq \frac{(d-c)(b-a)}{4\Lambda(1)\nabla(1)} \times \left(\int_0^1 \int_0^1 \left| \frac{\Lambda(t)}{2} - \frac{\Lambda(1)}{3} \right| \left| \frac{\nabla(s)}{2} - \frac{\nabla(1)}{3} \right| ds dt \right) \\
 &\times \left[\left| \frac{\partial^2}{\partial t \partial s} f(a, c) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a, d) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(b, c) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(b, d) \right| \right]
 \end{aligned}$$

where

$$\Lambda(t) = \int_0^t \frac{\varphi\left(\frac{b-a}{2}u\right)}{u} du, \quad \nabla(s) = \int_0^s \frac{\varphi\left(\frac{d-c}{2}u\right)}{u} du.$$

We hope to incorporate parameter settings, in line with the spirit of Theorem 2.4, so that the Simpson-Mercer inequality can yield results different from the traditional Simpson inequality with the addition of different parameters.

3. Main Results

Throughout this study, for briefly, we define

The aim of this paper is to obtain some Simpson type inequalities for coordinates involving generalized fractional integrals, we define the double generalized fractional integrals as follows:

$$\begin{aligned}
 {}_{a^+, c^+} I_{\varphi, \psi} f(x, y) &= \int_c^y \int_a^x f(t, s) \frac{\varphi(t-a)\psi(s-c)}{(t-a)(s-c)} dt ds, \\
 {}_{a^+, d^-} I_{\varphi, \psi} f(x, y) &= \int_y^d \int_a^x f(t, s) \frac{\varphi(t-a)\psi(d-s)}{(t-a)(d-s)} dt ds, \\
 {}_{b^-, c^+} I_{\varphi, \psi} f(x, y) &= \int_c^y \int_x^b f(t, s) \frac{\varphi(b-t)\psi(s-c)}{(b-t)(s-c)} dt ds,
 \end{aligned}$$

and

$${}_{b^-, d^-} I_{\varphi, \psi} f(x, y) = \int_y^d \int_x^b f(t, s) \frac{\varphi(b-t)\psi(d-s)}{(b-t)(d-s)} dt ds.$$

where $\varphi, \psi : [0, \infty) \rightarrow [0, \infty)$ satisfying $\int_0^1 \frac{\varphi(t)}{t} dt < \infty$ and $\int_0^1 \frac{\psi(t)}{t} dt < \infty$, respectively.

Lemma 3.1

Let $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ be a absolutely continuous function such that the partial derivative of order $\frac{\partial^2}{\partial t \partial s} f(t, s)$ is continuous and integrable on $[a, b] \times [c, d]$ and $a \leq m < n \leq b$, $c \leq k < l \leq d$. Then for $\sigma, \rho \geq 0$, the following equality holds:

$$\int_0^1 \int_0^1 p(t, s) \frac{\partial^2}{\partial t \partial s} f\left(a + b - [tm + (1-t)n], c + d - [sk + (1-s)l]\right) dt ds$$

$$\begin{aligned}
&= \frac{1}{(n-m)(l-k)} \\
&\times \left\{ 4(\sigma-\rho)^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
&+ 2\rho(\sigma-\rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) \\
&\times \left[f\left(a+b-\frac{m+n}{2}, c+d-l\right) + f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\
&+ \left. f\left(a+b-m, c+d-\frac{k+l}{2}\right) + f\left(a+b-n, c+d-\frac{k+l}{2}\right) \right] \\
&+ \rho^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) \\
&\times \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\
&+ \left. f(a+b-m, c+d-l) + f(a+b-m, c+d-k) \right] \\
&- \sigma(\sigma-\rho) \Lambda_1\left(\frac{1}{2}\right) \\
&\times \left[{}_{c+d-l^+} I_{\psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) + {}_{c+d-k^-} I_{\psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
&+ \left. {}_{c+d-l^+} I_{\psi} f\left(a+b-m, c+d-\frac{k+l}{2}\right) + {}_{c+d-k^-} I_{\psi} f\left(a+b-m, c+d-\frac{k+l}{2}\right) \right] \\
&- \sigma(\sigma-\rho) \Lambda_2\left(\frac{1}{2}\right) \\
&\times \left[{}_{a+b-n^+} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) + {}_{a+b-n^+} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\
&+ \left. {}_{a+b-m^-} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) + {}_{a+b-m^-} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right] \\
&- \sigma\rho \Lambda_1\left(\frac{1}{2}\right) \\
&\times \left[{}_{c+d-l^+} I_{\psi} f\left(a+b-n, c+d-\frac{k+l}{2}\right) + {}_{c+d-k^-} I_{\psi} f\left(a+b-n, c+d-\frac{k+l}{2}\right) \right. \\
&+ \left. {}_{c+d-l^+} I_{\psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) + {}_{c+d-k^-} I_{\psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right] \\
&- \sigma\rho \Lambda_2\left(\frac{1}{2}\right) \\
&\times \left[{}_{a+b-n^+} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-l\right) + {}_{a+b-n^+} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
&+ \left. {}_{a+b-m^-} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-l\right) + {}_{a+b-m^-} I_{\varphi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right] \\
&+ \sigma^2 \left[{}_{a+b-n^+, c+d-l^+} I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
&+ \left. {}_{a+b-n^+, c+d-k^-} I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right]
\end{aligned}$$

$$\begin{aligned}
 &+_{a+b-m^-,c+d-l^+} I_{\varphi,\psi} f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right) \\
 &+_{a+b-m^-,c+d-k^-} I_{\varphi,\psi} f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)
 \end{aligned}$$

where

$$p(t,s) = \begin{cases} \left[\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right], & (t,s) \in \left[0,\frac{1}{2}\right] \times \left[0,\frac{1}{2}\right] \\ \left[\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(1-s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right], & (t,s) \in \left[0,\frac{1}{2}\right] \times \left(\frac{1}{2},1\right] \\ \left[\sigma\Lambda_1(1-t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right], & (t,s) \in \left(\frac{1}{2},1\right] \times \left[0,\frac{1}{2}\right] \\ \left[\sigma\Lambda_1(1-t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(1-s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right], & (t,s) \in \left(\frac{1}{2},1\right] \times \left(\frac{1}{2},1\right] \end{cases}$$

and

$$\Lambda_1(t) = \int_0^t \frac{\varphi((b-a)\lambda)}{\lambda} d\lambda, \quad \Lambda_2(s) = \int_0^s \frac{\psi((d-c)\mu)}{\mu} d\mu.$$

proof:

By using integration by parts twice and changing variables by $x = a+b-[tm+(1-t)n]$ and $y = c+d-[sk+(1-s)l]$, we have

$$\begin{aligned}
 I_1 &:= \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left[\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right] \\
 &\times \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n],c+d-[sk+(1-s)l]\right) dt ds \\
 &= \frac{1}{(n-m)(l-k)} \times \left\{ (\sigma-\rho)^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right) \right. \\
 &+ \rho(\sigma-\rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2},c+d-l\right) \\
 &- \rho(\sigma-\rho) \Lambda_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} f\left(a+b-\frac{m+n}{2},c+d-[sk+(1-s)l]\right) \frac{\psi((d-c)s)}{s} ds \\
 &+ \rho(\sigma-\rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-n,c+d-\frac{k+l}{2}\right) \\
 &+ \rho^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f(a+b-n,c+d-l) \\
 &- \rho\sigma \Lambda_1\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} f\left(a+b-n,c+d-[sk+(1-s)l]\right) \frac{\psi((d-c)s)}{s} ds \\
 &+ \sigma(\sigma-\rho) \Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} f\left(a+b-[tm+(1-t)n],c+d-\frac{k+l}{2}\right) \frac{\varphi((b-a)t)}{t} dt \\
 &+ \rho\sigma \Lambda_2\left(\frac{1}{2}\right) \int_0^{\frac{1}{2}} f\left(a+b-[tm+(1-t)n],c+d-l\right) \frac{\varphi((b-a)t)}{t} dt \\
 &- \sigma^2 \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} f\left(a+b-[tm+(1-t)n],c+d-[sk+(1-s)l]\right)
 \end{aligned}$$

$$\begin{aligned}
 & \times \frac{\varphi((b-a)t) \psi((d-c)s)}{t \quad s} ds dt \\
 & = \frac{1}{(n-m)(l-k)} \\
 & \times \left\{ (\sigma - \rho)^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + \rho(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2}, c+d-l\right) \\
 & + \rho(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-n, c+d-\frac{k+l}{2}\right) \\
 & + \rho^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f(a+b-n, c+d-l) \\
 & - \sigma(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right)_{c+d-l^+} I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & - \sigma(\sigma - \rho) \Lambda_2\left(\frac{1}{2}\right)_{a+b-n^+} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & - \sigma\rho \Lambda_1\left(\frac{1}{2}\right)_{c+d-l^+} I_\psi f\left(a+b-n, c+d-\frac{k+l}{2}\right) \\
 & - \sigma\rho \Lambda_2\left(\frac{1}{2}\right)_{a+b-n^+} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-l\right) \\
 & \left. + \sigma^2_{a+b-n^+, c+d-l^+} I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right\}.
 \end{aligned}$$

By the same way, we get

$$\begin{aligned}
 I_2 & := \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left[\sigma \Lambda_1(t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(1-s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] \\
 & \times \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]\right) dt ds \\
 & = \frac{1}{(n-m)(l-k)} \\
 & \times \left\{ -(\sigma - \rho)^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & - \rho(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2}, c+d-k\right) \\
 & - \rho(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-n, c+d-\frac{k+l}{2}\right) \\
 & - \rho^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f(a+b-n, c+d-k) \\
 & + \sigma(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right)_{c+d-k^-} I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & + \sigma(\sigma - \rho) \Lambda_2\left(\frac{1}{2}\right)_{a+b-n^+} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & \left. + \sigma\rho \Lambda_1\left(\frac{1}{2}\right)_{c+d-k^-} I_\psi f\left(a+b-n, c+d-\frac{k+l}{2}\right) \right\}
 \end{aligned}$$

$$\begin{aligned}
 & +\sigma\rho\Lambda_2\left(\frac{1}{2}\right)_{a+b-n^+}I_\varphi f\left(a+b-\frac{m+n}{2},c+d-k\right) \\
 & -\sigma^2_{a+b-n^+,c+d-k^-}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\}, \\
 I_3 := & \int_0^{\frac{1}{2}}\int_{\frac{1}{2}}^1\left[\sigma\Lambda_1(1-t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(1-s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right] \\
 & \times\frac{\partial^2}{\partial t\partial s}f\left(a+b-[tm+(1-t)n],c+d-[sk+(1-s)l]\right)dtds \\
 = & \frac{1}{(n-m)(l-k)} \\
 & \times\left\{-(\sigma-\rho)^2\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & -\rho(\sigma-\rho)\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(a+b-\frac{m+n}{2},c+d-l\right) \\
 & -\rho(\sigma-\rho)\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(a+b-m,c+d-\frac{k+l}{2}\right) \\
 & -\rho^2\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f(a+b-m,c+d-l) \\
 & +\sigma(\sigma-\rho)\Lambda_1\left(\frac{1}{2}\right)_{c+d-l^+}I_\psi f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right) \\
 & +\sigma(\sigma-\rho)\Lambda_2\left(\frac{1}{2}\right)_{a+b-m^-}I_\varphi f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right) \\
 & +\sigma\rho\Lambda_1\left(\frac{1}{2}\right)_{c+d-l^+}I_\psi f\left(a+b-m,c+d-\frac{k+l}{2}\right) \\
 & +\sigma\rho\Lambda_2\left(\frac{1}{2}\right)_{a+b-m^-}I_\varphi f\left(a+b-\frac{m+n}{2},c+d-l\right) \\
 & \left.-\sigma^2_{a+b-m^-,c+d-l^+}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right\},
 \end{aligned}$$

and

$$\begin{aligned}
 I_4 := & \int_{\frac{1}{2}}^1\int_{\frac{1}{2}}^1\left[\sigma\Lambda_1(1-t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(1-s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right] \\
 & \times\frac{\partial^2}{\partial t\partial s}f\left(a+b-[tm+(1-t)n],c+d-[sk+(1-s)l]\right)dtds \\
 = & \frac{1}{(n-m)(l-k)} \\
 & \times\left\{(\sigma-\rho)^2\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & +\rho(\sigma-\rho)\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(a+b-\frac{m+n}{2},c+d-k\right) \\
 & +\rho(\sigma-\rho)\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f\left(a+b-m,c+d-\frac{k+l}{2}\right) \\
 & \left.+\rho^2\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)f(a+b-m,c+d-k)\right\}
 \end{aligned}$$

$$\begin{aligned}
 & -\sigma(\sigma - \rho)\Lambda_1\left(\frac{1}{2}\right)_{c+d-k^-} I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & -\sigma(\sigma - \rho)\Lambda_2\left(\frac{1}{2}\right)_{a+b-m^-} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-k\right) \\
 & -\sigma\rho\Lambda_1\left(\frac{1}{2}\right)_{c+d-k^-} I_\psi f\left(a+b-m, c+d-\frac{k+l}{2}\right) \\
 & -\sigma\rho\Lambda_2\left(\frac{1}{2}\right)_{a+b-m^-} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & +\sigma^2_{a+b-m^-, c+d-k^-} I_{\varphi,\psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right)\}.
 \end{aligned}$$

By adding $I_1 - I_2 - I_3 + I_4$, then we get

$$\begin{aligned}
 & \int_0^1 \int_0^1 p(t,s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]\right) dt ds \\
 & = \frac{1}{(n-m)(l-k)} \\
 & \times \left\{ 4(\sigma - \rho)^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + 2\rho(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) \\
 & \times \left[f\left(a+b-\frac{m+n}{2}, c+d-l\right) + f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\
 & \left. \left. + f\left(a+b-m, c+d-\frac{k+l}{2}\right) + f\left(a+b-n, c+d-\frac{k+l}{2}\right) \right] \right. \\
 & + \rho^2 \Lambda_1\left(\frac{1}{2}\right) \Lambda_2\left(\frac{1}{2}\right) \\
 & \times \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\
 & \left. + f(a+b-m, c+d-l) + f(a+b-m, c+d-k) \right] \\
 & - \sigma(\sigma - \rho) \Lambda_1\left(\frac{1}{2}\right) \\
 & \times \left[{}_{c+d-l^+} I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + {}_{c+d-k^-} I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & \left. + {}_{c+d-l^+} I_\psi f\left(a+b-m, c+d-\frac{k+l}{2}\right) + {}_{c+d-k^-} I_\psi f\left(a+b-m, c+d-\frac{k+l}{2}\right) \right] \\
 & - \sigma(\sigma - \rho) \Lambda_2\left(\frac{1}{2}\right) \\
 & \times \left[{}_{a+b-n^+} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) + {}_{a+b-n^+} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\
 & \left. + {}_{a+b-m^-} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) + {}_{a+b-m^-} I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & -\sigma\rho\Lambda_1\left(\frac{1}{2}\right) \\
 & \times \left[{}_{c+d-l^+}I_\psi f\left(a+b-n, c+d-\frac{k+l}{2}\right) + {}_{c+d-k^-}I_\psi f\left(a+b-n, c+d-\frac{k+l}{2}\right) \right. \\
 & + {}_{c+d-l^+}I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & \left. + {}_{c+d-k^-}I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right] \\
 & -\sigma\rho\Lambda_2\left(\frac{1}{2}\right) \\
 & \times \left[{}_{a+b-n^+}I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-l\right) + {}_{a+b-n^+}I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + {}_{a+b-m^-}I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-l\right) + {}_{a+b-m^-}I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \left. \right] \\
 & + \sigma^2 \left[{}_{a+b-n^+, c+d-l^+}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + {}_{a+b-n^+, c+d-k^-}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & + {}_{a+b-m^-, c+d-l^+}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & \left. + {}_{a+b-m^-, c+d-k^-}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right] \Big\}.
 \end{aligned}$$

Remark 3.2

If we assume $\varphi(t) = t, \psi(s) = s$ in Lemma 3.1, then we obtain the following equality:

$$\begin{aligned}
 & \int_0^1 \int_0^1 p(t, s) \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]\right) dt ds \\
 & = \frac{1}{(n-m)(l-k)} \\
 & \times \left\{ (\sigma-\rho)^2 (b-a)(d-c) f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + \rho(\sigma-\rho) \frac{(b-a)(d-c)}{2} \\
 & \times \left[f\left(a+b-\frac{m+n}{2}, c+d-l\right) + f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\
 & \left. + f\left(a+b-m, c+d-\frac{k+l}{2}\right) + f\left(a+b-n, c+d-\frac{k+l}{2}\right) \right] \\
 & + \rho^2 \frac{(b-a)(d-c)}{4} \\
 & \times \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\
 & \left. + f(a+b-m, c+d-l) + f(a+b-m, c+d-k) \right]
 \end{aligned}$$

$$\begin{aligned}
 & -\sigma(\sigma-\rho)\frac{(b-a)}{2}\times\left[2\int_{c+d-l}^{c+d-k}f\left(a+b-\frac{m+n}{2},s\right)ds\right] \\
 & -\sigma(\sigma-\rho)\frac{(d-c)}{2}\times\left[2\int_{a+b-n}^{a+b-m}f\left(t,c+d-\frac{k+l}{2}\right)dt\right] \\
 & -\sigma\rho\frac{b-a}{2}\times\left[\int_{c+d-l}^{c+d-k}f(a+b-n,s)ds+\int_{c+d-l}^{c+d-k}f(a+b-m,s)ds\right] \\
 & -\sigma\rho\frac{(d-c)}{2}\times\left[\int_{a+b-n}^{a+b-m}f(t,c+d-l)dt\int_{a+b-n}^{a+b-m}f(t,c+d-k)dt\right] \\
 & +\sigma^2\int_{a+b-n}^{a+b-m}\int_{c+d-l}^{c+d-k}f(t,s)dtds
 \end{aligned}$$

where

$$p(t,s)=\begin{cases} \left[\sigma t-\frac{\rho}{2}\right]\left[\sigma s-\frac{\rho}{2}\right], & (t,s)\in\left[0,\frac{1}{2}\right]\times\left[0,\frac{1}{2}\right] \\ \left[\sigma t-\frac{\rho}{2}\right]\left[\sigma(1-s)-\frac{\rho}{2}\right], & (t,s)\in\left[0,\frac{1}{2}\right]\times\left(\frac{1}{2},1\right] \\ \left[\sigma(1-t)-\frac{\rho}{2}\right]\left[\sigma s-\frac{\rho}{2}\right], & (t,s)\in\left(\frac{1}{2},1\right]\times\left[0,\frac{1}{2}\right] \\ \left[\sigma(1-t)-\frac{\rho}{2}\right]\left[\sigma(1-s)-\frac{\rho}{2}\right], & (t,s)\in\left(\frac{1}{2},1\right]\times\left(\frac{1}{2},1\right] \end{cases}$$

Corollary 3.3

In Remark 3.2, we let $\sigma=1, \rho=\frac{1}{3}, m=a, n=b, k=c, l=d$, then the equality reduced to the original Simpson’ inequalities on the coordinated [7] as follows:

$$\begin{aligned}
 & \int_0^1\int_0^1p(t,s)\frac{\partial^2}{\partial t\partial s}f\left(a+b-\left[tm+(1-t)n\right],c+d-\left[sk+(1-s)l\right]\right)dtds \\
 & =\left\{\frac{4}{36}f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right. \\
 & +\frac{1}{9}\times\left[f\left(\frac{a+b}{2},c\right)+f\left(\frac{a+b}{2},d\right)+f\left(a,\frac{c+d}{2}\right)+f\left(b,\frac{c+d}{2}\right)\right] \\
 & +\frac{1}{36}\times\left[f(a,c)+f(a,d)+f(b,c)+f(b,d)\right] \\
 & -\frac{1}{d-c}\frac{1}{3}\times\left[2\int_c^df\left(\frac{a+b}{2},s\right)ds\right] \\
 & -\frac{1}{b-a}\frac{1}{3}\times\left[2\int_a^bf\left(t,\frac{c+d}{2}\right)dt\right] \\
 & -\frac{1}{d-c}\frac{1}{6}\times\left[\int_c^df(a,s)ds+\int_c^df(b,s)ds\right] \\
 & -\frac{1}{b-a}\frac{1}{6}\times\left[\int_a^bf(t,c)dt+\int_a^bf(t,d)dt\right] \\
 & +\frac{1}{(b-a)(d-c)}\int_a^b\int_c^df(t,s)dtds
 \end{aligned}$$

where

$$p(t,s) = \begin{cases} \left[t - \frac{1}{6} \right] \left[s - \frac{1}{6} \right], & (t,s) \in \left[0, \frac{1}{2} \right] \times \left[0, \frac{1}{2} \right] \\ \left[t - \frac{1}{6} \right] \left[\frac{5}{6} - s \right], & (t,s) \in \left[0, \frac{1}{2} \right] \times \left(\frac{1}{2}, 1 \right] \\ \left[\frac{5}{6} - t \right] \left[s - \frac{1}{6} \right], & (t,s) \in \left(\frac{1}{2}, 1 \right] \times \left[0, \frac{1}{2} \right] \\ \left[\frac{5}{6} - t \right] \left[\frac{5}{6} - s \right], & (t,s) \in \left(\frac{1}{2}, 1 \right] \times \left(\frac{1}{2}, 1 \right] \end{cases}$$

In this section, we establish some new generalized inequalities for differentiable convex functions via generalized fractional integrals.

Theorem 3.4

Assume that the conditions of Lemma 3.1 hold. If the mapping $\left| \frac{\partial^2}{\partial t \partial s} f \right|$ is the convex function on the coordinates on $[a,b] \times [c,d]$, then the following inequality holds for generalized fractional integrals:

$$\begin{aligned} & \left\{ 4(\sigma - \rho)^2 f\left(a + b - \frac{m+n}{2}, c + d - \frac{k+l}{2}\right) \right. \\ & + \rho(\sigma - \rho) \left[f\left(a + b - \frac{m+n}{2}, c + d - l\right) + f\left(a + b - \frac{m+n}{2}, c + d - k\right) \right. \\ & \left. + f\left(a + b - n, c + d - \frac{k+l}{2}\right) + f\left(a + b - m, c + d - \frac{k+l}{2}\right) \right] \\ & + \rho^2 \left[f(a + b - n, c + d - l) + f(a + b - n, c + d - k) \right. \\ & \left. + f(a + b - m, c + d - l) + f(a + b - m, c + d - k) \right] \\ & - \rho(\sigma - \rho) \frac{1}{\Lambda_2\left(\frac{1}{2}\right)} \left[2 {}_{c+d-l^+} I_{\psi} f\left(a + b - \frac{m+n}{2}, c + d - \frac{k+l}{2}\right) \right. \\ & \left. + 2 {}_{c+d-k^-} I_{\psi} f\left(a + b - \frac{m+n}{2}, c + d - \frac{k+l}{2}\right) \right] \\ & - \sigma\rho \frac{1}{\Lambda_2\left(\frac{1}{2}\right)} \left[{}_{c+d-l^+} I_{\psi} f\left(a, c + d - \frac{k+l}{2}\right) + {}_{c+d-l^+} I_{\psi} f\left(a + b - m, c + d - \frac{k+l}{2}\right) \right. \\ & \left. + {}_{c+d-k^-} I_{\psi} f\left(a, c + d - \frac{k+l}{2}\right) + {}_{c+d-k^-} I_{\psi} f\left(a + b - m, c + d - \frac{k+l}{2}\right) \right] \\ & - \sigma(\sigma - \rho) \frac{1}{\Lambda_1\left(\frac{1}{2}\right)} \left[2 {}_{a+b-n^+} I_{\varphi} f\left(a + b - \frac{m+n}{2}, c + d - \frac{k+l}{2}\right) \right. \\ & \left. + 2 {}_{a+b-m^-} I_{\varphi} f\left(a + b - \frac{m+n}{2}, c + d - \frac{k+l}{2}\right) \right] \\ & - \sigma\rho \frac{1}{\Lambda_1\left(\frac{1}{2}\right)} \left[{}_{a+b-n^+} I_{\varphi} f\left(a + b - \frac{m+n}{2}, c + d - l\right) \right. \end{aligned}$$

$$\begin{aligned}
 &+ {}_{a+b-n^+}I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-k\right) \\
 &+ {}_{a+b-m^-}I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-l\right) + {}_{a+b-m^-}I_\varphi f\left(a+b-\frac{m+n}{2}, c+d-k\right) \Big] \\
 &+ 4\sigma^2 \frac{1}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)} \Big[{}_{a+b-n^+, c+d-l^+}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 &+ {}_{a+b-n^+, c+d-k^-}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 &+ {}_{a+b-m^-, c+d-l^+}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 &+ {}_{a+b-m^-, c+d-k^-}I_{\varphi, \psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \Big] \Big] \\
 &\leq \frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)} \left\{ \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right| \right. \\
 &+ \left. \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right| \right\} \\
 &\times \left(\int_0^{\frac{1}{2}} \left| \sigma \Lambda_1(t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right| dt \right) \left(\int_0^{\frac{1}{2}} \left| \sigma \Lambda_2(s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right| ds \right).
 \end{aligned}$$

Proof:

By taking Lemma 3.1, and the mapping $\left| \frac{\partial^2}{\partial t \partial s} f \right|$ is the convex on $[a, b] \times [c, d]$, we have

$$\begin{aligned}
 &\left\{ 4(\sigma - \rho)^2 f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 &+ \rho(\sigma - \rho) \\
 &\times \left[f\left(a+b-\frac{m+n}{2}, c+d-l\right) + f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\
 &+ \left. \left. f\left(a+b-n, c+d-\frac{k+l}{2}\right) + f\left(a+b-m, c+d-\frac{k+l}{2}\right) \right] \right. \\
 &+ \rho^2 \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\
 &+ \left. \left. f(a+b-m, c) + f(a+b-m, c+d-k) \right] \right. \\
 &- \frac{\rho(\sigma - \rho)}{\Lambda_2\left(\frac{1}{2}\right)} \left[2 {}_{c+d-l^+}I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 &+ \left. \left. 2 {}_{c+d-k^-}I_\psi f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right] \right. \\
 &- \frac{\sigma\rho}{\Lambda_2\left(\frac{1}{2}\right)} \left[{}_{c+d-l^+}I_\psi f\left(a+b-n, c+d-\frac{k+l}{2}\right) + {}_{c+d-l^+}I_\psi f\left(a+b-m, c+d-\frac{k+l}{2}\right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + {}_{c+d-k}I_{\psi}f\left(a+b-n, c+d-\frac{k+l}{2}\right) + {}_{c+d-k}I_{\psi}f\left(a+b-m, c+d-\frac{k+l}{2}\right) \\
 & - \frac{\sigma(\sigma-\rho)}{\Lambda_1\left(\frac{1}{2}\right)} \left[2 {}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + 2 {}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & \left. - \frac{\sigma\rho}{\Lambda_1\left(\frac{1}{2}\right)} \left[{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-l\right) \right. \right. \\
 & + {}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-k\right) \\
 & + {}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-l\right) + {}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-k\right) \left. \right] \\
 & + \frac{4\sigma^2}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)} \left[{}_{a+b-n^+, c+d-l^+}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + {}_{a+b-n^+, c+d-k^-}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & + {}_{a+b-m^-, c+d-l^+}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\
 & \left. \left. + {}_{a+b-m^-, c+d-k^-}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right] \right\} \\
 & \leq \frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)} \\
 & \times \int_0^1 \int_0^1 |p(t,s)| \left| \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]\right) \right| dt ds \\
 & \leq \frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)} \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left[\sigma\Lambda_1(t) - \rho\Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma\Lambda_2(s) - \rho\Lambda_2\left(\frac{1}{2}\right) \right] \right. \\
 & \times \left| \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]\right) \right| dt ds \\
 & + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left[\sigma\Lambda_1(t) - \rho\Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma\Lambda_2(1-s) - \rho\Lambda_2\left(\frac{1}{2}\right) \right] \\
 & \times \left| \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]\right) \right| dt ds \\
 & + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left[\sigma\Lambda_1(1-t) - \rho\Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma\Lambda_2(s) - \rho\Lambda_2\left(\frac{1}{2}\right) \right] \\
 & \left. \times \left| \frac{\partial^2}{\partial t \partial s} f\left(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]\right) \right| dt ds \right.
 \end{aligned}$$

$$\begin{aligned}
 & + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left[\left[\sigma \Lambda_1(1-t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(1-s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] \right] \\
 & \times \left[\frac{\partial^2}{\partial t \partial s} f(a+b-[tm+(1-t)n], c+d-[sk+(1-s)l]) \right] dt ds \Big\} \\
 & \leq \frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)} \left\{ \int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left[\left[\sigma \Lambda_1(t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] \right] \right. \\
 & \times \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right. \\
 & + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \\
 & \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right] dt ds \\
 & + \int_{\frac{1}{2}}^1 \int_0^{\frac{1}{2}} \left[\left[\sigma \Lambda_1(t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(1-s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] \right] \\
 & \times \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right. \\
 & + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \\
 & \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right] dt ds \\
 & + \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left[\left[\sigma \Lambda_1(1-t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] \right] \\
 & \times \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right. \\
 & + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \\
 & \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right] dt ds \\
 & + \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left[\left[\sigma \Lambda_1(1-t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(1-s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] \right] \\
 & \times \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right. \\
 & + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \\
 & \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right] dt ds \Big\} \\
 & \leq \frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)} \left\{ \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right| \right\}
 \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right| \Bigg\} \\
 & \times \left(\int_0^{\frac{1}{2}} \left| \sigma \Lambda_1(t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right| dt \right) \left(\int_0^{\frac{1}{2}} \left| \sigma \Lambda_2(s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right| ds \right).
 \end{aligned}$$

Remark 3.5

If we assume $\varphi(t) = t, \psi(s) = s$ in Theorem 3.4, then we obtain the following inequality:

$$\begin{aligned}
 & \left\{ 4(\sigma - \rho)^2 f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \right. \\
 & + \rho^2 [f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \\
 & + f(a+b-m, c+d-l) + f(a+b-m, c+d-k)] \\
 & + \rho(\sigma - \rho) \left[f\left(a+b - \frac{m+n}{2}, c+d-l\right) \right. \\
 & + f\left(a+b - \frac{m+n}{2}, c+d-k\right) \\
 & + f\left(a+b-n, c+d - \frac{k+l}{2}\right) \\
 & \left. + f\left(a+b-m, c+d - \frac{k+l}{2}\right) \right] \\
 & + \frac{16\sigma^2}{(n-m)(l-k)} \left[\int_{c+d-l}^{c+d-k} \int_{a+b-n}^{a+b-m} f(t, s) dt ds \right] \\
 & - \frac{4\rho(\sigma - \rho)}{l-k} \left[\int_{c+d-l}^{c+d-k} f\left(a+b - \frac{m+n}{2}, s\right) ds \right] \\
 & - \frac{4\sigma(\sigma - \rho)}{n-m} \left[\int_{a+b-n}^{a+b-m} f\left(t, c+d - \frac{k+l}{2}\right) dt \right] \\
 & - \frac{2\sigma\rho}{l-k} \left[\int_{c+d-l}^{c+d-k} f(a, s) ds + \int_{c+d-l}^d f(a+b-m, s) ds \right] \\
 & \left. - \frac{2\sigma\rho}{n-m} \left[\int_{a+b-n}^{a+b-m} f(t, c) dt + \int_{a+b-n}^{a+b-m} f(t, d) dt \right] \right\} \\
 & \leq 4(n-m)(l-k) \left\{ \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right| \right. \\
 & + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right| \Bigg\} \\
 & \times \left(\int_0^{\frac{1}{2}} \left| \sigma t - \frac{\rho}{2} \right| dt \right) \left(\int_0^{\frac{1}{2}} \left| \sigma s - \frac{\rho}{2} \right| ds \right).
 \end{aligned}$$

Corollary 3.6

In Theorem 3.4, we let $\sigma = 1, \rho = \frac{1}{3}, m = a, n = b, k = c, l = d$, then the equality reduced to the parametered Simpson' inequalities in Theorem 2.4.

Remark 3.7

If we assume $\varphi(t) = \frac{t^\alpha}{\Gamma(\alpha)}, \psi(s) = \frac{s^\beta}{\Gamma(\beta)}$ in Theorem 3.4, then we obtain the following new equality for the Riemann-Liouville fractional integral inequality:

$$\begin{aligned} & \left\{ 4(\sigma - \rho)^2 f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\ & + \rho(\sigma - \rho) \left[f\left(a+b-\frac{m+n}{2}, c+d-l\right) + f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\ & + f\left(a+b-n, c+d-\frac{k+l}{2}\right) + f\left(a+b-m, c+d-\frac{k+l}{2}\right) \left. \right] \\ & + \rho^2 \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\ & + f(a+b-m, c+d-l) + f(a+b-m, c+d-k) \left. \right] \\ & - \rho(\sigma - \rho) \frac{2^\beta \Gamma(\beta+1)}{(l-k)^\beta} \left[2_{c+d-l^+} J_\beta f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\ & + 2_{c+d-k^-} J_\beta f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \left. \right] \\ & - \sigma \rho \frac{2^\beta \Gamma(\beta+1)}{(l-k)^\beta} \left[2_{c+d-l^+} J_\beta f\left(a+b-n, c+d-\frac{k+l}{2}\right) + 2_{c+d-l^+} J_\beta f\left(b, c+d-\frac{k+l}{2}\right) \right. \\ & + 2_{c+d-k^-} J_\beta f\left(a+b-n, c+d-\frac{k+l}{2}\right) + 2_{c+d-k^-} J_\beta f\left(a+b-m, c+d-\frac{k+l}{2}\right) \left. \right] \\ & - \sigma(\sigma - \rho) \frac{2^\alpha \Gamma(\alpha+1)}{(n-m)^\alpha} \left[2_{a^+} J_\alpha f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\ & + 2_{b^-} J_\alpha f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \left. \right] \\ & - \sigma \rho \frac{2^\alpha \Gamma(\alpha+1)}{(n-m)^\alpha} \left[2_{a+b-n^+} J_\alpha f\left(a+b-\frac{m+n}{2}, c\right) \right. \\ & + 2_{a+b-n^+} J_\alpha f\left(a+b-\frac{m+n}{2}, c+d-k\right) \\ & + 2_{a+b-m^-} J_\alpha f\left(a+b-\frac{m+n}{2}, c\right) + 2_{a+b-m^-} J_\alpha f\left(\frac{a+b}{2}, c+d-k\right) \left. \right] \\ & + 4\sigma^2 \frac{2^{\alpha+\beta} \Gamma(\alpha+1) \Gamma(\beta+1)}{(n-m)^\alpha (l-k)^\beta} \left[2_{a+b-n^+, c+d-l^+} J_{\alpha, \beta} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\ & + 2_{a+b-n^+, c+d-k^-} J_{\alpha, \beta} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\ & + 2_{a+b-m^-, c+d-l^+} J_{\alpha, \beta} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \\ & + 2_{a+b-m^-, c+d-k^-} J_{\alpha, \beta} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \left. \right] \left. \right\} \\ & \leq (n-m)(l-k) \left\{ \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right| \right\} \end{aligned}$$

$$\begin{aligned}
 & + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right| + \left| \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right| \Bigg\} \\
 & \times \left(\int_0^{\frac{1}{2}} \left| \sigma t^\alpha - \rho \left(\frac{1}{2} \right)^\alpha \right| dt \right) \left(\int_0^{\frac{1}{2}} \left| \sigma s^\beta - \rho \left(\frac{1}{2} \right)^\beta \right| ds \right).
 \end{aligned}$$

Remark 3.8

If we assume $\varphi(t) = \frac{t^\alpha}{k\Gamma_k(\alpha)}$, $\psi(s) = \frac{s^\beta}{k\Gamma_k(\beta)}$ in Theorem 3.4, then we obtain the following new equality for the k -Riemann-Liouville fractional integral inequality:

$$\begin{aligned}
 & \left\{ 4(\sigma - \rho)^2 f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \right. \\
 & + \rho(\sigma - \rho) \left[f\left(a+b - \frac{m+n}{2}, c+d-l\right) + f\left(a+b - \frac{m+n}{2}, c+d-k\right) \right. \\
 & + f\left(a+b-n, c+d - \frac{k+l}{2}\right) + f\left(b, c+d - \frac{k+l}{2}\right) \Big] \\
 & + \rho^2 \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\
 & + f(a+b-m, c+d-l) + f(a+b-m, c+d-k) \Big] \\
 & - \rho(\sigma - \rho) \frac{2^{\frac{\beta}{k}} \Gamma(\beta+k)}{(l-k)^{\frac{\beta}{k}}} \left[{}_2_{c+d-l^+} J_{\beta;k} f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \right. \\
 & + 2_{{c+d-k}^-} J_{\beta;k} f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \Big] \\
 & - \sigma \rho \frac{2^{\frac{\beta}{k}} \Gamma(\beta+k)}{(l-k)^{\frac{\beta}{k}}} \left[{}_{c+d-l^+} J_{\beta;k} f\left(a+b-n, c+d - \frac{k+l}{2}\right) \right. \\
 & + {}_{c+d-l^+} J_{\beta;k} f\left(a+b-m, c+d - \frac{k+l}{2}\right) \\
 & + {}_{c+d-k}^- J_{\beta;k} f\left(a+b-n, c+d - \frac{k+l}{2}\right) \\
 & + {}_{c+d-k}^- J_{\beta;k} f\left(a+b-m, c+d - \frac{k+l}{2}\right) \Big] \\
 & - \sigma(\sigma - \rho) \frac{2^{\frac{\alpha}{k}} \Gamma(\alpha+k)}{(n-m)^{\frac{\alpha}{k}}} \left[2_{{a+b-n^+} J_{\alpha;k} f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \right. \\
 & + 2_{{a+b-m}^-} J_{\alpha;k} f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \Big] \\
 & - \sigma \rho \frac{2^{\frac{\alpha}{k}} \Gamma(\alpha+k)}{(n-m)^{\frac{\alpha}{k}}} \left[{}_{a+b-n^+} J_{\alpha;k} f\left(\frac{a+b}{2}, c\right) + {}_{a+b-n^+} J_{\alpha;k} f\left(\frac{a+b}{2}, c+d-k\right) \right.
 \end{aligned}$$

$$\begin{aligned}
 &+ {}_{a+b-m^-}J_{\alpha;k}f\left(a+b-\frac{m+n}{2},c\right)+ {}_{a+b-m^-}J_{\alpha;k}f\left(a+b-\frac{m+n}{2},c+d-k\right)\Big] \\
 &+4\sigma^2\frac{2^{\frac{\alpha}{k}+\frac{\beta}{k}}\Gamma(\alpha+k)\Gamma(\beta+k)}{(n-m)^{\frac{\alpha}{k}}(l-k)^{\frac{\beta}{k}}}\Big[{}_{a+b-n^+,c+d-l^+}J_{\alpha,\beta;k}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right) \\
 &+ {}_{a+b-n^+,c+d-k^-}J_{\alpha,\beta;k}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right) \\
 &+ {}_{a+b-m^-,c+d-l^+}J_{\alpha,\beta;k}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right) \\
 &+ {}_{a+b-m^-,c+d-k^-}J_{\alpha,\beta;k}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\Big]\Big] \\
 &\leq(n-m)(l-k)\left\{\left|\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-k)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-l)\right|\right. \\
 &+ \left.\left|\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-k)\right|+\left|\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-l)\right|\right\} \\
 &\times\left(\int_0^{\frac{1}{2}}\left|\sigma t^\alpha-\rho\left(\frac{1}{2}\right)^\alpha\right|dt\right)\left(\int_0^{\frac{1}{2}}\left|\sigma s^\beta-\rho\left(\frac{1}{2}\right)^\beta\right|ds\right).
 \end{aligned}$$

Theorem 3.9

Assume that the conditions of Lemma 3.1 hold. If $\left|\frac{\partial^2}{\partial t\partial s}f(t,s)\right|^r, r > 1$ is a convex mapping on $[a,b]\times[c,d]$, then we have the following inequality:

$$\begin{aligned}
 &\left\{4(\sigma-\rho)^2f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 &+ \rho(\sigma-\rho)\left[f\left(a+b-\frac{m+n}{2},c+d-l\right)+f\left(a+b-\frac{m+n}{2},c+d-k\right)\right. \\
 &+ \left.f\left(a+b-n,c+d-\frac{k+l}{2}\right)+f\left(a+b-m,c+d-\frac{k+l}{2}\right)\right] \\
 &+ \rho^2\left[f(a+b-n,c+d-l)+f(a+b-n,c+d-k)\right. \\
 &+ \left.f(a+b-m,c+d-l)+f(a+b-m,c+d-k)\right] \\
 &- \rho(\sigma-\rho)\frac{1}{\Lambda_2\left(\frac{1}{2}\right)}\left[{}_{c+d-l^+}I_\psi f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 &+ \left.2{}_{c+d-k^-}I_\psi f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right] \\
 &- \sigma\rho\frac{1}{\Lambda_2\left(\frac{1}{2}\right)}\left[{}_{c+d-l^+}I_\psi f\left(a+b-n,c+d-\frac{k+l}{2}\right)\right. \\
 &+ \left.{}_{c+d-l^+}I_\psi f\left(a+b-m,c+d-\frac{k+l}{2}\right)+{}_{c+d-k^-}I_\psi f\left(a+b-n,c+d-\frac{k+l}{2}\right)\right]
 \end{aligned}$$

$$\begin{aligned}
 & + {}_{c+d-k}I_{\psi}f\left(a+b-m, c+d-\frac{k+l}{2}\right) \\
 & - \sigma(\sigma-\rho)\frac{1}{\Lambda_1\left(\frac{1}{2}\right)}\left[2 {}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+ 2 {}_{a+b-m}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right)\right] \\
 & - \sigma\rho\frac{1}{\Lambda_1\left(\frac{1}{2}\right)}\left[{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-l\right)\right. \\
 & \left.+ {}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-k\right) + {}_{a+b-m}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-l\right)\right. \\
 & \left.+ {}_{a+b-m}I_{\varphi}f\left(a+b-\frac{m+n}{2}, c+d-k\right)\right] \\
 & + 4\sigma^2\frac{1}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)}\left[{}_{a+b-n^+, c+d-l^+}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+ {}_{a+b-n^+, c+d-k^-}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+ {}_{a+b-m^-, c+d-l^+}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+ {}_{a+b-m^-, c+d-k^-}I_{\varphi, \psi}f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right)\right] \Bigg\} \\
 & \leq \frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)}\left\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[\left[\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right]\right]dtds\right)^{1-\frac{1}{r}}\right. \\
 & \times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[ts\frac{\partial^2}{\partial t\partial s}f(a+b-m, c+d-k)\right.\right. \\
 & \left.\left.+ t(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-m, c+d-l)+(1-t)s\frac{\partial^2}{\partial t\partial s}f(a+b-n, c+d-k)\right.\right. \\
 & \left.\left.+ (1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-n, c+d-l)\right]dtds\right)^{\frac{1}{r}} \\
 & \left. + \left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left[\left[\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right]\left[\sigma\Lambda_2(1-s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right]\right]dtds\right)^{1-\frac{1}{r}}\right. \\
 & \times\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left[ts\frac{\partial^2}{\partial t\partial s}f(a+b-m, c+d-k)\right.\right. \\
 & \left.\left.+ t(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-m, c+d-l)+(1-t)s\frac{\partial^2}{\partial t\partial s}f(a+b-n, c+d-k)\right.\right. \\
 & \left.\left.+ (1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-n, c+d-l)\right]dtds\right)^{\frac{1}{r}} \Bigg\}
 \end{aligned}$$

$$\begin{aligned}
 & + \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left[\sigma \Lambda_1(1-t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] dt ds \right)^{\frac{1}{r}} \\
 & \times \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) \right. \right. \\
 & \left. \left. + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right. \right. \\
 & \left. \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right]^r dt ds \right)^{\frac{1}{r}} \\
 & + \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left[\sigma \Lambda_1(1-t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(1-s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] dt ds \right)^{\frac{1}{r}} \\
 & \times \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right. \right. \\
 & \left. \left. + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right. \right. \\
 & \left. \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right]^r dt ds \right)^{\frac{1}{r}} \Bigg\}.
 \end{aligned}$$

proof:

By the power mean inequality, then we have

$$\begin{aligned}
 & \left\{ 4(\sigma - \rho)^2 f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & + \rho(\sigma - \rho) \left[f\left(a+b-\frac{m+n}{2}, c+d-l\right) + f\left(a+b-\frac{m+n}{2}, c+d-k\right) \right. \\
 & \left. + f\left(a+b-n, c+d-\frac{k+l}{2}\right) + f\left(a+b-m, c+d-\frac{k+l}{2}\right) \right] \\
 & + \rho^2 \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\
 & \left. + f(a+b-m, c+d-l) + f(a+b-m, c+d-k) \right] \\
 & - \frac{\rho(\sigma - \rho)}{\Lambda_2\left(\frac{1}{2}\right)} \left[{}_{c+d-l^+} I_{\psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right. \\
 & \left. + 2 {}_{c+d-k^-} I_{\psi} f\left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2}\right) \right] \\
 & - \frac{\sigma\rho}{\Lambda_2\left(\frac{1}{2}\right)} \left[{}_{c+d-l^+} I_{\psi} f\left(a+b-n, c+d-\frac{k+l}{2}\right) + {}_{c+d-l^+} I_{\psi} f\left(a+b-m, c+d-\frac{k+l}{2}\right) \right. \\
 & \left. + {}_{c+d-k^-} I_{\psi} f\left(a+b-n, c+d-\frac{k+l}{2}\right) + {}_{c+d-k^-} I_{\psi} f\left(a+b-m, c+d-\frac{k+l}{2}\right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right]^r dt ds \Bigg)^{\frac{1}{r}} \\
 & + \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left[\sigma \Lambda_1(1-t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] dt ds \right)^{1-\frac{1}{r}} \\
 & \times \left(\int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right. \right. \\
 & \left. \left. + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right. \right. \\
 & \left. \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right]^r dt ds \right)^{\frac{1}{r}} \\
 & + \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left[\sigma \Lambda_1(1-t) - \rho \Lambda_1\left(\frac{1}{2}\right) \right] \left[\sigma \Lambda_2(1-s) - \rho \Lambda_2\left(\frac{1}{2}\right) \right] dt ds \right)^{1-\frac{1}{r}} \\
 & \times \left(\int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^1 \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right. \right. \\
 & \left. \left. + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right. \right. \\
 & \left. \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right]^r dt ds \right)^{\frac{1}{r}} \Bigg\}.
 \end{aligned}$$

Theorem 3.10

Assume that the conditions of Lemma 3.1 hold. If $\left| \frac{\partial^2}{\partial t \partial s} f(t, s) \right|^q$, $q > 1$ is a convex mapping on $[a, b] \times [c, d]$, then we have the following inequality:

$$\begin{aligned}
 & \left\{ 4(\sigma - \rho)^2 f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \right. \\
 & + \rho(\sigma - \rho) \left[f\left(a+b - \frac{m+n}{2}, c+d-l\right) + f\left(a+b - \frac{m+n}{2}, c+d-k\right) \right. \\
 & \left. + f\left(a+b-n, c+d - \frac{k+l}{2}\right) + f\left(a+b-m, c+d - \frac{k+l}{2}\right) \right] \\
 & + \rho^2 \left[f(a+b-n, c+d-l) + f(a+b-n, c+d-k) \right. \\
 & \left. + f(a+b-m, c+d-l) + f(a+b-m, c+d-k) \right] \\
 & - \frac{\rho(\sigma - \rho)}{\Lambda_2\left(\frac{1}{2}\right)} \left[2_{c+d-l^+} I_{\psi} f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \right. \\
 & \left. + 2_{c+d-k^-} I_{\psi} f\left(a+b - \frac{m+n}{2}, c+d - \frac{k+l}{2}\right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{\sigma\rho}{\Lambda_2\left(\frac{1}{2}\right)}\left[{}_{c+d-l^+}I_{\psi}f\left(a,c+d-\frac{k+l}{2}\right)+{}_{c+d-l^+}I_{\psi}f\left(b,c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{c+d-k^-}I_{\psi}f\left(a,c+d-\frac{k+l}{2}\right)+{}_{c+d-k^-}I_{\psi}f\left(a,c+d-\frac{k+l}{2}\right)\right] \\
 & -\frac{\sigma(\sigma-\rho)}{\Lambda_1\left(\frac{1}{2}\right)}\left[2{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+2{}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right] \\
 & -\frac{\sigma\rho}{\Lambda_1\left(\frac{1}{2}\right)}\left[{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2},c\right)+{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2},d\right)\right. \\
 & \left.+{}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2},c\right)+{}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2},d\right)\right] \\
 & +\frac{4\sigma^2}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)}\left[{}_{a+b-n^+,c+d-l^+}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{a+b-n^+,c+d-k^-}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{a+b-m^-,c+d-l^+}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{a+b-m^-,c+d-k^-}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right]\Bigg| \\
 & \leq\frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)}\left\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right|^p\left|\sigma\Lambda_2(s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right|^p dt ds\right)^{\frac{1}{p}}\right. \\
 & \times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[ts\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-k)+t(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-l)\right.\right. \\
 & \left.\left.+(1-t)s\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-k)\right.\right. \\
 & \left.\left.+(1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-l)\right]^q dt ds\right)^{\frac{1}{q}} \\
 & \left.+\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right|^p\left|\sigma\Lambda_2(1-s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right|^p dt ds\right)^{\frac{1}{p}}\right. \\
 & \times\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left[ts\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-k)\right.\right. \\
 & \left.\left.+t(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-l)+(1-t)s\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-k)\right.\right.
 \end{aligned}$$

$$\begin{aligned}
 & -\frac{\sigma\rho}{\Lambda_2\left(\frac{1}{2}\right)}\left[{}_{c+d-l^+}I_{\psi}f\left(a+b-n,c+d-\frac{k+l}{2}\right)+{}_{c+d-l^+}I_{\psi}f\left(b,c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{c+d-k^-}I_{\psi}f\left(a+b-n,c+d-\frac{k+l}{2}\right)+{}_{c+d-k^-}I_{\psi}f\left(b,c+d-\frac{k+l}{2}\right)\right] \\
 & -\frac{\sigma(\sigma-\rho)}{\Lambda_1\left(\frac{1}{2}\right)}\left[2{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+2{}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right] \\
 & -\frac{\sigma\rho}{\Lambda_1\left(\frac{1}{2}\right)}\left[{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2},c\right)+{}_{a+b-n^+}I_{\varphi}f\left(a+b-\frac{m+n}{2},d\right)\right. \\
 & \left.+{}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2},c\right)+{}_{a+b-m^-}I_{\varphi}f\left(a+b-\frac{m+n}{2},d\right)\right] \\
 & +\frac{4\sigma^2}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)}\left[{}_{a+b-n^+,c+d-l^+}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{a+b-n^+,c+d-k^-}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{a+b-m^-,c+d-l^+}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right. \\
 & \left.+{}_{a+b-m^-,c+d-k^-}I_{\varphi,\psi}f\left(a+b-\frac{m+n}{2},c+d-\frac{k+l}{2}\right)\right]\Bigg\} \\
 & \leq\frac{(n-m)(l-k)}{\Lambda_1\left(\frac{1}{2}\right)\Lambda_2\left(\frac{1}{2}\right)}\left\{\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left|\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right|^p\left|\sigma\Lambda_2(s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right|^p dt ds\right)^{\frac{1}{p}}\right. \\
 & \times\left(\int_0^{\frac{1}{2}}\int_0^{\frac{1}{2}}\left[ts\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-k)\right.\right. \\
 & \left.\left.+t(1-s)\frac{\partial^2}{\partial t\partial s}f(b,c+d-l)+(1-t)s\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-k)\right.\right. \\
 & \left.\left.+ (1-t)(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-l)\right]^q dt ds\right)^{\frac{1}{q}} \\
 & \left.+\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left|\sigma\Lambda_1(t)-\rho\Lambda_1\left(\frac{1}{2}\right)\right|^p\left|\sigma\Lambda_2(1-s)-\rho\Lambda_2\left(\frac{1}{2}\right)\right|^p dt ds\right)^{\frac{1}{p}}\right. \\
 & \times\left(\int_{\frac{1}{2}}^1\int_0^{\frac{1}{2}}\left[ts\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-k)\right.\right. \\
 & \left.\left.+t(1-s)\frac{\partial^2}{\partial t\partial s}f(a+b-m,c+d-l)+(1-t)s\frac{\partial^2}{\partial t\partial s}f(a+b-n,c+d-k)\right.\right.
 \end{aligned}$$

$$\begin{aligned}
 & + {}_{a+b-n^+, c+d-k^-} J_{\alpha, \beta; k} f \left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2} \right) \\
 & + {}_{a+b-m^-, c+d-l^+} J_{\alpha, \beta; k} f \left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2} \right) \\
 & + {}_{a+b-m^-, c+d-k^-} J_{\alpha, \beta; k} f \left(a+b-\frac{m+n}{2}, c+d-\frac{k+l}{2} \right) \Big] \Bigg| \\
 & \leq (n-m)(l-k) \left\{ \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left| \sigma t^\alpha - \rho \left(\frac{1}{2} \right)^\alpha \right|^p \left| \sigma s^\beta - \rho \left(\frac{1}{2} \right)^\beta \right|^p dt ds \right)^{\frac{1}{p}} \right. \\
 & \times \left(\int_0^{\frac{1}{2}} \int_0^{\frac{1}{2}} \left[ts \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-k) \right. \right. \\
 & \left. \left. + t(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-m, c+d-l) + (1-t)s \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-k) \right. \right. \\
 & \left. \left. + (1-t)(1-s) \frac{\partial^2}{\partial t \partial s} f(a+b-n, c+d-l) \right]^q dt ds \right)^{\frac{1}{q}} \Bigg\}
 \end{aligned}$$

By the same way, we can get some remarks about Theorem 3.9 and Theorem 3.10 with Riemann-Liouville, k-Riemann-Liouville, and other generalized fractional integrals.

4. Conclusions

In this paper, we have introduced and studied parameterized Simpson-Mercer-type inequalities for twice partially differentiable functions of two variables via generalized fractional integral operators. By deriving suitable integral identities for the mixed partial derivatives and exploiting co-ordinated convexity together with Hölder-type techniques, we obtained several families of Simpson-Mercer-type bounds involving pairs of free parameters and double generalized fractional integrals.

The parameterized nature of our inequalities offers additional flexibility for both theoretical investigations and practical applications. On the theoretical side, different parameter choices generate a continuum of bounds with varying strengths, which can be adapted to the geometry of the domain and to the regularity of the function under consideration. On the applied side, the derived estimates may be used to obtain error bounds for numerical quadrature formulas and to study qualitative properties of solutions to fractional differential and integral equations defined on rectangular domains.

Possible directions for further research include extending the present methodology to other families of hybrid inequalities, such as Hermite-Hadamard-Mercer and Ostrowski-Mercer-type results, and exploring applications to more general classes of convexity as well as to fractional models arising in applied sciences.

Data Availability Statement

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Funding Statement

This research received no external funding.

Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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