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Generalized Inverses in Jordan Algebras

Mustapha Laayouni

Department of Mathematics, Faculty of Sciences and Technology, University Moulay Ismail, Errachidia, Morocco
Email: mustaphalaayouni@yahoo.fr

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Abstract

The article introduces the concept of generalized inverses in Jordan algebras as an analog to the classical generalized inverses in the absence of the associativity. We define nonassociative generalized inverses based on quadartic operator U_a, explore properties of these notion (e.g., existence, characterization, coincidence with the notion of generalized inverse when Jordan algebra is special or associative, compactness of the generalized spectrum introduced here), and prove results analogous to those in associative algebras, including a Jordan version of the generalized resolvent and a conorm.

Keywords

Associative Algebra, Jordan Algebra, Generalized Inverse, Generalized Spectrum, Conorm

1. Introduction

A complete theory of Jordan's algebras was found in Gauss's unpublished papers. Their first appearance in history seems to be in the early 1930s when the theory burst into maturity from the minds of Pascual Jordan, John von Neumann, and Eugene Wigner in their 1934 paper, On an algebric generalization of the quantum mechanical formalism [1].

Jordan took as his axioms the existence of a bilinear product $x \circ y$ on a complex vector space satisfying the identities:

$$(J1)\ \ x \circ y = y \circ x\ \ (\text{commutative law}),$$

$$(J2)\ \ \left(x^2 \circ y\right) \circ x = x^2 \circ \left(y \circ x\right)\ \ (\text{Jordan identity}).$$

A better example of Jordan algebras is obtained by replacing the product of an associative algebra A xy by the Jordan's product $x \circ y = \frac{1}{2}(xy + yx)$, for all $x, y \in A$. The Jordan algebra obtained will be noted A^+. Any Jordan algebra isomorphic to a sub-algebra of such an algebra A^+ is called a special Jordan algebra. Otherwise, it is called exceptional Jordan algebra. A deep and useful result in the theory of Jordan algebras is the following theorem, due to Shirshov and Cohn ([2], Theorem 3.1.55).

Theorem 1.1 *Any Jordan algebra generated by two elements* (*and 1*, *if unital*) *is special.*

The extension of any notion, in particular generalized inverse, known in the associative case to Jordan's algebras must take into consideration this gateway from A to A^+. For this purpose, we will use the quadratic operator:

For an element a in a Jordan algebra J, we denote by U_a the mapping $b \mapsto a \circ (a \circ b) - a^2 \circ b$ from J to J and note that when J is special, $U_a(b) = aba$. This operator satisfies the fundamental formula $U_{U_a(b)} = U_a U_b U_a$ for Jordan algebras which was derived from the so-called Macdonald's theorem.

It is well known that invertibility plays a fundamental role in functional analysis. Hence, the importance of developing the notions of generalized inverse and generalized spectral theory in the Jordan algebras.

2. Generalized Inverse in a Jordan Algebra

In order to give, in the Jordan case, an adequate definition of generalized invertibility, we must remember two prerequisites: First, this definition will allow us to find the one already known in the associative case, since a large class of Jordan algebras come from associative algebras by replacing the initially associative product by that of Jordan. Second, it must simplify the extension of spectral analysis when the algebra in question is endowed with an Jordan-Banach norm. We are also inspired by the works [3] and [4] where the authors used the quadratic operator instead of the multiplication operator.

Definition 2.1 *Let* J *be a Jordan algebra and let* a *be in* J. *An element* $c \in J$ *is called a generalized inverse of* a *if the equality* $a = U_a(c)$ *holds.*

For example, if b is the inverse of a, then b is a generalized inverse of a. If $a = a^2$ is a projection, then $a = U_a(c)$ holds with $c = a$.

In view of the above fundamental formula, this definition emphasizes that, the operator U_a satisfies

$$U_a = U_{U_a(c)} = U_a U_c U_a$$

The above definition generalizes the associative forerunner: Suppose that J is a special Jordan algebra A^+. Then the definition 1.1 is formulated here as follows.

$$a = U_a(b) = aba$$

which is the well-known generalized inversibility of a in the associative algebra A (see [5]-[8]). Note also that $U_a(b) = a$ is actually expressed by saying that a is *von Neumann regular.*

Proposition 2.1 *If* c *is a generalized inverse of* $a \in J$ *then the set of all generalized inverses of* a *consists of all elements of the form* $x = c + u - U_c U_a(u)$, *where* u *is arbitrary in* J.

Proof. Suppose that $a = U_a(c)$ and $x = c + u - U_c U_a(u)$ for some $u \in J$ then

$$\begin{aligned} U_a(x) &= U_a(c + u - U_c U_a(u)) \\ &= U_a(c) + U_a(u) - U_a(U_c U_a(u)) \\ &= U_a(c) + U_a(u) - U_{U_a(c)}(u) \\ &= U_a(c) + U_a(u) - U_a(u) \\ &= U_a(c) \\ &= a \end{aligned}$$

Conversely, if $U_a(x) = a$ then the operator $P = U_c U_a$ is a projector, so $J = Im(P) \oplus Ker(P)$ with

$$P(x - c) = U_c U_a(x - c) = U_c(U_a(x) - U_a(c)) = U_c(a - a) = 0$$

It follows that the element $u = x - c$ satisfies $x = c + u - U_c U_a(u)$.

Proposition 2.2 *Let* a *be an element of* J *admitting a generalized inverse* c *then there exists an element* b *in* J *such that*

$$a = U_a(b) \quad \text{and} \quad b = U_b(a)$$

Proof. It is a direct consequence of the fundamental formula.

Throughout the following, $B(J)$ denotes the algebra of operators of J.

Proposition 2.3 *If an element* a *of* J *admits a generalized inverse* b *then* U_b *is a generalized inverse of* U_a *in the associative algebra* $B(J)$

Proof. Suppose that $a, b \in J$ satisfy $a = U_a(b)$. Then

$$U_a U_b U_a = U_{U_a(b)} = U_a$$

Then U_b is a generalized inverse of U_a in $B(J)$.

In order to state a converse of this proposition, we have:

Proposition 2.4 *If* $a \in J$ *and* $T \in B(J)$ *satisfy* $U_a T U_a = U_a$ *and* $T U_a T = T$ *then* $V = U_a T$ (*respectively,* $W = T U_a$) *is a projection of* J *upon* $Im(V)$ (*respectively,* $Im(W)$) *in the direction of* $Ker(V)$ (*respectively,* $Ker(W)$).

Proof. Obvious.

Now we need to involve in our development a deep result in the theory of generalized invertibility. This is to unify the different approaches to generalized inverses in the algebraic context, which reads as follows.

Proposition 2.5 *If* $a \in J$ *and* $T \in B(J)$ *satisfy* $U_a T U_a = U_a$ *and* $T U_a T = T$ *then* T *is the unique operator of* J *solution of the system* $U_a X = U_a T$, $X U_a = T U_a$ *and* $X U_a X = X$.

Proof. It is enough to apply ([7], Theorem 1.3(a)) with $P = I - T U_a$ and $Q = U_a T$ to realize that conclusion in the proposition is fulfilled. We have, $P^2 = (I - W)^2 = I - 2W + W^2 = I - W = P$ and $U_a(P(x)) = U_a(x) - U_a T U_a(x) = 0$ for all $x \in J$. Then $P(J) \subset Ker(U_a)$, likewise $Q^2 = Q$ and $Im(Q) = Im(U_a)$. So, the conditions of ([7], Theorem 1.3(a)) that we want to use for our proof are verified.

Theorem 2.1 *Let* a *be not a divisor of zero in* J *and suppose the operator* U_a *admits a generalized inverse* T *in* $B(J)$. *Then the following assertions are satisfied*:

i) a admits a generalized inverse $b = T(a)$.

ii) $T = U_{T(a)}$.

Proof. Proposition 2.4 allows us to obtain that $a = U_a T(u) + v$ and $a = TU_a(y) + z$, for unique quadruple (u, v, y, z) in J with $v \in Ker(U_a T)$ and $z \in Ker(TU_a)$.

Hence $T(a) = T(U_a T(u) + v) = TU_a T(u) + T(v) = T(u) + T(v)$ and $U_a T(a) = U_a T(u) + U_a T(v) = U_a T(u) + 0 = U_a T(u)$. So, $a = U_a T(a) + v$, $v \in Ker(U_a T)$. On the other hand, $T(a) = T(U_a T(a) + v) = TU_a T(a) + T(v) = T(a) + T(v)$ then $T(v) = 0$.

Similarly, $a = TU_a(a) + z$ with $U_a(z) = 0$. Since a is not a divisor of zero then $z = 0$ and $a = TU_a(a)$.

Now, by considering $R = 2I - V - W + T + U_a \in B(J)$, we have.

$$\begin{aligned} R(a) &= 2I(a) - V(a) - W(a) + T(a) + U_a(a) \\ &= 2a - a + v - W(P(a)) + W(v) + T(a) + U_a(a) \\ &= a + v - W(P(a)) + W(v) + T(a) + U_a(a) \end{aligned} \tag{2.1}$$

$$\begin{aligned} R(V(a)) &= 2I(V(a)) - V^2(a) - Q(V(a)) + T(V(a)) + U_a(V(a)) \\ &= V(a) - W(V(a)) + T(V(a)) + U_a(V(a)) \\ &= a - v - W(V(a)) + T(a) - T(v) + U_a(a) - U_a(v) \\ &= a - v - W(V(a)) + T(a) + U_a(a) - U_a(v) \end{aligned} \tag{2.2}$$

$$\begin{aligned} R(v) &= 2v - V(v) - W(v) + T(v) + U_a(v) \\ &= 2v - W(v) + U_a(v) \end{aligned}$$

Then

$$\begin{aligned} & a + v - W(V(a)) + W(v) + T(a) + U_a(a) \\ &= R(a) = R(V(a)) + R(v) \\ &= a - v - W(V(a)) + T(a) + U_a(a) - U_a(v) + 2v - W(v) + U_a(v) \\ &= a + v - W(V(a)) + T(a) + U_a(a) - W(v) \end{aligned}$$

So, $W(v) = -W(v)$ and $W(v) = 0$. Since $U_a = U_a TU_a = U_a W$, then

$$U_a(v) = (U_a W)(v) = U_a(W(v)) = 0$$

Since a is not a divisor of zero then $v = 0$ and $a = U_a(T(a))$. Then $c = T(a)$ is a generalized inverse of a. We claim that

$$U_a U_{T(a)} U_a = U_a \text{ and } U_{T(a)} U_a U_{T(a)} = U_{T(a)}$$

To prove the claim, the equality $U_a(T(a)) = a$ and the fundamental formula give us $U_a U_{T(a)} U_a = U_a$ and, multiplying on the right by $U_{T(a)}$, we have

$U_a\left(U_{T(a)}U_aU_{T(a)}\right)=\left(U_aU_{T(a)}U_a\right)U_{T(a)}=U_aU_{T(a)}$. Since U_a is injective, we have $U_{T(a)}U_aU_{T(a)}=U_{T(a)}$, as desired.

Finally, in order to prove assertion (ii), note that it suffices to show that $U_aU_{T(a)}(x)=U_aT(x)$, for every $x\in J$. The conclusion follows from the fact that U_a is injective. Assume at first that $x=U_a(\alpha)\in Im(U_a)$. Then

$$\begin{aligned}U_aT(x)&=U_aTU_a(\alpha)\\&=U_a(\alpha)\\&=U_aU_{T(a)}U_a(\alpha)\\&=U_a\left(U_{T(a)}U_a(\alpha)\right)\\&=U_aU_{T(a)}(x)\end{aligned}$$

To conclude the proof, we must show that the same conclusion holds for an arbitrary element $x\in J$. Since U_aT and $U_aU_{T(a)}$ are projectors satisfying

$$Im(U_aT)=Im\left(U_aU_{T(a)}\right)=Im(U_a)$$

$$J=Im(U_aT)\oplus Ker(U_aT)\ \text{ and }\ J=Im\left(U_{T(a)}\right)\oplus Ker\left(U_{T(a)}\right)$$

Decomposing x into these two direct sums respectively, $x=\alpha+v$ and $x=\beta+w$, we have:

$$U_aT(x)=U_aT(\alpha)=\alpha$$

$$U_aU_{T(a)}(x)=U_aU_{T(a)}(\beta)=\beta$$

It follows from the first case that $U_aT(x)=U_aT(\beta)+U_aT(w)=\beta+U_aT(w)$ and $U_aU_{T(a)}(x)=U_aU_{T(a)}(\alpha)+U_aU_{T(a)}(v)=\alpha+U_aU_{T(a)}(v)$. So,

$$\begin{cases}\alpha=\beta+U_aT(w)\\ \beta=\alpha+U_aU_{T(a)}(v)\end{cases}$$

then

$$\begin{cases}U_aT(w)+U_aU_{T(a)}(v)=0\\ 2(\alpha-\beta)=U_aT(w)-U_aU_{T(a)}(v)\end{cases}$$

So, $\alpha-\beta=U_aT(w)$ which leads to $w-v=U_aT(w)$ and $x=\alpha+w-U_aT(w)$. Furthermore, if we set $P=U_aT$ and $Q=U_aU_{T(a)}$ and if we apply the operator $Z=2I-P-Q+U_{T(a)}+T$ to the two decompositions of x, then we will obtain

$$\begin{aligned}Z(x)&=Z\left(\alpha+w-P(w)\right)\\&=Z(\alpha)+Z(w)-Z\left(P(w)\right)\\&=U_{T(a)}(\alpha)+T(\alpha)+Z(w)+U_{T(a)}\left(P(w)\right)+T\left(P(w)\right)\end{aligned}$$

and

$$
\begin{aligned}
Z(x) &= Z(\beta + w) \\
&= Z(\beta) + Z(w) \\
&= U_{T(a)}(\beta) + T(\beta) + Z(w)
\end{aligned}
$$

Then $U_{T(a)}(\alpha)+T(\alpha)+Z(w)+U_{T(a)}(P(w))+T(P(w))=U_{T(a)}(\beta)+T(\beta)$ $+Z(w)$ and $U_{T(a)}(\alpha)+T(\alpha)+U_{T(a)}(P(w))+T(P(w))=U_{T(a)}(\beta)$ $+T(\beta)$. Finally, by applying the linear operator U_a to both members of these last equality we derive that

$$U_aU_{T(a)}(\alpha)+U_aT(\alpha)+U_aU_{T(a)}U_aT(w)+U_aTU_aT(w)=U_aU_{T(a)}(\beta)+U_aT(\beta)$$

and consequently

$$2\alpha + 2P(w) = 2\beta$$

Hence, we obtain that

$$\alpha - \beta = -P(w)$$

Thus, $P(w) = -P(w) = 0$, and then $U_aT(x) = \alpha = \beta = U_aU_{T(a)}(x)$ as desired. Then $T = U_{T(a)}$.

Keeping in mind the arguments and notions above, the following theorem follows easily.

Theorem 2.2 *Let* J *be a Jordan algebra over* K ($K = \mathbb{R}$ *or* $K = \mathbb{C}$) *and let* a *be an element in* J. *We have:*

(i) If a possesses a generalized inverse b in J, then the operator U_b is a generalized inverse of U_a in $B(J)$.

(ii) If a is not a divisor of zero in J and suppose the operator U_a admits a generalized inverse T in $B(J)$. Then $T(a)$ is a generalized inverse of a in J.

Note that (ii) of the theorem 2.2 shows that generalized inverses of a verifies the well-known equality satisfied by the inverse a^{-1} when it exists in the Jordan-algebra J: $U_a^{-1} = U_{a^{-1}}$ (see [2], Theorem 4.1.3).

Proposition 2.6 *Let* $J = A^+$ *be a special Jordan algebra. For every* $a \in A$ *the following conditions are equivalent:*

i) a admits a generalized inverse in *A*.

ii) a admits a generalized inverse in *J*.

Proof. Let $a \in A$, by the definition of the operator U_a in J, we have

a has a generalized inverse in A

$\Leftrightarrow$ there existes $b \in A$ such that $aba = a$ and $bab = b$

$\Leftrightarrow$ there existes $b \in J$ such that $U_a(b) = a$ and $U_b(a) = b$

$\Leftrightarrow$ a has a generalized inverse in J

Proposition 2.7 *If an element* a *non-divisor of zero in* J *admits a generalized inverse then the operator* U_a *is left invertible in* $B(J)$.

Proof. It follows from Proposition 2.2 that there exists $b \in J$ such that $a = U_a(b)$ and $U_b(a) = b$. Which is said in terms of operators that

$U_aU_bU_a = U_a$ and $U_bU_aU_b = U_b$. So the projector U_bU_a satisfies $J = Im(U_bU_a) \oplus Ker(U_bU_a)$ and $Ker(U_bU_a) = Ker(U_a) = \{0_J\}$. Then we have that U_bU_a is invertible, that is to say, if we write $TU_bU_a = U_bU_aT = Id_J$, from which we deduce that U_a is left invertible.

In the following section, J denotes a complex unital Jordan-Banach algebra with unit 1_J. For $a \in J$, $\sigma(a)$, $\varrho(a)$, and $r(a)$ denote the spectrum, the resolvent set and the spectral radius of a, respectively (see [2], Chapiter 4).

Recall that

$$\lambda \in \varrho(a) \Leftrightarrow a - \lambda 1_J \text{ is invertible}$$

Note that $R(a,\lambda) = (a - \lambda 1_J)^{-1}$ the resolvent of a at point λ is analytic on the open $\varrho(a)$.

3. Generalized Spectral Theory in a Jordan-Banach Algebra

Spectrum theory and spectral analysis play a fundamental role in functional analysis. Thus, we would like to define the generalized spectrum where the notion of inverse is replaced by the generalized inverse. Given an element a of J, the natural idea is to define the generalized spectrum of a by considering

$$S(a) = \{\lambda \in \mathbb{K} : a - \lambda 1_J \text{ does not admit a generalized inverse in } J\}$$

But this definition does not preserve the main properties of classical spectral analysis even in associative cases ([5], p. 70). The same defects are obtained by considering the author's special Jordan algebra. In other words, $S(a)$ can be empty for an element a of J and especially the famous spectral mapping theorem is missing. In the following, we extend to Jordan-Banach algebras the notion of generalized spectral analysis defined and studied in associative algebras by numerous authors.

Definition 3.1 *Let Ω be a part of $\mathbb{C}$, we will say that a admits a generalized resolvent $Rg(a,\mu)$ in Ω if for all $\mu \in \Omega$, there exists $Rg(a,\mu)$ in J such that*

$$U_{(a-\mu I)}(Rg(a,\mu)) = a - \mu I \quad \textit{and} \quad U_{Rg(a,\mu)}(a - \mu I) = Rg(a,\mu)$$

Definition 3.2 *An element a of J admits a generalized resolvent $Rg(a,\lambda)$ in an open U of $\mathbb{C}$ if*

$U_{a-\lambda 1_J}(Rg(a,\lambda)) = a - \lambda 1_J$ *and* $U_{Rg(a,\lambda)}(a - \lambda 1_J) = Rg(a,\lambda)$ *for every* $\lambda \in U$.

In this definition, the condition (which may seem tedious) of the existence of an open U on which $a - \lambda 1_J$ admits a generalized resolvent is automatically ensured within the framework of classical invertibility. Indeed, the set of invertible elements in J (denoted $Inv(J)$) is open and the mapping $x \mapsto x^{-1}$ from $Inv(J)$ to J is differentiable at any point $a \in Inv(J)$, with derivative equal to the mapping $-U_{a^{-1}}$ (See [2], Theorem 4.1.7).

We will also use the following.

Definition 3.3 *The generalized resolvent set or the regular set of $a \in J$ (denoted by $Rg(a)$) is the subset of $\mathbb{C}$ formed of numbers λ such that a ad-*

mits an analytical generalized resolvent $Rg(a,\mu)$ *in a neighborhood* Ω *of* λ*. which means*

$$\lambda \in Rg(a) \Leftrightarrow a \text{ admits an analytical generalized resolvent in a neighborhood } \Omega \text{ of } \lambda$$

Example 3.1 Let A be the Banach algebra of operators of a Hilbert space H, J is the special Jordan-Banach algebra A^+. It follows from Proposition2.6 above and ([5], p. 71) that λ is in $Rg(a)$ if and only if

$$Im(A-\lambda I) \text{ is closed and } \ker(A-\lambda I)^n \subset Im(A-\lambda I)$$

Definition 3.4 *The generalized spectrum of an element* a *of* J (*relative to* J) *which will be denoted by* $\sigma_g(a)$, *is the complement in* $\mathbb{C}$ *of* $Rg(a)$.

As the complement of an open, $\sigma_g(a)$ is closed. Obviously $\sigma_g(a)$ is contained in the compact $\sigma(a)$, so $\sigma_g(a)$ is a compact (since we will show that it is never empty).

As another relation between these two spectra of an element a of J we have the following:

Proposition 3.1 *Let* a *in* J *and* $\partial\sigma(a)$ *denotes the boundary of the spectrum* $\sigma(a)$ *then*

$$\partial\sigma(a) \subset \sigma_g(a)$$

Proof. Assume, to derive a contradiction, that the proposition is not true. Then there exists $\lambda_0 \in \partial\sigma(a) \cap Rg(a)$. So, a admits an analytical generalized resolvent $Rg(a,\mu)$ in a neighborhood Ω of λ_0 which coincides with the analytical resolvent $R(a,\mu)=(a-\mu)^{-1}$ on the non-empty open set $\Omega \cap R(a)$. So, $(a-\lambda_0)^{-1} = R(a,\lambda_0)$ exists, which is a contradiction.

Tow easy, but not so straightforward results, are the following.

Corollary 3.1 *For each* $a \in J$, *the generalized spectrum* $\sigma_g(a)$ *of* a *is not empty.*

Corollary 3.2 *If for* $a \in J$ *its generalized spectrum is at most countable then its two spectrums are equal:* $\sigma(a) = \sigma_g(a)$.

Proof. Since $\sigma_g(a) \subset \sigma(a)$, it suffices to prove the reciprocal inclusion. Now $\sigma(a)$ is at most countable, then it coincides with its boundary which is contained in $\sigma_g(a)$, as desired.

Proposition 3.2 *Let* a *in* J *and* $\mathcal{K}$ *denotes a connected component of* $Rg(a)$. *Then*

$$\mathcal{K} \cap R(a) \neq \varnothing \Rightarrow \mathcal{K} \subset R(a)$$

Proof. Assume the existence of $\lambda_0 \in \mathcal{K} \cap R(a)$.

So, a admits an analytical generalized resolvent $Rg(a,\mu)$ in a neighborhood Ω of λ_0 which coincides with the analytical resolvent $R(a,\mu)=(a-\mu)^{-1}$ on the non-empty open set $\Omega \cap R(a)$. So, $(a-\lambda_0)^{-1} = R(a,\lambda_0)$ exists, which is a contradiction.

Proposition 3.3 *Let* a *in* J *and assume that* $\sigma_g(a)$ *is at most countable.*

Then

$$\sigma_g(a) = \sigma(a)$$

Proof. The fact that $\sigma_g(a)$ is at most countable implies $Rg(a)$ is connected. it is enough to replace $\mathcal{K}$ with $Rg(a)$ in proposition 3.2 to realize that $Rg(a) = \varrho(a)$ and then $\sigma_g(a) = \sigma(a)$.

Now we define the conorm in a Jordan-Banach algebra, along with some associated results.

Definition 3.5 *The conorm* $\gamma(a)$ *of* $a \in J$ *is defined by:*

$$\gamma(a) = \inf\left\{\|U_a(x)\| : d(x, \ker(U_a)) = 1\right\}, \text{ if } a \neq 0$$

and

$$\gamma(0) = +\infty$$

Note that the connorm of a in J defined here coincides with the classical connorm of the operator U_a in the Banach algebra $B(J)$.

Proposition 3.4 *If* a *is invertible in* J *then* $\gamma(a) = \|U_a^{-1}\|^{-1}$.

Proof. According to Definition 3.5, we have

$$\begin{aligned}
\gamma(a) &= \inf\left\{\|U_a(x)\| : d(x, \ker(U_a)) = 1\right\} \\
&= \inf\left\{\|U_a(x)\| : \|x\| = 1\right\} \\
&= \inf\left\{\frac{\|U_a(x)\|}{\|x\|} : x \neq 0\right\} \\
&= \inf\left\{\frac{\|U_a(x)\|}{\|U_a^{-1}(U_a(x))\|} : x \neq 0\right\} \\
&= \left[\sup\left\{\frac{\|U_a^{-1}(y)\|}{\|y\|} : y \neq 0\right\}\right]^{-1} \\
&= \|U_a^{-1}\|^{-1}
\end{aligned}$$

Theorem 3.1 *If an element* a *of* J *has a generalized inverse* b *in* J *then:*

(i) $U_a(J) = \{U_a(x) : x \in J\}$ is closed;

(ii) $\gamma(a) > 0$:

(iii) if $b \in J$ is a generalized inverse of a, then

$$\frac{1}{\gamma(a)} \leq \|U_b\|$$

Proof. Let $b \in J$ such that $U_a(b) = a$. Then $U_a = U_{U_a(b)} = U_a U_b U_a$. Put $P = U_a U_b$ and $Q = I - U_b U_a$. Then it is easy to see that $P^2 = P$ and $Q^2 = Q$. Since $U_a(J) = U_a U_b U_a(J) = PU_a(J)$ then

$$PU_a(J) \subset U_a(J) \subset P(J) \subset U_a(J)$$

and $U_a(J)=P(J)$ is closed. This shows (i).

To prove assertion (ii) it is enough to show that $\ker(U_a)=Q(J)$. Use (i) and [9] (Satz 55.2) to see that (ii) holds.

Now take $x\in J$ such that $d(x,\ker(U_a))=1$. Since $U_a(x)=U_aU_bU_a(x)$, $x-U_bU_a(x)\in\ker(U_a)$, thus

$$1=d(x,\ker(U_a))=d(U_bU_a(x),\ker(U_a))\le\|U_b\|\|U_a(x)\|$$

Hence

$$\frac{1}{\|U_b\|}\le\|U_a(x)\|\text{ for all }x\in J\text{ with }d(x,\ker(U_a))=1$$

This gives $\frac{1}{\|U_b\|}\le\gamma(a)$.

Since $\|U_b\|\le3\|b\|^2$, it follows that $\frac{1}{3\|b\|^2}\leqslant\gamma(a)$, for all generalized inverse b of a.

It follows from the above proposition that $\gamma(a)=\|U_a^{-1}\|^{-1}$ when a is invertible, this shows that the mapping $a\to\gamma(a)$ is not continuous. This justifies the hypothesis of the existence of the limit in the following.

Proposition 3.5 *Let* a *in* J *and* λ_0 *be a complex number. Assume that* $\lim\limits_{\lambda\to\lambda_0}\gamma(a-\lambda 1_J)$ *exists. Then the following assertions are equivalent:*

(i) $\lambda_0\in Rg(a)$

(ii) $\lim\limits_{\lambda\to\lambda_0}\gamma(a-\lambda 1_J)>0$

Proof. If $\lambda_0\in Rg(a)$, then there exists $r_0>0$ such that the open disc $D(\lambda_0,2r_0)$ is contained in $Rg(a)$. Using the remark $\gamma(a-\lambda 1_J)=d(\lambda,\sigma_g(a))$ mentioned above, therefore $\gamma(a-\lambda 1_J)\ge r_0$ for all $\lambda\in D(\lambda_0,r_0)$. Hence $\lim\limits_{\lambda\to\lambda_0}\gamma(a-\lambda 1_J)>0$.

Conversely, if $r=\lim\limits_{\lambda\to\lambda_0}\gamma(a-\lambda 1_J)>0$, then the open disc $D\left(\lambda_0,\frac{r}{2}\right)$ is contained in $Rg(a)$, and the implication (ii) $\Rightarrow$ (i) is proved.

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Conflicts of Interest

The author declares no conflicts of interest regarding the publication of this paper.

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